Quantum Field Theory
2014
TEST SOLUTIONS.

(1) (i)

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^* \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*$$

E-L: $$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\Leftrightarrow \quad \partial_\mu \partial^\mu \phi^* = -m^2 \phi^*$$

similarly

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi \qquad \frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) = \frac{\partial \mathcal{L}}{\partial \phi^*}$$

$$\Leftrightarrow \quad \partial_\mu \partial^\mu \phi = -m^2 \phi$$

//

(ii) $$\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi^*$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot\phi^*} = \dot\phi$$

$$\mathcal{H} = \pi \dot\phi + \pi^* \dot\phi^* - \mathcal{L}$$
$$= \pi \pi^* + \pi^* \pi - \dot\phi \dot\phi^* + \underline{\nabla} \phi \cdot \underline{\nabla} \phi^* + m^2 \phi \phi^*$$
$$= \pi \pi^* + \underline{\nabla} \phi \cdot \underline{\nabla} \phi^* + m^2 \phi \phi^*$$

$$H = \int d^3x \, \mathcal{H}$$

(iii)

$$\partial_\mu T^{\mu\nu} = \partial^2\phi\,\partial^\nu\phi^* + \partial^\mu\phi\,\cancel{\partial_\mu}\partial^\nu\phi^* + \partial^2\phi^*\,\partial^\nu\phi$$
$$+ \partial^\mu\phi^*\,\cancel{\partial_\mu}\partial^\nu\phi - \underbrace{\partial_\nu\left[\partial_\lambda\phi\,\partial^\lambda\phi^* - m^2\phi^*\phi\right]}$$
$$- \partial_\nu\partial_\lambda\cancel{\phi}\,\partial^\lambda\phi^* - \partial_\lambda\phi\,\cancel{\partial_\nu}\partial^\lambda$$
$$+ m^2\partial_\nu\phi^*\phi + m^2\phi^*\partial_\nu\phi$$

$$= \left(\partial^2\phi + m^2\phi\right)\partial^\nu\phi^* + \left(\partial^2\phi^* + m^2\phi^*\right)\partial^\nu\phi$$

$= 0$ using the equations of motion

The symmetry is translation invariance in space and time.

(iv) $P^\mu = \int d^3x\, T^{0\mu}$

$$\dot{P}^\mu = \partial_0 P^\mu = \int d^3x\, \partial_0 T^{0\mu}$$
$$= -\int d^3x\, \partial_i T^{i\mu}$$
$$= -\int_\infty d^2S_i\, T^{i\mu}$$
$= 0$ Assuming fields fall off fast enough at spatial infinity.

$$P^0 = \int d^3x \left\{ \dot\phi \dot\phi^* + \dot\phi^* \dot\phi - \left[\dot\phi^* \dot\phi - \nabla\phi \cdot \nabla\phi^* - m^2 \phi\phi^* \right] \right\}$$

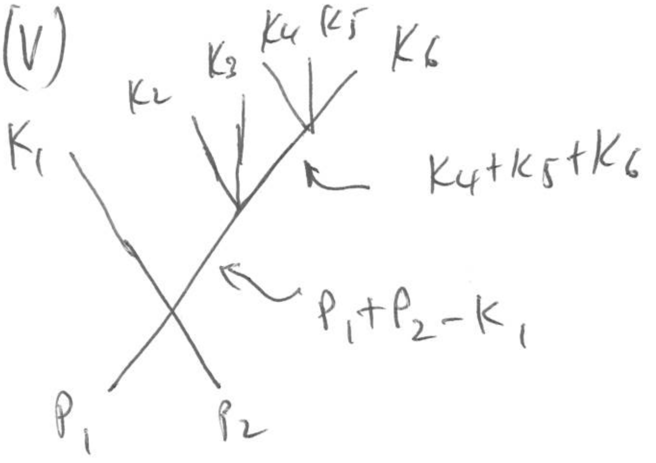
$$= \int d^3x \left\{ \dot\phi \dot\phi^* + \nabla\phi \cdot \nabla\phi^* + m^2 \phi \phi^* \right\}$$

$$= \int d^3x \left\{ \pi \pi^* + \nabla\phi \cdot \nabla\phi^* + m^2 \phi \phi^* \right\}$$

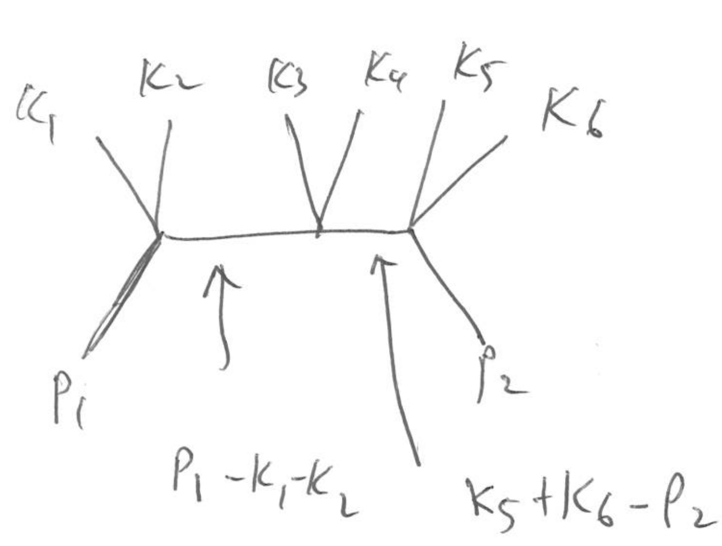
$$= H$$

p^i is the spatial momentum of the field.

(V)



$$(-i\lambda)^3 \frac{i}{(p_1+p_2-k_1)^2 - m^2 + i\varepsilon} \times \frac{i}{(k_4+k_5+k_6)^2 - m^2 + i\varepsilon}$$



$$(-i\lambda)^3 \frac{i}{(p_1-k_1-k_2)^2 - m^2 + i\varepsilon} \times \frac{i}{(k_5+k_6-p_2)^2 - m^2 + i\varepsilon}$$

Yes, energy is conserved.

(2)

(i) $\gamma^0 \gamma^\mu \gamma^0$

$\mu=0$: $\gamma^0\gamma^0\gamma^0 = \gamma^0$

$\mu=i$: $\gamma^0\gamma^i\gamma^0 = -\gamma^0\gamma^0\gamma^i = -\gamma^i$

$(\gamma^\mu)^\dagger$

$\mu=0$ $(\gamma^0)^\dagger = \gamma^0$

$\mu=i$ $(\gamma^i)^\dagger = -\gamma^i$

$\Rightarrow$ $\boxed{\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger}$

(ii) $\left.\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\gamma^\mu \partial_\mu - m)\psi \\ \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \end{array}\right\}$ E-L $\Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0$ *

$\left.\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi} \\ \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi} i\gamma^\mu \end{array}\right\}$ E-L $\Rightarrow i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$

To show that these are the same take the dagger of the Dirac equation *

$$\psi^\dagger(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m) = 0$$

$$\Rightarrow \psi^\dagger\gamma^0\gamma^0(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m) = 0$$

$$\Rightarrow \bar{\Psi}(-i\gamma^0\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m\gamma^0) = 0$$

$$\Rightarrow \bar{\Psi}(-i\gamma^0\gamma^{\mu\dagger}\gamma^0\overleftarrow{\partial}_\mu - m\gamma^0\gamma^0) = 0$$

$$\Rightarrow \bar{\Psi}(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0$$

$$\Rightarrow i\partial_\mu\bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0 \quad //$$

see later for (iii)

(ii) $\mathcal{L}(x) = \bar{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$

NOTE No transformation of x!

$$\to \bar{\Psi}(x')(i\gamma^\mu\partial_\mu - m)\psi(x')$$

$$= \underbrace{\psi^\dagger(\Lambda^{-1}x)\Lambda^\dagger_{1/2}\gamma^0}(i\gamma^\mu\partial_\mu - m)\Lambda_{1/2}\psi(\Lambda^{-1}x)$$

$$\underbrace{\psi^\dagger(\Lambda^{-1}x)\gamma^0\gamma^0\Lambda^\dagger_{1/2}\gamma^0}_{\bar{\Psi}(\Lambda^{-1}x)\Lambda^{-1}_{1/2}}$$

$$= \bar{\Psi}(\Lambda^{-1}x)\left[i\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2}\partial_\mu - \Lambda^{-1}_{1/2}\Lambda_{1/2}m\right]\psi(\Lambda^{-1}x)$$

$$= \bar{\Psi}(\Lambda^{-1}x)\left[i\Lambda^\mu{}_\nu\gamma^\nu\partial_\mu - m\right]\psi(\Lambda^{-1}x)$$

Define $y^\mu = \Lambda^{-1\,\mu}{}_\nu x^\nu$

then $\dfrac{\partial}{\partial x^\mu} = \dfrac{\partial y^\nu}{\partial x^\mu}\dfrac{\partial}{\partial y^\nu} = (\Lambda^{-1})^\nu{}_\mu\dfrac{\partial}{\partial y^\nu}$

Hence

$$\mathcal{L}(x) \longrightarrow \bar{\psi}(y)\left[i \Lambda^{\mu}{}_{\nu} \gamma^{\nu} (\Lambda^{-1})^{\rho}{}_{\mu} \frac{\partial}{\partial y^{\rho}} - m \right] \psi(y)$$

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$$= \bar{\psi}(y)\left[i \gamma^{\nu} \frac{\partial}{\partial y^{\nu}} - m \right] \psi(y)$$

$$= \mathcal{L}(y)$$

$$= \mathcal{L}(\Lambda^{-1} x)$$

i.e. $\mathcal{L}$ transforms as a scalar

(iv) $\psi^{(c)}(x) = C\psi^{*}(x)$

Hence under a Lorentz transformation

$$\psi^{(c)}(x) \rightarrow C\left[\Lambda_{1/2} \psi(\Lambda^{-1}x) \right]^{*}$$

$$= C \Lambda_{1/2}^{*} \psi^{*}(\Lambda^{-1}x)$$

$$= C \Lambda_{1/2}^{*} C^{\dagger} C \psi^{*}(\Lambda^{-1}x)$$

$$= C \Lambda_{1/2}^{*} C^{\dagger} \psi^{c}(\Lambda^{-1}x)$$

$\Rightarrow \psi^{c}(x)$ transforms as a spinor provided that $C \Lambda_{1/2}^{*} C^{\dagger} = \Lambda_{1/2}$

Now show this infintesimally

(iii) $i\not\partial\psi = m\psi$

$\Rightarrow i\not\partial\, i\not\partial\psi = m^2\psi$

$\Rightarrow -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\psi = m^2\psi$

$\Rightarrow -\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu\psi = +m^2\psi$

$\Rightarrow (\partial^2 + m^2)\psi = 0.$

(v) ~~con~~

$$\gamma^0\psi(x) \to \gamma^0\Lambda_{1/2}\psi(\Lambda^{-1}x)$$

$$= (\gamma^0\Lambda_{1/2}\gamma^0)\gamma^0\psi(\Lambda^{-1}x)$$

from (i) $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$

$$\gamma^0\Lambda_{1/2}\gamma^0 = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\gamma^0\right)$$

$$\gamma^0 S^{\mu\nu}\gamma^0 = \frac{i}{4}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0$$

$$= \frac{i}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}] \equiv \bar{S}^{\mu\nu}$$

$$\Rightarrow \gamma^0\psi(x) \to \bar{\Lambda}_{1/2}\gamma^0\psi(\Lambda^{-1}x)$$

$$\bar{\Lambda}_{1/2} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{S}^{\mu\nu}\right)$$

Now

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\Rightarrow \{\gamma^{\mu+}, \gamma^{\nu+}\} = 2\eta^{\mu\nu}$$

Hence $\bar{S}^{\mu\nu}$ provides a rep'n. of Lorentz Lie algebra

& hence $\exp(-\frac{i}{2}\omega_{\mu\nu}\bar{S}^{\mu\nu})$ provides a rep'n

of Lorentz Lie group.

Hence $\gamma^0\psi$ transforms as a spinor.

③ i) $|0\rangle$ defined by $a^s_{\underset{\sim}{p}}|0\rangle = b^s_{\underset{\sim}{p}}|0\rangle = 0$ 1

$$\forall\, s, \underset{\sim}{p}$$

$\underset{\sim}{p}$ labels momentum of particle

e^- labels particle state

e^+ labels anti-particle state

s labels <u>two</u> spin states $s=1,2$

$$|\underset{\sim}{p}, s, e^-\rangle \equiv a^{s\dagger}_{\underset{\sim}{p}}|0\rangle$$

$$|\underset{\sim}{p}, s, e^+\rangle \equiv b^{s\dagger}_{\underset{\sim}{p}}|0\rangle$$

$$\begin{aligned}|\underset{\sim}{p}, s, e^-; \underset{\sim}{p}', s', e^-\rangle &= a^{s\dagger}_{\underset{\sim}{p}} a^{s'\dagger}_{\underset{\sim}{p}'}|0\rangle \\ &= -a^{s'\dagger}_{\underset{\sim}{p}'} a^{s\dagger}_{\underset{\sim}{p}}|0\rangle \\ &= -|\underset{\sim}{p}', s', e^-; \underset{\sim}{p}, s, e^-\rangle\end{aligned}$$

Hence fermions

Similarly $|\underset{\sim}{p}, s, e^+; p', s' e^+\rangle = -|p', s', e^+; p s, e^+\rangle$

(ii)

$$a^{r\dagger}_{\underset{\sim}{q}} a^r_{\underset{\sim}{q}}|p, s, e^-\rangle = a^{r\dagger}_{\underset{\sim}{q}} a^r_{\underset{\sim}{q}} a^{s\dagger}_{\underset{\sim}{p}}|0\rangle$$

$$= a^{r\dagger}_{\underset{\sim}{q}}\left[\{a^r_{\underset{\sim}{q}}, a^{s\dagger}_{\underset{\sim}{p}}\} - \cancel{a^{s\dagger}_{\underset{\sim}{p}} a^r_{\underset{\sim}{q}}}\right]|0\rangle$$

$$= (2\pi)^3\delta^3(\underset{\sim}{p}-\underset{\sim}{q})\,\delta^{rs} a^{r\dagger}_{\underset{\sim}{q}}|0\rangle$$

$$= (2\pi)^3\delta^3(\underset{\sim}{p}-\underset{\sim}{q})\,\delta^{rs}|\underset{\sim}{q}, r, e^-\rangle$$

similarly

$$b^{r+}_{\tilde{q}}\, b^{r}_{\tilde{q}}\, |p, s, e^+\rangle = b^{r+}_{\tilde{q}}\, b^{r}_{\tilde{q}}\, b^{s+}_{\tilde{p}}\, |0\rangle$$

$$= b^{r+}_{\tilde{q}} \{ b^{r}_{\tilde{q}}, b^{s+}_{\tilde{p}} \} |0\rangle$$

$$= (2\pi)^3\, \delta^3(\tilde{p} - \tilde{q})\, \delta^{rs}\, |\tilde{q}, r, e^+\rangle$$

$$Q|0\rangle = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} \left(a^{s+}_{\tilde{p}}\, a^{s}_{\tilde{p}} - b^{s+}_{\tilde{p}}\, b^{s}_{\tilde{p}} \right) |0\rangle$$

$$= 0 \qquad \Rightarrow \quad 0 \text{ eigenvalue}$$

$$Q|\tilde{p}, s, e^-\rangle = \int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} \left(a^{r+}_{\tilde{q}}\, a^{r}_{\tilde{q}} - b^{r+}_{\tilde{q}}\, b^{r}_{\tilde{q}} \right) |p$$

$$= \int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} a^{r+}_{\tilde{q}}\, a^{r}_{\tilde{q}}\, |\tilde{p}, s, e^-\rangle$$

since $b^{r}_{\tilde{q}}\, |p, s, e^-$

$$= \int \frac{d^3q}{\cancel{(2\pi)^3}} \sum_{r=1}^{2} \cancel{(2\pi)^3}\, \delta^{rs}\, \delta^3(\tilde{p} - \tilde{q})\, |\tilde{p}, s,$$

$$= |p, s, e^-\rangle$$

$$\Rightarrow +1 \text{ eigenvalue.}$$

$$Q\,|\,\underset{\sim}{p}, s, e^+\rangle = \int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} \left(a_{\underset{\sim}{q}}^{r\dagger} a_{\underset{\sim}{q}}^{r} - b_{\underset{\sim}{q}}^{r\dagger} b_{\underset{\sim}{q}}^{r} \right) |\underset{\sim}{p}, s, e^+)$$

$$= -\int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} b_q^{r\dagger} b_q^r \,|\underset{\sim}{p}, s, e^+)$$

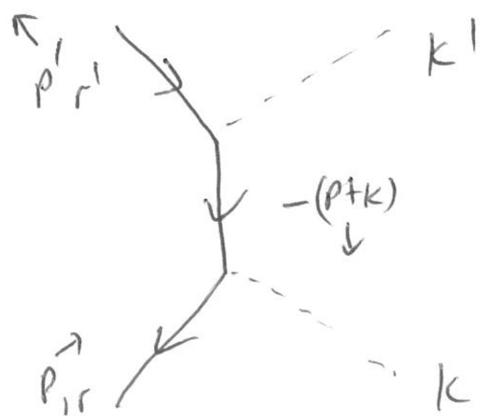
since $a_{\underset{\sim}{q}}^r |\underset{\sim}{p}, s, e^+) = 0$

$$= -\int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} (2\pi)^3 \delta^{rs} \delta^3(\underline{p}-\underline{q}) \,|\underline{p}, s, e^+)$$

$$= -\,|\underset{\sim}{p}, s, e^+)$$

$\Rightarrow$ eigenvalue $= -1.$ //

(iii) $e^+\phi \to e^+\phi$



not sign

$$(-ig)^2 \quad \bar{v}^r(p) \quad \frac{i\left[-(\cancel{p}+\cancel{k}) + m\right]}{\left((p+k)^2 - m^2 + i\varepsilon\right)} \, v^{r'}(p')$$

2 vertices — ingoing positron — Dirac propagator — outgoing positron

(iv) No!

eg $\phi\phi \to e^+e^-$ is possible

The correct statement is that the <u>net</u> number of fermions − antifermions is the same in initial & final states (thus is due to a symmetry)