Solutions 2: Classical Field Theory

1. Non-Linear equations of motion

The Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \tag{1}$$

Given the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi).(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \quad - \lambda\phi^4$ we have that

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \quad - 4\lambda\phi^3, \quad \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial^\mu \phi \tag{2}$$

Thus the equation of motion is

$$\partial^\mu \partial_\mu \phi + m^2 \phi + 4\lambda\phi^3 = 0 \tag{3}$$

as given.

Suppose ϕ_1 and ϕ_2 are two solutions of (4) so that

$$\partial^\mu \partial_\mu \phi_i + m^2 \phi_i + 4\lambda\phi_i^3 = 0, \quad i = 1, 2 \tag{4}$$

Then substituting $a\phi_1 + b\phi_2$ into (4) we have that

$$\begin{aligned} 0 &= \partial^\mu \partial_\mu (a\phi_1 + b\phi_2) + m^2(a\phi_1 + b\phi_2) + 4\lambda(a\phi_1 + b\phi_2)^3, && (5) \\ &= a\left(\partial^\mu \partial_\mu \phi_1 - m^2\phi_1\right) + b\left(\partial^\mu \partial_\mu \phi_2 - m^2\phi_2\right) \\ &\quad +4a^3\lambda\phi_1^3 + 4b^3\lambda\phi_2^3 + 12a^2b\lambda\phi_1^2\phi_2 + 4ab^2\lambda\phi_1\phi_2^2, && (6) \end{aligned}$$

and using the solutions (4) we have that

$$0 = 4a(1-a^2)\lambda\phi_1^3 + 4b(1-b^2)\lambda\phi_2^3 + 12a^2b\lambda\phi_1^2\phi_2 + 4ab^2\lambda\phi_1\phi_2^2, \tag{7}$$

which is clearly non-zero for general solutions unless $\lambda = 0$.

Thus the terms higher than quadratic in the fields in the Lagrangian, higher than liner in the resulting equations of motion, do indeed spoil linearity. They represent the interactions, particle creation/destruction terms. On the other hand the free parts can be treated as linear, so particles far apart, where interactions are essentially negligible, can be treated as linear. We can add as many free particle solutions together to represent initial or final states provided we think of them as separated by distances much greater than the characteristic interaction length scale.

2. Complex Scalar Field Equation of Motion

(i) We have

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i \phi_i\right\} = \frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2\phi_1^2\right) + \frac{1}{2}\left(\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2\phi_2^2\right) . \tag{8}$$

That is we have two copies of the usual Lagrangian for a single real scalar field. The Euler–Langrange equations for multiple fields (in a relativistic context) are

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \tag{9}$$

That is for each independent field component, here each value of the index $i = 1, 2$, we apply the usual relativistic form of the Euler-Lagrange equations to find

$$\partial^2\phi_1 + m^2\phi_1 = 0\,, \qquad \partial^2\phi_2 + m^2\phi_2 = 0 \tag{10}$$

since

$$\frac{\partial\mathcal{L}}{\partial\phi_i} = -m^2\phi_i \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = \partial^\mu\phi_i \tag{11}$$

(ii) We have that

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right)\,, \qquad \Phi^*(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) - i\phi_2(x)\right)\,. \tag{12}$$

We can invert (add and subtract the two equations) to see that

$$\phi_1(x) = \frac{1}{\sqrt{2}}\left(\Phi(x) + \Phi^*(x)\right)\,, \qquad \phi_2(x) = \frac{1}{\sqrt{2}i}\left(\Phi(x) - \Phi^*(x)\right)\,. \tag{13}$$

Substitute and you find

$$\mathcal{L} = (\partial_\mu\Phi^*)(\partial^\mu\Phi) - m^2\Phi^*\Phi\,. \tag{14}$$

(iii) Treating Φ and Φ^* as independent variables we have

$$\frac{\partial\mathcal{L}}{\partial\Phi} = -m^2\Phi^*\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = \partial^\mu\Phi^*\,, \qquad \frac{\partial\mathcal{L}}{\partial\Phi^*} = -m^2\Phi\,, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\,, = \partial^\mu\Phi \tag{15}$$

so that the Euler–Lagrange equations read

$$(\partial^2 + m^2)\Phi = 0\,, \qquad (\partial^2 + m^2)\Phi^* = 0 \tag{16}$$

(iv) Adding and subtracting the equations (16) gives the same equations as we found for the pair of real fields in (11).

(v) If we write $\dot{\Phi} = \partial_0\Phi$ then

$$\mathcal{L} = \dot{\Phi}^*\dot{\Phi} - \boldsymbol{\nabla}\Phi^*\cdot\boldsymbol{\nabla}\Phi - m^2\Phi^*\Phi \tag{17}$$

Then by definition

$$\Pi := \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \dot{\Phi}^* \qquad \Pi^* := \frac{\partial\mathcal{L}}{\partial\dot{\Phi}^*} = \dot{\Phi} \tag{18}$$

and the Hamiltonian density is given by

$$\begin{aligned} \mathcal{H} &= \Pi\dot{\Phi} + \Pi^*\dot{\Phi}^* - \mathcal{L} && (19)\\ &= \Pi^*\Pi + \boldsymbol{\nabla}\Phi^*\cdot\boldsymbol{\nabla}\Phi + m^2\Phi^*\Phi && (20) \end{aligned}$$

3. Conserved currents: Complex Scalar Field

The dynamics of a complex scalar (spin 0) field $\Phi(x)$ is described by the Lagrangian density

$$\mathcal{L} = (\partial^\mu\Phi^*)(\partial_\mu\Phi) - V(\Phi^*\Phi) \tag{21}$$

where V is some arbitrary potential.

(i) First $|\Phi|^2$ is invariant (even if θ not a constant)

$$\Phi \to \Phi' = e^{i\theta}\Phi \Rightarrow \Phi^*\Phi \to \Phi'^*\Phi' = (e^{i\theta}\Phi)^*.e^{i\theta}\Phi = e^{-i\theta}\Phi^*.e^{i\theta}\Phi = \Phi^*.\Phi\,. \tag{22}$$

The derivative term is invariant only if $\partial_\mu\theta = 0$

$$\begin{aligned}\partial_\mu\Phi \to \partial_\mu\Phi' &= \partial_\mu(e^{i\theta}\Phi) = (\partial_\mu e^{i\theta})\Phi + e^{i\theta}(\partial_\mu\Phi) & (23)\\ &= i(\partial_\mu\theta)e^{i\theta}\Phi + e^{i\theta}(\partial_\mu\Phi) = e^{i\theta}\left(i(\partial_\mu\theta)\Phi + (\partial_\mu\Phi)\right) = e^{i\theta}(\partial_\mu\Phi) & (24)\end{aligned}$$

Hence $\partial^\mu\Phi \to e^{i\theta}(\partial^\mu\Phi)$ and so $\partial^\mu\Phi^* \to e^{-i\theta}(\partial^\mu\Phi^*)$ so together we have that $(\partial_\mu\Phi^*)(\partial^\mu\Phi)$ is invariant.

(ii) The Euler-Lagrange equations are given by

$$0 = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi^*} - \frac{\partial\mathcal{L}}{\partial\Phi^*}\,. \tag{25}$$

Here we have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\Phi^*} &= \Phi\left.\frac{dV(z)}{dz}\right|_{z=|\phi|^2} & (26)\\ \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi^*} &= \partial^\mu\Phi & (27)\end{aligned}$$

so the equations of motion are

$$0 = \partial_\mu\partial^\mu\Phi - V'(|\Phi|^2)\Phi\,. \tag{28}$$

Variation with respect to Φ and its derivative give the complex conjugate of this equation of motion.

(iii) Consider $\partial^\mu J_\mu$ where the fields satisfy the equations of motion (28) then

$$\begin{aligned}\partial^\mu J_\mu &= \partial^\mu\left(\Phi^*(\partial_\mu\Phi)\right) - (\text{c.c.}) & (29)\\ &= (\partial^\mu\Phi^*)(\partial_\mu\Phi) + \Phi^*(\partial^\mu\partial_\mu\Phi) - (\text{c.c.}) & (30)\\ &= (\partial^\mu\Phi^*)(\partial_\mu\Phi) + \Phi^*(V'(|\Phi|^2)\Phi) - (\text{c.c.}) & (31)\\ &= (\partial^\mu\Phi^*)(\partial_\mu\Phi) + |\Phi|^2.V'(|\Phi|^2) - (\text{c.c.}) & (32)\end{aligned}$$

using the equation of motion (28) to remove the double derivative. Now the explicit terms are clearly real so they are cancelled by the complex conjugate terms (those in the (c.c.) bracket), so we have that $\partial^\mu J_\mu = 0$ for any potential V.