QFT New Year Test 2016 Solutions[1]

27/1/2016

1. In this question we consider a free field theory work in the Heisenberg picture using natural units ($\hbar = c = 1$).

For simplicity the H subscript and hats on operators may be dropped when answering in this question.

The time evolution of any operator $\widehat{\mathcal{O}}_{\rm H}$ in the Heisenberg picture is given by

$$\mathcal{O}_{\rm H}(t) \quad = \quad \exp\{+i\widehat{H}_{\rm H}t\}\, \mathcal{O}_{\rm H}(t=0)\, \exp\{-i\widehat{H}_{\rm H}t\}\,. \tag{1}$$

while states are time-invariant in the Heisenberg picture. The operator $\widehat{H}_{\rm H}$ is the Hamiltonian in the Heisenberg picture and it is independent of time.

Note that in answering this question you should be very careful to specify what is a three- or four-vector and what value the zero-th index of an energy-momentum four-vector might have, that is do you have some general real values p_0 or is $p_0 = \omega_{\boldsymbol{p}}$ for some four-vector p^μ. Is px short for $p^\mu p_\mu$ or $\boldsymbol{p}.\boldsymbol{x}$? Define your notation carefully, use $\vec{p}$ or $\underline{p}$ rather than the boldface $\mathbf{p}$ I often use for three-vectors Students were very loose in their notation when answering this and lost marks.

(i) A free real scalar field in the Heisenberg picture is given by

$$\hat{\phi}_{\rm H}(t, \boldsymbol{x}) \quad = \quad \int \frac{{\rm d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(\hat{a}_{\boldsymbol{p}} {\rm e}^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{p}} {\rm e}^{i\omega_{\boldsymbol{p}} t - i\boldsymbol{p}\cdot\boldsymbol{x}} \right) , \tag{2}$$

$$\text{with } \omega_p = \left| \sqrt{\boldsymbol{p}^2 + m^2} \right| \geq 0\,. \tag{3}$$

The annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}} \right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q})\,, \qquad [\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}} \right] = 0\,. \tag{4}$$

Since $\pi = \dot{\phi} = \partial_t \phi$ classically, we try this on the operator (2) to find the momentum operator $\hat{\pi}(t, \boldsymbol{x})$ conjugate to the field $\hat{\phi}(t, \boldsymbol{x})$ of (2).

$$\hat{\pi}(t, \boldsymbol{x}) \quad = \quad -i \int \frac{{\rm d}^3 p}{(2\pi)^3} \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left(\hat{a}_{\boldsymbol{p}} {\rm e}^{-i\omega_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}} - \hat{a}^\dagger_{\boldsymbol{p}} {\rm e}^{i\omega_{\boldsymbol{p}} t - i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \tag{5}$$

The equal time commutation relations for a real scalar field operator $\hat{\phi}$ and its conjugate momentum $\hat{\pi}$ are

$$\left[\hat{\phi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{y}) \right] = i\hbar \delta^3(\boldsymbol{x} - \boldsymbol{y})\,, \qquad \left[\hat{\phi}(t, \boldsymbol{x}), \hat{\phi}(t, \boldsymbol{y}) \right] = [\hat{\pi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{y})] = 0\,. \tag{6}$$

though we work with $\hbar = 1$ in this question.

To show that this real scalar field and its conjugate momentum satisfy the equal time commutation relations *at any time*[2]

[1] LaTeX'd January 27, 2016

[2] The question specifically asked for a commutator that depended on time t. It was not acceptable to show the result for one time, say $t = 0$.

The first field commutation relation is then (setting $p_0 = \omega_p$, $q_0 = \omega_q$, and $x_0 = y_0 = t$)

$$\left[\hat{\phi}(t,\boldsymbol{x}), \hat{\pi}(t,\boldsymbol{y})\right] = \left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}), \int \frac{d^3q}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{q}}}{2}}(\hat{a}_{\boldsymbol{q}}e^{-iqx} - \hat{a}^\dagger_{\boldsymbol{q}}e^{+iqx})\right] \quad (7)$$

$$= (-i)\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{4\omega_{\boldsymbol{q}}}} \times \left(-\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right]e^{-i(\omega_p-\omega_q)t+i\boldsymbol{p}.x-i\boldsymbol{q}.y} + \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right]e^{+i(\omega_p-\omega_q)t-i\boldsymbol{p}.x+i\boldsymbol{q}.y}\right) \quad (8)$$

where we have used the factor that

$$[A+B, C+D] = [A,C] + [A,D] + [B,C] + [B,D] \quad (9)$$

and the fact that two of the commutators are zero by (4). This leaves us with

$$[\phi(t=0,\boldsymbol{x}), \pi(t=0,\boldsymbol{y})] = (-i)\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{4\omega_{\boldsymbol{q}}}} \quad (10)$$

$$\times \left(-(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})e^{-i(\omega_p-\omega_q)t+i\boldsymbol{p}.x-i\boldsymbol{q}.y} + (2\pi)^3(-\delta^3(\boldsymbol{p}-\boldsymbol{q}))e^{+i(\omega_p-\omega_q)t-i\boldsymbol{p}.x+i\boldsymbol{q}.y}\right) \quad (11)$$

$$= i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left(e^{i\boldsymbol{p}.(x-y)} + e^{-i\boldsymbol{p}.(x-y)}\right) = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \quad (12)$$

This is the desired value from (6).

(ii) The commutator of the free scalar fields $\hat{\phi}$ of (2) at different space-time points x and y is

$$\Delta_{\mathrm{C}}(x-y) = \left[\hat{\phi}(x), \hat{\pi}(y)\right]$$

$$= \left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}), \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}(\hat{a}_{\boldsymbol{q}}e^{-iqy} + \hat{a}^\dagger_{\boldsymbol{q}}e^{+iqy})\right] \quad (13)$$

where we set $p_0 = |\omega_{\boldsymbol{p}}|$ and $q_0 = |\omega_{\boldsymbol{q}}|$. So we find

$$\Delta_{\mathrm{C}}(x-y) = \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}}\left[(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}), (\hat{a}_{\boldsymbol{q}}e^{-iqy} + \hat{a}^\dagger_{\boldsymbol{q}}e^{+iqy})\right] \quad (14)$$

$$= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}}\left(e^{-ipx+iqy}\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] + e^{+ipx-iqy}\left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}\right]\right) \quad (15)$$

$$= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}} \times \left(e^{-ipx+iqy}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q}) - e^{+ipx-iqy}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\right) \quad (16)$$

So finally we have

$$\Rightarrow \Delta_{\mathrm{C}}(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(e^{-ip(x-y)} - e^{+ip(x-y)}\right) \quad (17)$$

as required.

From (17) we have that the commutator of the field at equal times is

$$\Delta_{\mathrm{C}}(t=0,\boldsymbol{x}) = \left[\hat{\phi}(t,\boldsymbol{x}), \hat{\pi}(t,0)\right] = I_1 - I_2\,, \quad (18)$$

$$I_1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(e^{+i\boldsymbol{p}.\boldsymbol{x}}\right)\,, \quad (19)$$

$$I_2 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(e^{-i\boldsymbol{p}.\boldsymbol{x}}\right)\,, \quad (20)$$

where $\boldsymbol{x}$ is a three-space position and $\boldsymbol{p}$ a three-momentum. Taking the second term I_2 we have three integrals of the form

$$I_2 = \left(\prod_{i=1,2,3} \int_{-\infty}^{+\infty} \frac{dp_i}{(2\pi)} e^{-ip_i . x_i} \right) \frac{1}{2\omega_{\boldsymbol{p}}} \,. \tag{21}$$

Changing integration variables to $p'_i = -p_i$ we have

$$I_2 = \left(\prod_{i=1,2,3} \int_{+\infty}^{-\infty} \frac{-dp'_i}{(2\pi)} e^{+ip'_i . x_i} \right) \frac{1}{2\omega_{\boldsymbol{p}'}} \tag{22}$$

where we use the fact that ω is only a function of $|\boldsymbol{p}|$ and note there is a change in the range of integration and a factor of -1 in the integration measure. Those cancel each other to leave the second term of (18) as

$$I_2 = \left(\prod_{i=1,2,3} \int_{-\infty}^{+\infty} \frac{dp'_i}{(2\pi)} e^{+ip'_i . x_i} \right) \frac{1}{2\omega_{\boldsymbol{p}'}} \tag{23}$$

which is exactly the same as the first term in (18). The commutator Δ_C is the difference of these two terms giving us zero for the equal time case $\Delta_\mathrm{C}(t = 0, \boldsymbol{x})) = 0$. This is indeed consistent with the ETCR.

(iii) From the expression for the commutator, Δ_C of (17), we have that

$$\begin{aligned} D_A(x) &= -\theta(-x^0) \langle 0|[\phi(x), \phi(0)]|0\rangle = -\theta(-x^0)\Delta_C(x) & (24)\\ &= -\theta(-x^0) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left(\frac{1}{2\omega_{\boldsymbol{p}}} \mathrm{e}^{-ip\cdot x}\Big|_{p^0=\omega_{\boldsymbol{p}}} + \frac{1}{(-2\omega_{\boldsymbol{p}})} \mathrm{e}^{-ip\cdot x}\Big|_{p^0=-\omega_{\boldsymbol{p}}} \right) & (25) \end{aligned}$$

We now need to introduce a p_0 integration and rewrite the expression in terms of a contour integration. Here we shift the poles of the integrand, introducing a small positive infinitesimal ϵ into the integrand which is taken to zero (from the positive side) at the end of the calculation. This is the approach[3] used in the lectures and it is common practice to use this notation, especially in the case of the time-ordered (Feynman) propagator.

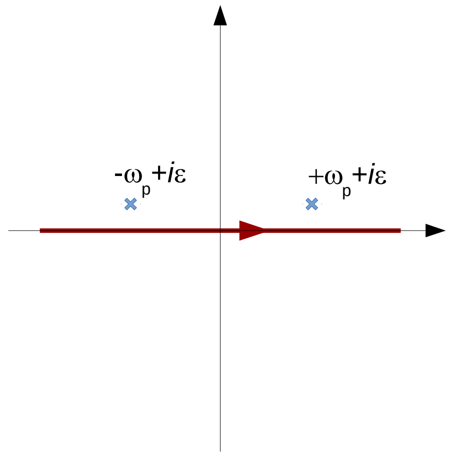
Consider

$$\begin{aligned} I_1(x) &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i}{(p_0 - i\epsilon)^2 - \boldsymbol{p}^2 - m^2} \mathrm{e}^{-ip\cdot x} & (26)\\ &= -\int \frac{\mathrm{d}^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi i} \frac{1}{(p^0 - \omega_{\boldsymbol{p}} - i\epsilon)(p^0 + \omega_{\boldsymbol{p}} - i\epsilon)} \mathrm{e}^{-ip\cdot x} & (27) \end{aligned}$$

where, as usual, $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. The p_0 integration is along the real axis with poles in the

[3] A second approach is to make small distortions in the contour away from the real p_0 axis near the poles. This is used by Tong in his derivation of the Feynman propagator (sec.2.7.1 page 38) though Tong reverts to the first and standard notation later on (see Tong equation (3.37)). Both methods are equivalent in the $\epsilon \to 0^+$ limit.

integrand as shown here



(28)

The $\mathrm{d}p^0$ integrand

$$f(p^0, \boldsymbol{p}) = \frac{1}{(p^0 - \omega_{\boldsymbol{p}} - i\epsilon)(p^0 + \omega_{\boldsymbol{p}} - i\epsilon)} \, \mathrm{e}^{-ip\cdot x} \tag{29}$$

has simple poles at

$$p^0 = \pm\omega_{\boldsymbol{p}} + i\epsilon. \tag{30}$$

Near these poles the integrand looks like $f \approx R_\pm/(p^0 \mp \omega_{\boldsymbol{p}} + i\epsilon)$ with residues $R_\pm$ given by

$$R_\pm = \pm\frac{1}{2\omega_{\boldsymbol{p}}} \, \mathrm{e}^{-ip\cdot x}\big|_{p^0=\pm\omega_{\boldsymbol{p}}+i\epsilon} \tag{31}$$

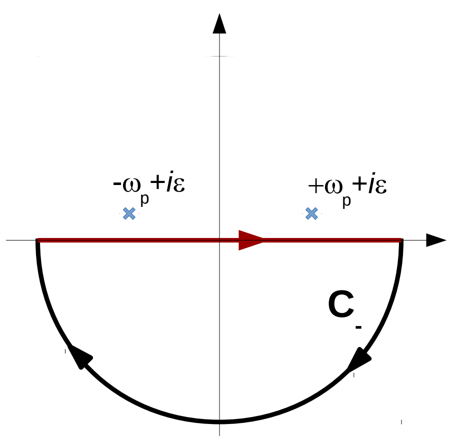
The idea is that we think of our expression for the advanced propagator in (25) as being of the form

$$D_A(x) = -\theta(-x^0) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(R_+ + R_-\right) . \tag{32}$$

In order for this to match $I_1(x)$ of (27) we need to find a closed contour C such that by using the residue theorem we can deduce that

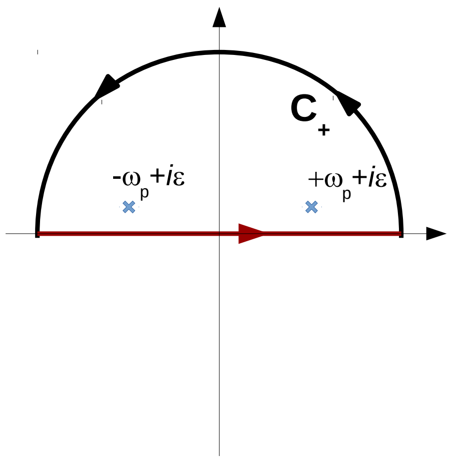
$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = \theta(-x^0) \left(R^+ + R^-\right) \tag{33}$$

If $x^0 > 0$ then $\mathrm{e}^{-ip^0 x^0} \to 0$ as $p^0 \to -i\infty$. This means that an integration of this integrand f round a large semi-circle running around the lower half plane is equal to zero. We can therefore add this integration of f to our p_0 integration along the real axis in I_1 without changing the result for I_1. So we produce an expression for I_1 which uses a closed contour for the p_0 integration by adding this lower semi-circle. Now no poles are enclosed within this closed contour so the residue theorem tells us the result is zero



$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = 0 \qquad \text{if } x^0 > 0 \, . \tag{34}$$

If $x^0 < 0$ then $\mathrm{e}^{-ip^0 x^0} \to 0$ as $p^0 \to i\infty$. This means that an integration of this integrand f round a large semi-circle running around the upper half plane will give zero. We can therefore add this to our existing p_0 integration along the real axis in I_1 without changing the result. So we produce a closed contour by adding the semi-circle above and now the residue theorem tells us that we pick up contributions from both poles. This gives us



$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = R_+ + R_- \qquad \text{if } x^0 < 0\,. \tag{35}$$

Putting the two cases together gives us the desired result

$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = \theta(-x^0)\left(R^+ + R^-\right)\,. \tag{36}$$

2. The full Hamiltonian $\widehat{H}$ in any picture is split into two parts: $\widehat{H} = \widehat{H}_0 + \widehat{H}_{\rm int}$ where $\widehat{H}_0$ is the free Hamiltonian and $\widehat{H}_{\rm int}$ is the interaction Hamiltonian. The relationship between the Schrödinger (subscript S) and Interaction pictures (subscript I) for any state ψ and for any operator $\widehat{\mathcal{O}}$ is given by

$$\begin{aligned} |\psi, t\rangle_{\rm I} &= \exp\{+iH_{0,{\rm S}}t\}\,|\psi, t\rangle_{\rm S}\,, & (1)\\ \widehat{\mathcal{O}}_{\rm I}(t) &= \exp\{+iH_{0,{\rm S}}t\}\,\widehat{\mathcal{O}}_{\rm S}\exp\{-iH_{0,{\rm S}}t\}\,. & (2) \end{aligned}$$

(i) The Schrödinger equation,

$$i\frac{d}{dt}|\psi, t\rangle_{\rm S} = \widehat{H}_{\rm S}\,|\psi, t\rangle_{\rm S}\,, \tag{3}$$

gives the time evolution of Schrödinger picture states as

$$|\psi, t\rangle_{\rm S} = \exp\{-i\widehat{H}_{\rm S}t\}\,|\psi, t=0\rangle_{\rm S}\,. \tag{4}$$

We can check (4) by substituting into (3). We may now insert (4) into (1) to find

$$|\psi, t\rangle_{\rm I} = \exp\{+iH_{0,{\rm S}}t\}\,|\psi, t\rangle_{\rm S} = \exp\{+i\widehat{H}_{0,{\rm S}}t\}\exp\{-i\widehat{H}_{\rm S}t\}\,|\psi, t=0\rangle_{\rm S}\,. \tag{5}$$

Here $\widehat{H}_{\rm S} = \widehat{H}_{0,{\rm S}} + \widehat{H}_{{\rm int},{\rm S}}$ is split of the Hamiltonian into free and interacting parts in the Schrödinger picture.

Use the relation between Schrödinger and Interaction states (1) to replace the former in the Schrödinger equation (3) to find

$$i\frac{d}{dt}\exp\{-iH_{0,{\rm S}}t\}|\psi, t\rangle_{\rm I} = \widehat{H}_{{\rm int},{\rm I}}\;\exp\{-iH_{0,{\rm S}}t\}|\psi, t\rangle_{\rm I}\,. \tag{6}$$

Looking at the right hand side shows that

$$i\frac{d}{dt}\left(\exp\{-iH_{0,\mathrm{S}}t\}|\psi,t\rangle_{\mathrm{I}}\right) = i\frac{d}{dt}\left(\exp\{-iH_{0,\mathrm{S}}t\}\right)|\psi,t\rangle_{\mathrm{I}} + i\exp\{-iH_{0,\mathrm{S}}t\}\frac{d}{dt}\left(|\psi,t\rangle_{\mathrm{I}}\right) \quad (7)$$

$$= H_{0,\mathrm{S}}\exp\{-iH_{0,\mathrm{S}}t\}|\psi,t\rangle_{\mathrm{I}} + i\exp\{-iH_{0,\mathrm{S}}t\}\frac{d}{dt}\left(|\psi,t\rangle_{\mathrm{I}}\right) \quad (8)$$

Putting this back into (6) and rearranging gives

$$i\exp\{-iH_{0,\mathrm{S}}t\}\frac{d}{dt}\left(|\psi,t\rangle_{\mathrm{I}}\right) = -H_{0,\mathrm{S}}\exp\{-iH_{0,\mathrm{S}}t\}|\psi,t\rangle_{\mathrm{I}} + \widehat{H}_{\mathrm{int,I}}\ \left(\exp\{-iH_{0,\mathrm{S}}t\}|\psi,t\rangle_{\mathrm{I}}\right). \quad (9)$$

$$= \left(\widehat{H}_{\mathrm{S}} - H_{0,\mathrm{S}}\right)\exp\{-iH_{0,\mathrm{S}}t\}|\psi,t\rangle_{\mathrm{I}} \quad (10)$$

Premultiplying by $\exp\{+iH_{0,\mathrm{S}}t\}$ and using $\exp\{-iH_{0,\mathrm{S}}t\}\exp\{+iH_{0,\mathrm{S}}t\}$ (proof not needed here, but BCH and operators in exponents commute does it) gives

$$i\frac{d}{dt}|\psi,t\rangle_{\mathrm{I}} = \exp\{+iH_{0,\mathrm{S}}t\}\widehat{H}_{\mathrm{int,S}}\exp\{-iH_{0,\mathrm{S}}t\}|\psi,t\rangle_{\mathrm{I}} \quad (11)$$

$$= \widehat{H}_{\mathrm{int,I}}(t)|\psi,t\rangle_{\mathrm{I}} \quad (12)$$

where we use the relationship between Schrödinger and Interaction picture operators (2) to reach the last line which is what we require.

(ii) To solve this look at an infinitesimal step in time, $|\epsilon| \ll 1$, so that (12) becomes

$$\frac{i}{\epsilon}\left(|\psi,t+\epsilon\rangle_{\mathrm{I}} - |\psi,t\rangle_{\mathrm{I}}\right) = \widehat{H}_{\mathrm{int,I}}(t)|\psi,t\rangle_{\mathrm{I}} \quad (13)$$

$$\Rightarrow \quad |\psi,t+\epsilon\rangle_{\mathrm{I}} = (1-i\epsilon)\widehat{H}_{\mathrm{int,I}}(t)|\psi,t\rangle_{\mathrm{I}} \approx \exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t)\}|\psi,t\rangle_{\mathrm{I}} \quad (14)$$

Repeating this we get

$$|\psi,t+2\epsilon\rangle_{\mathrm{I}} \approx \exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t+\epsilon)\}\exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t)\}|\psi,t\rangle_{\mathrm{I}}\,. \quad (15)$$

$$|\psi,t+n\epsilon\rangle_{\mathrm{I}} \approx \exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t+(n-1)\epsilon)\}\ldots\exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t+\epsilon)\}\exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t)\}|\psi,t\rangle_{\mathrm{I}} \quad (16)$$

It is important to remember here that the $\widehat{H}_{\mathrm{int,I}}(t)$ is not invariant over time so the time argument of each $\widehat{H}_{\mathrm{int,I}}(t)$ must be carefully noted. In addition the $\widehat{H}_{\mathrm{int,I}}(t)$ need not commute at different times so the order of operators must be carefully preserved.

So far the analysis is valid for any infinitesimal time step ϵ. However now we come to an important choice missed by many answering this question and not always in the text books. There are now two choices: (A) $\epsilon > 0$ and $\lim \epsilon \to 0^+$, (B) $\epsilon < 0$ and $\lim \epsilon \to 0^-$.

We were asked for an evolution operator from an early time t_1 to a later time t_2. This will explain the condition $t_2 > t_1$ highlighted in this question. For that reason we need small positive steps forward in time, so $\epsilon > 0$ and we will consider case (A). We will comment on case (B) at the end.

Choosing case (A) and keeping $\epsilon > 0$, what we then have is that the operators are time-ordered where the T time-ordering operator puts operators in the order of their time argument (largest times to the left). That is

$$|\psi,t+n\epsilon\rangle_{\mathrm{I}} \approx \mathrm{T}\left(\prod_{j=0}^{n-1}\exp\{-i\epsilon\widehat{H}_{\mathrm{int,I}}(t+j\epsilon)\}\right)|\psi,t\rangle_{\mathrm{I}}\,. \quad (17)$$

$$\approx \mathrm{T}\left(\exp\{-i\sum_{j=0}^{n-1}\epsilon\widehat{H}_{\mathrm{int,I}}(t+j\epsilon)\}\right)|\psi,t\rangle_{\mathrm{I}}\,. \quad (18)$$

Note that we can only replace the product of exponentials of operators by the exponential of the sum of operators because the operator ordering is fixed by the T operator. That is we are using the given identity that $\mathrm{T}\left(\exp\{\widehat{A}\}\exp\{\widehat{B}\}\right) = \mathrm{T}\left(\exp\{\widehat{A}+\widehat{B}\}\right)$ (we normally get corrections as given by BCH for such an operation). Now we can take the $\epsilon \to 0^+$ limit

$$|\psi, t_2\rangle_\mathrm{I} \;=\; \mathrm{T}\left(\exp\{-i\int_t^{t_2} dt'\ \widehat{H}_{\mathrm{int,I}}(t')\}\right)|\psi, t\rangle_\mathrm{I}\,,\ \text{where } t_2 > t_1\,. \tag{19}$$

Note the $t_2 > t_1$ follows here from the fact that ϵ is positive.
As this is true for any state, by comparing with

$$|\psi, t_2\rangle_\mathrm{I} \;=\; \widehat{U}(t_2, t_1)\,|\psi, t_1\rangle_\mathrm{I} \tag{20}$$

we see that we have

$$\widehat{U}(t_2, t_1) = \mathrm{T}\left(\exp\{-i\int_{t_1}^{t_2} dt'\ \widehat{H}_{\mathrm{int,I}}(t')\}\right)\ \text{if } t_2 > t_1\,. \tag{21}$$

This next part is not needed for this question but is provided for completeness. Now if you follow case (B) where $\epsilon < 0$, the logic is exactly the same but we see we are now evolving backwards in time, we have anti-time ordering denoted by an the operator $\tilde{\mathrm{T}}$ which puts the operators in time order with earliest (latest) times to the left (right). The solution of U is the same but with anti-time ordering and $t_2 < t_1$ required. That means the true full solution for U is in fact

$$\widehat{U}(t_2, t_1) \;=\; \theta(t_2 - t_1)\mathrm{T}\left(\exp\{-i\int_{t_1}^{t_2} dt'\ \widehat{H}_{\mathrm{int,I}}(t')\}\right) \tag{22}$$

$$+\theta(t_1 - t_2)\tilde{\mathrm{T}}\left(\exp\{-i\int_{t_1}^{t_2} dt'\ \widehat{H}_{\mathrm{int,I}}(t')\}\right)\,. \tag{23}$$

(iii) Wick's theorem for a theory with a single real scalar field states that the time ordered product of such fields is equal to a sum over normal ordered products of the field where fields are contracted in all possible ways. That is

$$\begin{aligned}
\mathrm{T}\left(\hat\phi_1\hat\phi_2\ldots\hat\phi_n\right) \;=\;& \mathrm{N}\left(\hat\phi_1\hat\phi_2\ldots\hat\phi_n\right) \\
&+\sum_{(i,j)}\mathrm{N}\left(\hat\phi_1\hat\phi_2\ldots\overbrace{\hat\phi_i\ldots\hat\phi_j}\ldots\hat\phi_n\right) \\
&+\sum_{(i,j),(k,l)}\mathrm{N}\left(\hat\phi_1\hat\phi_2\ldots\overbrace{\hat\phi_i\ldots\hat\phi_j}\ldots\overbrace{\hat\phi_k\ldots\hat\phi_l}\ldots\hat\phi_n\right) \\
&+\qquad\vdots
\end{aligned} \tag{24}$$

Here

- $\hat\phi_j = \hat\phi(x_j)$,
- T(fields) orders field operators according to their time with the latest time on the left,
- N(fields) is normal ordering of fields where, for a given split of fields $\hat\phi_i = \hat\phi_i^+ + \hat\phi_i^-$, $\hat\phi_i^+$ are moved to the right of all $\hat\phi_i^-$,
- the contraction may be defined by the case of two fields

$$\overbrace{\hat\phi_1\hat\phi_2} = \Delta_{12} = \mathrm{T}(\hat\phi_1\hat\phi_2) - \mathrm{N}(\hat\phi_1\hat\phi_2)\,. \tag{25}$$

Many students did not give these definitions when answering the question despite the explicit wording of the question.

Sometimes students gave the definition in terms of a split into annihilation and creation operators, a common definition used for normal ordering but not the most general. The more limited definition was not requested here, the discussion was meant to be general. Note that many texts may talk as if there is only one definition of normal ordering, but such texts are just misleading.

Likewise several students also gave the definition of the contraction as the Feynman propagator (the usual $i/(p^2 - m_i^2 \epsilon)$ in momentum space) which is not in general correct and did not get full marks. While the Feynman propagator is often equal to the contraction, it depends on both the expectation value being considered and on the choice of normal ordering. In this question at this point, neither was specified. For the purposes of Wicks theorem the fundamental definition, the only self-consistent definition, is that given here in terms of the two field case of Wick's theorem. The content of Wick's theorem is that all the higher order versions with $N > 2$ fields, are be reexpressed in term of the two-field Wick's theorem result which serves to define the contraction.

Consider a theory with a single real scalar field of mass m and an interaction Hamiltonian of the form $\widehat{H}_{\text{int}} = (\lambda/4!) \int d^3\boldsymbol{x}\, \hat{\phi}^4$. The expression for the two-point Greens function

$$G(y,z) = \langle 0|\mathrm{T}\hat{\phi}_{\mathrm{I}}(y)\hat{\phi}_{\mathrm{I}}(z)\hat{S}|\,0\rangle \tag{26}$$

up to and including first order in λ is

$$\begin{aligned} G(y,z) &= \langle 0|\mathrm{T}\left(\hat{\phi}(y)\hat{\phi}(z)\hat{S}\right)|\,0\rangle && (27)\\ &\approx \langle 0|\mathrm{T}\left(\hat{\phi}(y)\hat{\phi}(z)\right)|\,0\rangle \\ &\quad -i\frac{\lambda}{4!}\int dt \int d^3\boldsymbol{x} \langle 0|\mathrm{T}\left(\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\right)|\,0\rangle + O(\lambda^2) && (28) \end{aligned}$$

Here $|\,0\rangle$ is the vacuum of the free (non-interacting) theory and the S matrix is $\hat{S} = \widehat{U}(+\infty, -\infty)$ and we have dropped the I subscript as all quantities are in the Interaction picture.

We now choose normal ordering so that $\langle 0|\mathrm{N}\,(\text{fields})\,|\,0\rangle = 0$. This is an *essential* step but many students forget to specify this. In fact we did not require anything else in this question. You did not need to define what split achieves this, nor the value of the contraction Δ when we make this choice (the answer is given in terms of Δ). Of course you may know what is required in this case and you could also specify your choice. What you could not do was not mention this.

Then the lowest order term comes from Wick's theorem for two fields, the definition of a contraction in (25), that is

$$G_0(y,z) = \overbrace{\hat{\phi}(x)\hat{\phi}(y)} = \Delta(x-y)\,. \tag{29}$$

Taking the vacuum expectation value of this confirms that the propagator is

$$\Delta(x-y) = \langle 0|\mathrm{T}\left(\hat{\phi}(y)\hat{\phi}(z)\right)|\,0\rangle\,. \tag{30}$$

The $O(\lambda)$ term is

$$\begin{aligned} G_1(y,z) &= -i\frac{\lambda}{4!}\int dt \int d^3\boldsymbol{x} \langle 0|\mathrm{T}\left(\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\right)|\,0\rangle && (31)\\ &= -i\frac{\lambda}{4!}\int dt \int d^3\boldsymbol{x} \sum_{\text{three contractions}} \left(\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\right) && (32) \end{aligned}$$

There are two types of term here.
First we can contract $\hat{\phi}(y)$ with $\hat{\phi}(z)$ which leaves us with three[4] ways to contract the $\hat{\phi}(x)$ fields. One example is

$$\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\,. \tag{33}$$

These all give the same factor and so adding them up we find a total contribution of the form

$$\begin{aligned} G_{1\mathrm{vac}}(y,z) &= -3i\frac{\lambda}{4!}\int dt \int d^3\boldsymbol{x}\Delta(y-z)\Delta(x-x)\Delta(x-x) && (34)\\ &= -i\frac{\lambda}{8}(VT)\Delta(y-z)[\Delta(0)]^2 && (35)\end{aligned}$$

where (VT) is a space-time volume factor associated with a vacuum diagram contribution. Note the 8 in the denominator is the symmetry factor of the vacuum subdiagram.
The only alternative is to contract $\hat{\phi}(y)$ one of the $\hat{\phi}(x)$ terms (four ways to do this), then we contract $\hat{\phi}(z)$ with one of the three remaining $\hat{\phi}(x)$ terms (three ways to do this) which leaves us then with no choice but to contract the last two $\hat{\phi}(z)$. One example of this second type of term is

$$\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\,. \tag{36}$$

These all give the same factor and all such terms summed up give an overall contribution of

$$\begin{aligned} G_{1\mathrm{se}}(y,z) &= -(4.3)i\frac{\lambda}{4!}\int dt \int d^3\boldsymbol{x}\Delta(y-x)\Delta(z-x)\Delta(x-x) && (37)\\ &= -i\frac{\lambda}{2}\Delta(0)\int d^4x\ \Delta(y-x)\Delta(z-x) && (38)\end{aligned}$$

Note the propagators are symmetric so $\Delta(y-x) = \Delta(x-y)$ etc. Note the 2 in the denominator is the symmetry factor of the tadpole self-energy diagram.

3. The scalar Yukawa theory for real scalar field ϕ of mass m and complex scalar field ψ with mass M has a cubic interaction with real coupling constant g (a measure of the interaction strength) with Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu\psi^\dagger)(\partial^\mu\psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x)\,. \tag{1}$$

In the interaction picture, the field operators take the form

$$\begin{aligned} \phi(x) &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \quad p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+m^2}\right|, && (2)\\ \psi(x) &= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}}e^{ipx}\right), \quad p_0 = \Omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+M^2}\right|, && (3)\end{aligned}$$

where the annihilation and creation operators obey their usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \qquad [\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{4}$$

Both the $\hat{b}$, $\hat{b}^\dagger$ pair and the $\hat{c}$ and $\hat{c}^\dagger$ pair of annihilation and creation operators obey similar commutation relations to those of the $\hat{a}$ and $\hat{a}^\dagger$ pair. Different types of annihilation and creation operator always commute e.g. $\left[\hat{a}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}\right] = 0$.

[4] Many students and my own first draft of these answers had six here. Think again. Yes symmetry factors are difficult.

We are considering the the case of $\psi\psi \to \psi\psi$ scattering with incoming ψ particles of three-momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ while the outgoing ψ particles have three-momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. The matrix element is $\mathcal{M}$ where

$$\mathcal{M} = \langle f|S|\, i\rangle = \langle q_1, q_2|S|\, p_1, p_2\rangle = \sum_{n=0}^{\infty} \mathcal{M}_n\,, \qquad \mathcal{M} \sim O(g^n)\,. \tag{5}$$

(i) Substituting in the form (3) for the field ψ, you find that

$$\left(\int d^3\boldsymbol{y}\, \exp\{-ipy\} 2\Omega(\boldsymbol{p})\right) \hat{\psi}(y)|\,0\rangle = \int \frac{đ^3\boldsymbol{k}}{\sqrt{2\Omega(\boldsymbol{k})}} 2\Omega(\boldsymbol{p}) \int d^3\boldsymbol{y}\, e^{-i(p-k)y} \tag{6}$$

$$= \int \frac{đ^3\boldsymbol{k}}{\sqrt{2\Omega(\boldsymbol{k})}} 2\Omega(\boldsymbol{p}) \delta^3(\boldsymbol{p}-\boldsymbol{k})\, e^{-i(\Omega(p)-\Omega(k))t} \hat{a}^\dagger_{\boldsymbol{k}}|\,0\rangle \tag{7}$$

$$= \sqrt{2\omega(\boldsymbol{p})}|\,\psi(\boldsymbol{p})\rangle = |\,\psi(p)\rangle \tag{8}$$

where $t = y^0$. The $|\,\psi(p)\rangle$ state is the one ψ particle state with the appropriate normalisation for relativistic calculations while $|\,\psi(\boldsymbol{p})\rangle = \hat{b}^\dagger_{\boldsymbol{k}}|\,0\rangle$ has the standard normalisation usually encountered when first looking at QHO.

We can repeat the process for the second ψ particle but now we need to introduce labels on the momenta of incoming particles, p_1 and p_2.

$$|\,\psi(p_1), \psi(p_2)\rangle = \sqrt{2\omega(\boldsymbol{p}_1)}\sqrt{2\omega(\boldsymbol{p}_2)}|\,\psi(\boldsymbol{p}_1), \psi(\boldsymbol{p}_2)\rangle \tag{9}$$

$$= \left(\int d^3\boldsymbol{y}_1\, \exp\{-ip_1y_2\} 2\Omega(\boldsymbol{p}_1)\right)\left(\int d^3\boldsymbol{y}_2\, \exp\{-ip_2y_2\} 2\Omega(\boldsymbol{p}_2)\right) \hat{\psi}(y_1)\hat{\psi}(y_2)|\,0\rangle \tag{10}$$

$$= \prod_{i=1,2}\left(\int d^3\boldsymbol{y}_i\, \exp\{-ip_iy_i\} 2\Omega(\boldsymbol{p}_i)\right) \hat{\psi}(y_1)\hat{\psi}(y_2)|\,0\rangle \tag{11}$$

Taking the hermitian conjugate will give us the final state in this case, provided we also switch the labels for momenta and coordinates appropriately. That is we have

$$\langle\psi(q_1), \psi(q_2)| = \prod_{f=1,2}\left(\int d^3\boldsymbol{z}_f\, \exp\{+iq_fz_f\} 2\omega(\boldsymbol{q}_f)\right) \langle 0|\hat{\psi}(z_1)\hat{\psi}(z_2) \tag{12}$$

From our expression (5), we then have that the relationship between the matrix element $\mathcal{M}$ and the relevant Green function for this $\psi\psi \to \psi\psi$ scattering process in Scalar Yukawa theory is just

$$\mathcal{M}(\phi \to \bar{\psi}\psi) = \prod_{f=1,2}\left(\int d^3\boldsymbol{z}_f\, \exp\{+iq_fz_f\} 2\omega(\boldsymbol{q}_f)\right) \prod_{i=1,2}\left(\int d^3\boldsymbol{y}_i\, \exp\{-ip_iy_i\} 2\Omega(\boldsymbol{p}_i)\right) \times G(z_1, z_2, y_1, y_2) \tag{13}$$

$$G(z_1, z_2, y_1, y_2) = \langle 0|\mathrm{T}\psi(z_1)\psi(z_2)\psi(y_1)\psi(y_2)S|\,0\rangle \tag{14}$$

where the order you write the operators in the vacuum expectation value is irrelevant as that is fixed by the time ordering.

(ii) The Feynman rules to calculate the Green functions in coordinate space for the scalar Yukawa theory of (1) are as follows.

Rule zero is that we draw all topologically distinct graphs (i.e. Feynman diagrams) where edges (lines) run between pairs of vertices subject to the rules below. Do not forget this rule (many did when answering this). The others are

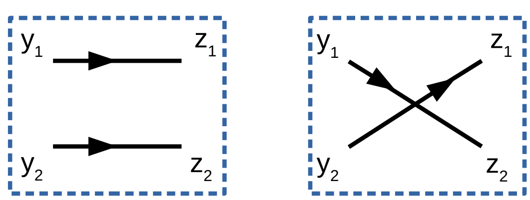
1. Each line is associated with a factor of $\Delta_f(x-y)$ for a field f of mass m_f where $\Delta_f(x-y)$ is the relevant Feynman propagator (in momentum space this is $\Delta(p) = i(p^2 - m_f^2 + i\epsilon)^{-1}$) and x and y are the coordinates associated with vertices at the end of each line.
   (a) For field ϕ this is a scalar propagator $\Delta_\phi(x-y)$ with mass $m_f = m$ and will be denoted by a dashed line with no arrows as ϕ is its own anti-particle.
   (b) For field ψ this is a scalar propagator $\Delta_\psi(x-y)$ with mass $m_f = M$ and will be denoted by a solid line with an arrow from a $\bar{\psi}$ field at a vertex to a ψ field at another vertex (the alternative convention also works) due to the distinction between ψ and $\bar{\psi}$.
2. There is one type of internal vertex associated with a coordinate x_i and a factor of $-ig\int d^4x_i$. This vertex has three legs: one ϕ propagator leg, one ψ propagator leg (arrow in), and one $\bar{\psi}$ propagator leg (arrow out) to be consistent with convention on lines.
3. An external vertex just carries one of the coordinates of the Green function (usually associated with an initial or final state particle in the corresponding matrix element). Note there is no integration over these coordinates.
4. We divide by the symmetry factor $\mathcal{S}$ for the diagram. This is the number of permutations of internal lines which leave the diagram invariant.

The diagrammatic elements are (diagrams are always good in an answer)

(15)

(iii) In terms of Feynman diagrams, the lack of $O(g^n)$ diagrams contributing to $\psi\psi \to \psi\psi$ scattering when n is odd is encoded in the rules we have for joining vertices by propagators. If we had a diagram at $O(g^n)$ it means we have $n$one vertices. That means it has n ϕ legs. However, this can not end anywhere but must be paired with another ϕ leg. If n is odd then there will always be one leg left unpaired. There are no ϕ in the initial or final states which could be linked to this last ϕ leg. So we can not construct a legal (non-zero) diagram at this order if we follow the rules for diagrams in this theory.

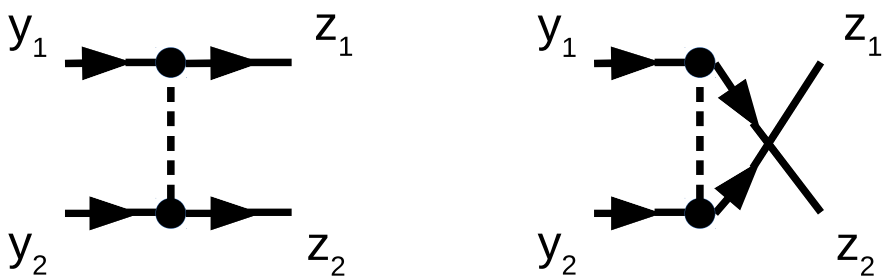
(iv) For reference there are two Feynman diagrams for G at lowest order in g which contribute to $\mathcal{M}_0$. They are



(16)

The Feynman diagrams which contribute to the $\psi\psi \to \psi\psi$ scattering matrix element $\mathcal{M}_2$ at $O(g^2)$ are shown in the following equations 17, 18 and 19. Further similar diagrams are indicated in the accompanying text, are found by permuting the z_1 and z_2 labels on the diagrams.

Diagrams representing first non-trivial contribution ($O(g^2)$) in the perturbative expansion for the $\psi\psi \to \psi\psi$ scattering process in scalar Yukawa theory of (1) are shown in equation (17).



17a 17b

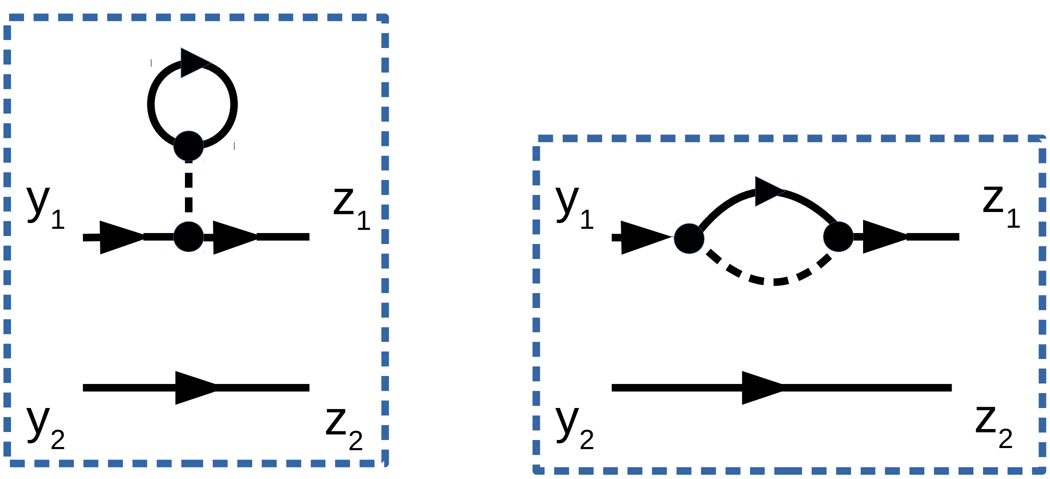
(17)

For these diagrams (17), the symmetry factors are $\mathcal{S} = 1$, no symmetries under exchange of internal lines.

In (17), the number of internal edges is $I = 2$, there are two vertices so $V = 2$ and there is just one component so $C = 1$. Using $L = I - V + C$ we find there are no loop momenta. Visually this is fairly obvious in the diagrams of (17).

Note that these two diagrams are distinct because you must fix the labels on external lines/vertices. Here the second is found by permuting the $z_1 \leftrightarrow z_2$ labels on the first. Most students noted this example. However many students missed similar permutations on the other diagrams below.

The diagrams in (18) also contribute at $O(g^2)$ to the $\psi\psi \to \psi\psi$ scattering process in scalar Yukawa theory of (1) but consist of self-energy contributions to the ψ propagator so do not change the non-trivial interactions. There are also the same diagrams but with y_1 and y_2 switched round and with the self-energy contribution on the z_2 leg not on the z_1 leg. Most students did not note these extra cases when answering this question. This gives us 8 distinct diagrams in total of this self-energy type. You can just state the existence of these permutations, writing them out in full is not needed.



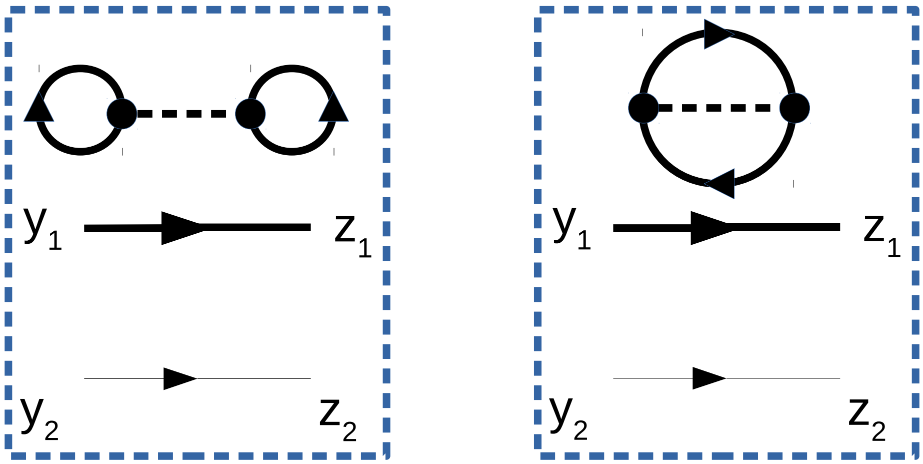
18a 18b

(18)

The symmetry factors in the diagrams of (18) are $\mathcal{S} = 1$, no symmetries under exchange of internal lines. Note that the first diagram, 18a, is a tadpole contribution. Such tadpole contributions are considered in the question on vacuum expectation values of ϕ.

Considering the non-trivial connected subdiagram of (18) (the top part with y_1 and z_1 external legs) we see the number of internal edges is $I = 2$, there are two vertices so $V = 2$ and there

is just one component in that subdiagram, $C = 1$. Using $L = I - V + C$ we find there is one loop momenta. Again this is fairly obvious visually in the diagrams of (18).
These last type of diagrams at $O(g^2)$ are



(19)

19a 19b

These are not usually included when calculating contributions at $O(g^2)$ to the $\psi\psi \to \psi\psi$ scattering process in scalar Yukawa theory of (1). The disconnected parts are vacuum diagrams which capture the difference between the vacuum in free theory ($|\,0\rangle$) and in the fully interacting theory[5] ($|\,\Omega\rangle$). These contributions are cancelled when the normalisation factor ($Z = \langle 0|S|\,0\rangle$) is included which is used to express the execration value in the vacuum of the full interacting theory ($|\,\Omega\rangle$) in terms of free vacuum expectation values $\langle 0|T(\text{fields})|\,0\rangle$.
There are two further diagrams which are identical except the z_1 and z_2 are swapped making a total of 4 distinct diagrams with vacuum contributions. Again no need to write these out in full but most students did not note this when answering this question.
The vacuum diagrams in (19) have symmetry factors $\mathcal{S} = 2$ coming from the interchange of the vertices as there are no external legs pinning these down. I had to use Wick's theorem to see this.
Each of the vacuum subdiagrams in (19) has two loop momenta since the number of internal edges is $I = 3$, there are two vertices so $V = 2$ and there is just one component in that subdiagram, $C = 1$. Using $L = I - V + C = 2$ we find there are two loop moments, certainly obvious in the diagram 19a, perhaps less so in 19b.
We have for the first diagram in 17

$$G(z_1, z_2, y_1, y_2) = \\ -g^2 \int d^4x_1 \int d^4x_2 \; \Delta_M(y_1 - x_1)\Delta_M(x_1 - z_1)\Delta_m(x_1 - x_2)\Delta_M(y_2 - x_2)\Delta_M(x_2 - z_2) \quad (20)$$

The symmetry factor is 1 in each case. The second diagram is the same expression except that z_1 (labelled as q_1 in the diagram 17b) switched is switched with with z_2.
We have for the first diagram with self-energy contributions, shown in 18a, that

$$G_{2A}(z_1, z_2, y_1, y_2) = \\ -g^2 \int d^4x_1 \int d^4x_2 \; \Delta_M(y_1 - x_1)\Delta_M(x_1 - z_1)\Delta_m(x_1 - x_2)\Delta_M(x_2 - x_2)\Delta_M(y_2 - z_2) \\ -g^2 \int d^4x_1 \int d^4x_2 \; \Delta_M(y_1 - x_1)\Delta_m(y_2 - x_1)\Delta_M(x_1 - z_2)\Delta_m(x_1 - x_2)\Delta_M(x_2 - x_2) \quad (21)$$

[5] The vacuum in free theory $|\,0\rangle$ is the state vacuum annihilated by the free creation operators used in the definition of the interaction picture fields. This is distinct from the vacuum in the fully interacting theory which we denote by $|\,\Omega\rangle$.

where again the symmetry factor is 1. There is another term where the self-energy contribution is on the other leg, so basically where we switch the 1 and 2 labels on y_i and z_f (not on the x's) to give a total of two terms. These two terms each have a partner where just y_1 (labelled here as q_1) is switched with y_2 (labelled here as q_2), leaving us with four terms of the same form but with various permutations of the coordinates (momenta) of the external lines, i.e. of the final and initial states.

For the second diagram with a self-energy contribution in 18 we have that

$$G_{2A}(z_1, z_2, y_1, y_2) = -g^2 \int d^4x_1 \int d^4x_2 \; \Delta_M(y_1 - x_1)\Delta_m(x_1 - x_2)\Delta_M(x_1 - x_2)\Delta_M(x_2 - z_1)\Delta_M(y_2 - z_2) \tag{22}$$

where again the symmetry factor is 1. There is another term where the self-energy contribution is on the other leg, so basically where we switch the 1 and 2 labels on y_i and z_f (not on the x's) to give a total of two terms. If we switch just the labels on the two external legs alone, we get two more contributions. This leaves us with four terms of the same form but with various permutations of the external leg coordinates, y_1, y_2, z_1 and z_2 (likewise with external momenta if you work in momentum space).

For the diagrams with a vacuum contribution we always pick up the same contribution as given above for the $O(g^0)$, multiplied by a vacuum diagram contribution. The vacuum diagram in the both cases has a symmetry factor is 2 (see the next question to see how I found that for 19a).