Imperial College London

QFFF MSc TEST January 2015

This paper is also taken for the relevant Examination for the Associateship

QUANTUM FIELD THEORY — TEST

For QFFF MSc Students

Monday, 12th January 2015: 14:00 to 16:00

Answer ALL questions.
Please answer each question in a separate book.

This test does not contribute to the grade for this course. It is only to provide feedback to students.

Operators may be written without 'hats' so you will need to deduce what is an operator from the context.

(10/2/2015) *Version with post-test corrections and mark scheme revisions.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. In this question we will work in the Heisenberg picture. A free real scalar field in the Heisenberg picture, $\hat{\phi}(t, \boldsymbol{x})$, is defined to be the operator

$$\hat{\phi}(t, \boldsymbol{x}) \quad = \quad \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(\hat{a}_{\boldsymbol{p}} \mathrm{e}^{-i\omega_p t + i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{p}} \mathrm{e}^{i\omega_p t - i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \,. \tag{1}$$

$$\text{where } \omega_p = \left| \sqrt{\boldsymbol{p}^2 + m^2} \right| \geq 0 \tag{2}$$

(i) Show that $\hat{\phi}^\dagger = \hat{\phi}$. [3 marks]

(ii) Show that if we define the conjugate momentum to be $\Pi = \partial\phi/\partial t$ then in the Heisenberg picture we have

$$\widehat{\Pi}(t, \boldsymbol{x}) \quad = \quad -i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left(\hat{a}_{\boldsymbol{p}} \mathrm{e}^{-i\omega_p t + i\boldsymbol{p}\cdot\boldsymbol{x}} - \hat{a}^\dagger_{\boldsymbol{p}} \mathrm{e}^{i\omega_p t - i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \tag{3}$$

[3 marks]

(iii) Assume that the annihilation and creation operators obey the commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}} \right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \,, \qquad \left[\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}} \right] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}} \right] = 0 \,. \tag{4}$$

Show that the ϕ field and its conjugate Π obey their canonical commutation relations

$$\left[\hat{\phi}(t, \boldsymbol{x}), \widehat{\Pi}(t, \boldsymbol{y}) \right] \quad = \quad i\delta^3(\boldsymbol{x} - \boldsymbol{y}) \,, \tag{5}$$

$$\left[\hat{\phi}(t, \boldsymbol{x}), \hat{\phi}(t, \boldsymbol{y}) \right] = \left[\widehat{\Pi}(t, \boldsymbol{x}), \widehat{\Pi}(t, \boldsymbol{y}) \right] \quad = \quad 0 \,. \tag{6}$$

[9 marks]

(iv) Show that if the operator for energy, the Hamiltonian $\widehat{H}$, is

$$\widehat{H} = \int d^3\boldsymbol{x} \left(\frac{1}{2} \widehat{\Pi}^2 + \frac{1}{2} (\nabla \hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi}^2 \right) \tag{7}$$

then

$$\widehat{H} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left(\omega_k \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} \right) + (\text{constant}) \,. \tag{8}$$

[9 marks]

(v) Given the Hamiltonian of a free field (8) and the commutation relations for the annihilation and creation operators (4)show that

$$\left[\widehat{H}, \hat{a}_{\boldsymbol{p}} \right] = -\omega_{\boldsymbol{p}} \hat{a}_{\boldsymbol{p}} \,. \tag{9}$$

[4 marks]

(vi) Using the commutator (9), prove by induction that

$$(\widehat{H})^n a_{\boldsymbol{p}} = \hat{a}_{\boldsymbol{p}} (\widehat{H} - \omega_{\boldsymbol{p}})^n \,. \tag{10}$$

You may assume that if $\widehat{A}$ and $\widehat{B}$ are two operators that commute, then $e^{\widehat{A}} e^{\widehat{B}} = e^{\widehat{A} + \widehat{B}}$ (special case of the Baker-Campbell-Hausdorf identity). [4 marks]

[This question continues on the next page . . .]

(vii) Show that

$$e^{i\widehat{H}t}\hat{a}_{\boldsymbol{p}}e^{-i\widehat{H}t} = \hat{a}_{\boldsymbol{p}}e^{-i\omega_p t}\,. \tag{11}$$

[4 marks]

(viii) Operators in the Schrödinger picture ($\mathcal{O}_{\mathrm{S}}$) are related to operators in the Heisenberg picture ($\mathcal{O}_{\mathrm{H}}(t)$) by $\mathcal{O}_{\mathrm{H}}(t) = e^{i\widehat{H}t}\mathcal{O}_{\mathrm{S}}e^{-i\widehat{H}t}$. Show that the form (1) is consistent with this equation. [4 marks]

[Total 40 marks]

2. The retarded propagator (two-point Green function) for the real scalar field $\phi(x)$ is defined as

$$D_{\mathrm{R}}(x) = \theta(x^0)\langle 0|\left[\hat{\phi}(x), \hat{\phi}(0)\right]|0\rangle \tag{1}$$

Consider this propagator in the Heisenberg picture for a free field theory where the field is given as

$$\hat{\phi}(t,\boldsymbol{x}) \;=\; \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(\hat{a}_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\,, \tag{2}$$

$$\text{where } \omega_p = \left|\sqrt{\boldsymbol{p}^2+m^2}\right| \geq 0 \tag{3}$$

(i) Starting from the definition of the retarded propagator $D_{\mathrm{R}}(x)$ given in (1), show this may be rewritten as

$$D_{\mathrm{R}}(x) = \int \frac{d^4 p}{(2\pi)^4}\,\frac{i}{(p_0+i\epsilon)^2-(\omega_{\boldsymbol{p}})^2}e^{-ip_0 t+i\boldsymbol{p}\cdot\boldsymbol{x}}\,, \quad \text{where } \omega_{\boldsymbol{p}}^2 = \boldsymbol{p}^2+m^2\,. \tag{4}$$

The integrals are from $-\infty$ to $+\infty$ along the real axis and ϵ is a real, positive but infinitesimal quantity. [10 marks]

(ii) Show that

$$(\partial^2+m^2)D_{\mathrm{R}}(x) = -i\delta^4(x)$$

[10 marks]

[Total 20 marks]

3. In this question all fields are in the **Interaction picture**.

The scalar Yukawa theory for real scalar field ϕ of mass m and complex scalar field ψ with mass M has a cubic interaction

$$H_{\text{int}} = g \int d^3\mathbf{x}\, \psi^\dagger(x)\psi(x)\phi(x) \tag{1}$$

where the coupling constant g is real and a measure of the interaction strength.

Consider the Green function in coordinate space which contributes to $\psi\psi \to \psi\psi$ scattering, namely

$$G(y_1, y_2, z_1, z_2) = \langle 0|\mathrm{T}\left(\psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2)S\right)|0\rangle = \sum_{n=0}^{\infty} G_n\,, \qquad G_n \sim O(g^n) \tag{2}$$

where $S = \mathrm{T}(\exp\{-i\int dt H_{\text{int}}\})$ is the S matrix, T(fields) is the time ordering operator, and G_n is the $O(g^n)$ term in the perturbative expansion of this Green function.

(i) State the Feynman rules for this theory needed when calculating Green functions in coordinate space. [5 marks]

(ii) Derive an expression for the $O(g^0)$ term G_0.
Draw the two Feynman diagrams which correspond to G_0. [5 marks]

(iii) Explain why $G_n = 0$ if n is odd using Wick's theorem. (*Hint* no detailed calculations are needed here).
Interpret this in terms of Feynman diagrams. [5 marks]

(iv) Draw all the Feynman diagrams which contribute to G_2, the $O(g^2)$ term in the expansion of G. There are three distinct types of diagram in G_2.

(a) Identify which of the diagrams for G_2 contain vacuum diagrams.

(b) Identify which of the diagrams for G_2 represent self-energy contributions to a full propagator.

(c) Hence identify the remaining two diagrams which are the only ones describing non-trivial scattering in the quantum theory at this order.

[20 marks]

(v) (originally the previous part was split over two parts). [0 marks]

(vi) Write down the contribution to G_2 coming from the two diagrams found in part (c) above which are the only ones describing non-trivial scattering. This should be an algebraic expression in terms of g, the propagators and should contain any appropriate symmetry factors. [5 marks]

[Total 40 marks]