MSc in Quantum Fields and Fundamental Forces

Quantum Field Theory Test

Monday, 7th January 2013, 14:00 to 16:00

Please answer all three questions.

Each question is worth 20 marks.

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

1. Consider the Lagrangian density for a free real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

(i) Show that the Euler-Lagrange equations give rise to the Klein-Gordon equation

$$(\partial^2 + m^2)\phi = 0$$

[2 marks]

(ii) Show that the momentum density field π is given by $\pi = \dot{\phi}$ and derive the Hamiltonian, H, of the theory, making sure that it is expressed as a functional of ϕ and π. [3 marks]

(iii) The energy momentum-tensor is given by

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

Show that the energy momentum tensor is conserved, i.e. $\partial_\mu T^{\mu\nu} = 0$, using the Euler-Lagrange equations of motion. [4 marks]

(iv) Show that the currents defined by

$$\mathcal{M}^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}$$

are also conserved. Define the corresponding Noether charges $J^{\nu\rho}$, explaining how many independent charges there are and what they correspond to physically. [5 marks]

(v) Consider now the quantum theory of the scalar field ϕ. Write down the equal time commutation relations satisfied by the Heisenberg operators ϕ and π. Use them to calculate

$$[J^{ij}, \phi(t, \boldsymbol{x})], \qquad i, j = 1, 2, 3$$

Briefly comment on the result. [6 marks]

[Total 20 marks]

2. The free Dirac field in the Heisenberg picture can be expanded as

$$\psi(x) = \int \frac{d^3p}{2\pi^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left(a^s_{\boldsymbol{p}}u^s(p)e^{-ip\cdot x} + b^{s\dagger}_{\boldsymbol{p}}v^s(p)e^{ip\cdot x}\right)$$
$$\bar{\psi}(x) = \int \frac{d^3p}{2\pi^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left(b^s_{\boldsymbol{p}}\bar{v}^s(p)e^{-ip\cdot x} + a^{s\dagger}_{\boldsymbol{p}}\bar{u}^s(p)e^{ip\cdot x}\right)$$

where $p^0 = E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. In addition

$$(\not{p} - m)u^r(p) = 0, \qquad (\not{p} + m)v^r(p) = 0,$$

$$u^{r\dagger}(p)u^s(p) = 2E_{\boldsymbol{p}}\delta^{rs}, \qquad v^{r\dagger}(p)v^s(p) = 2E_{\boldsymbol{p}}\delta^{rs}$$

and

$$u^{r\dagger}(p^0, \boldsymbol{p})v^s(p^0, -\boldsymbol{p}) = 0, \qquad v^{r\dagger}(p^0, \boldsymbol{p})u^s(p^0, -\boldsymbol{p}) = 0$$

The non-vanishing anti-commutation relations involving the $a^s_{\boldsymbol{p}}$ and $b^r_{\boldsymbol{q}}$ operators are $\{a^r_{\boldsymbol{p}}, a^{s\dagger}_{\boldsymbol{q}}\} = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q})\delta^{rs}$ and $\{b^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}}\} = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q})\delta^{rs}$.

(i) Define the Hamiltonian for the free Dirac field by

$$H = N\left[\int d^3x\bar{\psi}(x)\left(-i\gamma^i\nabla_i + m\right)\psi(x)\right]$$

where $N[...]$ denotes normal ordering. Show that H can be written in the form

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s E_{\boldsymbol{p}}\left(a^{s\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}} + b^{s\dagger}_{\boldsymbol{p}}b^s_{\boldsymbol{p}}\right)$$

[12 marks]

(ii) Define the vacuum state, a single fermion state and a single anti-fermion state. Calculate the eigenvalues of these states with respect to H. Finally, explain why these particles obey Fermi-Dirac statistics. [8 marks]

[Total 20 marks]

3. This question concerns ϕ^4-theory. The scattering matrix $\mathcal{S}$ can be written

$$\mathcal{S} = T[exp(i\int d^4x\mathcal{L}'(x))]$$

with interaction Lagrangian density

$$\mathcal{L}' = -\frac{\lambda}{4!}\int d^4x\phi^4(x)$$

where $\phi(x)$, in the interaction picture, can be written $\phi(x) = \phi^+(x) + \phi^-(x)$ where

$$\phi^+(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}a_{\boldsymbol{p}}\mathrm{e}^{-\mathrm{i}p\cdot x},$$

$$\phi^-(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}a^\dagger_{\boldsymbol{p}}\mathrm{e}^{\mathrm{i}p\cdot x},$$

where $E_{\boldsymbol{p}} = p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$ and $[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{q})$.

(i) Show that

$$\left[\phi^+(x), \phi^-(y)\right] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

[2 marks]

(ii) Show that

$$T(\phi(x)\phi(y)) = N[\phi(x)\phi(y)] + D_F(x-y)$$

where $N[...]$ denotes normal ordering and the Feynman propagator is given by

$$D_F(z) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\left(\theta(z^0)\mathrm{e}^{-\mathrm{i}p\cdot z} + \theta(-z^0)\mathrm{e}^{\mathrm{i}p\cdot z}\right).$$

Also, state, without proof, Wicks theorem. [4 marks]

(iii) Consider the scattering of 3 particles with momenta k_1, k_2, k_3 into n particles with momenta $p_1, \ldots, p_n$. The scattering amplitude $\mathrm{i}\mathcal{M}$ can be written as

$$\langle p_1, \ldots, p_n|\mathrm{i}\mathcal{T}|k_1, k_2k_3\rangle = (2\pi)^4\delta^{(4)}(X)\ \mathrm{i}\mathcal{M}$$

where $\mathcal{T}$ is defined via $\mathcal{S} = \mathbb{1} + \mathrm{i}\mathcal{T}$. Why is the $\mathbb{1}$ contribution not included? What is the argument, X, of the delta function. What is the physical meaning of the δ-function? [3 marks]

(iv) Write down the Feynman rules in momentum space for the calculation of the connected amputated diagrams contributing to $i\mathcal{M}$. (You do not need to discuss possible symmetry factors). [5 marks]

(v) Consider the scattering of 3 particles into 3 particles at order λ^2. Write down two different Feynman diagrams (which cannot be obtained by relabelling, separately, the ingoing and the outgoing momenta) contributing to $i\mathcal{M}$ at this order and use the Feynman rules in momentum space to calculate the contribution for both diagrams. [6 marks]

[Total 20 marks]

End of examination paper