Solutions 5: Bosonic Interacting Quantum Field Theory

1. Delta Functions

The Dirac delta function is defined through

\[
\int_{-\infty}^{+\infty} dx \, \delta(x - x_0) f(x) = f(x_0).
\]

You should always start from this equation when using a delta function.

(i) Consider

\[
J = \int dy \, \delta(g(y)) f(y).
\]

Assume that the zero’s of \( g \) are at \( Z = \{ y_0 | g(y_0) = 0 \} \) and are widely spaced. Then the only places where the integral (2) has a non-zero contribution is in the region of one of these zeros as we need the argument of the delta function to be zero from (1). So we can write \( J \) as

\[
J = \sum_{y_0 \in Z} \int_{y_0 - \epsilon}^{y_0 + \epsilon} dy \, \delta(g(y)) f(y).
\]

So consider one of these zeros, say \( y_0 \), and expand the function \( g \) around this zero to find that \( g(y) = g(y_0) + (y - y_0)g'(y_0) + O((y - y_0)^2) \). By definition \( G(y_0) = 0 \) so we have for small \( \epsilon \) that

\[
J = \sum_{y_0 \in Z} \int_{y_0 - \epsilon}^{y_0 + \epsilon} dy \, \delta((y - y_0)g'(y_0)) f(y).
\]

Now change variable to \( x = (y - y_0)g'(y_0) \) to match the form given in the definition of the delta function (1). The change of variables is gives us

\[
J = \sum_{y_0 \in Z} \int_{-\eta'}^{\eta} \frac{dx}{g'(y_0)} \, \delta(x) f(y), \quad \eta = |g'(y_0)| \epsilon.
\]

Now in order to apply the formula (1) we note that we must be running from below the zero of the argument of the delta function, from below \( x_0 \) in (1), to above it. For the case of \( g'(y_0) > 0 \) there is no problem as \( \eta \) is positive and we have the same form as (1). The range of integration can be extended to infinity without any problem. When \( g'(y_0) < 0 \) however, \( \eta \) is negative and we are running past the zero in the wrong direction. However easy to switch direction but we get an overall minus sign in this case. This then cancels the negative sign of \( g'(y_0) \) in the denominator. Thus we have that

\[
J = \sum_{y_0 \in Z} \int_{-\eta'}^{\eta'} \frac{dx}{|g'(y_0)|} \, \delta(x) f(y), \quad \eta' = |g'(y_0)| \epsilon.
\]

Now we apply (1) to find that

\[
J = \int dy \, \delta(g(y)) f(y) = \sum_{y_0 \in Z} \frac{f(y_0)}{|g'(y_0)|},
\]

(ii) By inspection

\[
I = \int \frac{d^4k}{(2\pi)^4} \delta(k^2 - m^2) f(k^2, p^2, (k - p)^2)
\]
must be Lorentz invariant if $k$ and $p$ are four vectors as the expression is made up of Lorentz scalars. The arguments $k^2$, $p^2$, $(k-p)^2$ and $m^2$ are all Lorentz scalars. The measure, $d^4k$ is Lorentz invariant because any boost to new variables $k^\nu = \Lambda^\nu_\mu k^\mu$ produces a Jacobian factor in the transformation $|\Lambda|$ but this is 1 by definition of the Lorentz transformations. Thus all elements of this integral are invariant.

There is only one variable $p$ in this problem so we can only be a function of the only remaining variable $p$ (and of course it can depend on $m$ or other constant parameters). The only invariant we can build out of this is $p^2$ so the result must be a function of $p^2$.

The result

$$I = \sum_{k_0 = \pm \omega} \int \frac{d^3k}{2\omega} f(m^2, p^2, (k-p)^2), \quad \omega = |\sqrt{k^2 - m^2}|. \quad (9)$$

follows from (7). Consider the $k_0$ integration where we have that $g(k_0) = (k_0)^2 - \omega^2$. This has two zeros at $k_0 = \pm \omega$. We find that $g'(k_0) = 2k_0$ so at the zeros we have $g'(k_0) = 2|k_0| = 2\omega$.

Since $I$ is Lorentz invariant as a whole, and all the other terms in the form (9) are Lorentz invariant, we deduce that $d^3k/(2\omega)$ is also Lorentz invariant. You could also prove this directly by changing variables to a boosted frame $k^\mu = \Lambda^\mu_\nu k^\nu$.

(iii) Consider the integral

$$K = \int_{-\infty}^{+\infty} d(p_0) f(p_0) \left( \frac{i}{p_0 - \omega + i\epsilon} - \frac{i}{p_0 - \omega - i\epsilon} \right) \quad (10)$$

The poles and contours used in the two terms are shown in figure 1.

For the first term we can distort the integration path so that near the pole at $p_0 - \omega + i\epsilon$ we run along a semi circle centred on $\omega$ of radius $\eta > 0$ running above the axis (positive imaginary part), see figure 2. We will assume $\eta$ is infinitesimal so that we do not encounter any poles in the function $f$. We run along the real axis for the rest of the path. We can now take the limit $\epsilon \to 0$ to place the pole on the real axis as for $\eta > 0$ the path does not run through the pole.

$$K_+ = \int_{C_+} dp_0 f(p_0) \frac{i}{p_0 - \omega} \quad (11)$$

For the second term we move the contour of integration so near its pole at $p_0 - \omega - i\epsilon$ so that now this path runs along the real axis except for a semi-circle centred on $\omega$ of radius $\eta > 0$ but this time the semi-circle is below the axis (negative imaginary part), see figure 2.

$$K_- = -\int_{C_-} dp_0 f(p_0) \frac{i}{p_0 - \omega} \quad (12)$$

If we reverse the direction of integration in this second case we absorb the overall minus sign. Since the whole result is $K = K_+ + K_-$ when we put the two together, the contributions coming from two integrations along the real axes now cancel. The only part remaining is an integration around a small circle of radius $\eta$ centred on the pole at $k_0 = \omega$, see figure 3. Note we are running round this pole in a negative sense so that the residue there tells us this integral is equal to $-2\pi i$ times the residue. The residue is simply $if(p_0 = \omega)$ with the the factor of $i$ coming from the numerator. Thus we find that we have

$$K = 2\pi f(p_0 = \omega) \quad (13)$$
However using the definition of the Dirac delta function (1) we also have that this may be written as

$$K = 2\pi \int_{-\infty}^{+\infty} dp_0 f(p_0) \delta(p_0 - \omega)$$

(14)

Now comparing (10) and (14) we can identify that

$$2\pi \delta(p_0 - \omega) = \frac{i}{p_0 - \omega + i\epsilon} - \frac{i}{p_0 - \omega - i\epsilon}$$

(15)

\[\text{Figure 1: Figure showing two contours used for two terms in the delta function representation } K. \text{ Here the poles are placed off the real axis and contours run along the real line.}\]

\[\text{Figure 2: Figure showing two contours used for two terms in the delta function representation, } K_+ \text{ of (11) uses } C_+ \text{ shown on the left while on the right is } C_- \text{ used by } K_- \text{ of (12). The poles of the integrand are now on the real axis but the contours follow semicircles above or below the real line to avoid the the poles.}\]
Figure 3: Absorbing the minus sign of the second term of (10) by reversing the direction of the countour $C_-$, the only non-zero contribution now comes from a small circle around the pole running in the negative direction.

2. The Interaction Picture Evolution Operator $U(t_2, t_1)$

The relationship between the Schrödinger and Interaction pictures is given by

\[ |\psi, t\rangle_I = \exp\{+iH_{0,S}t\} |\psi, t\rangle_S, \]
\[ O_I(t) = \exp\{+iH_{0,S}t\} O_S \exp\{-iH_{0,S}t\}. \]

The evolution of interaction picture states is found through the interaction picture evolution operator $U(t_2, t_1)$ where

\[ |\psi, t_2\rangle_I = U(t_2, t_1) |\psi, t_1\rangle_I. \]

(i) Show that the $U$ operator satisfies the following conditions:

(a) This is clearly needed as $|\psi, t\rangle_I = |\psi, t\rangle_1$.

\[ U(t, t) = 1. \]

(b) From the definition of $U$ in (18) we have

\[ |\psi, t_2\rangle_I = U(t_2, t_1) |\psi, t_1\rangle_I, \]
\[ |\psi, t_1\rangle_I = U(t_1, t_0) |\psi, t_0\rangle_I, \]
\[ |\psi, t_2\rangle_I = U(t_2, t_0) |\psi, t_0\rangle_I. \]

Substituting (20) into (21) and comparing to (22) gives us the result

\[ U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0). \]

(c) We need the normalisation of states to be time independent so that $1\langle \psi, t_2 | \psi, t_2 \rangle_I = 1\langle \psi, t_1 | \psi, t_1 \rangle_I$. Taking the hermitian conjugate of (18) we have that

\[ 1\langle \psi, t_2 | = 1\langle \psi, t_1 | [U(t_2, t_1)]^\dagger \]

so that

\[ 1\langle \psi, t_2 | \psi, t_2 \rangle_I = 1\langle \psi, t_1 | [U(t_2, t_1)]^\dagger U(t_2, t_1) |\psi, t_1\rangle_I = 1\langle \psi, t_1 | \psi, t_1 \rangle_I. \]
This demands the unitary condition that
\[
[U(t_2, t_1)]^\dagger . U(t_2, t_1) = 1 .
\]  
(26)

(d) If we set \( t_2 = t_0 \) in (23) we get that \( U(t_0, t_1)U(t_1, t_0) = U(t_0, t_0) = 1 \) using the identity (19). Comparing this with (26) we arrive at
\[
[U(t_2, t_1)]^\dagger = U(t_1, t_2) .
\]  
(27)

(ii) The Schrödinger equation defines all the time-evolution in the Schrödinger picture which lies in their states
\[
i \frac{d}{dt} |\psi, t\rangle_S = H_S |\psi, t\rangle_S .
\]  
(28)

This tells us that
\[
|\psi, t\rangle_S = \exp\{-iHt\} |\psi, t = 0\rangle_S ,
\]  
(29)
as we can check by substituting this into (28). Substituting (29) into (16) gives us
\[
|\psi, t\rangle_1 = \exp\{+iH_{0,s}t\} \exp\{-iH_s t\} |\psi, t = 0\rangle_S ,
\]  
(30)

From this we deduce that
\[
|\psi, t\rangle_1 = \exp\{-iH_{\text{int},s}t\} |\psi, t = 0\rangle_S \quad \text{iff} \quad [H_S, H_{0,s}] = 0
\]  
(31)

but this is not generally true. (EFS: What is happening if this is true?)

(iii) To make progress we have to do this in infinitesimal steps which we can do by producing a differential equations. Substitute in \( |\psi, t\rangle_S = \exp\{-iH_{0,s}t\} |\psi, t\rangle_1 \) from (16) to the left hand side of (28) to find
\[
i \frac{d}{dt} |\psi, t\rangle_S = i \frac{d}{dt} (\exp\{-iH_{0,s}t\} |\psi, t\rangle_1)
\]  
(32)
\[
= H_{0,s} e^{-iH_{0,s} t} |\psi, t\rangle_1 + ie^{-iH_{0,s} t} \frac{d}{dt} |\psi, t\rangle_1
\]  
(33)

The right hand side of (28) gives us \( H_S |\psi, t\rangle_S = H_S \exp\{-iH_{0,s}t\} |\psi, t\rangle_1 \) noting the order carefully as \( H_S \) and \( H_{0,s} \) do not commute. Put these two sides of the Schrödinger equation together gives
\[
ie^{-iH_{0,s}t} \frac{d}{dt} |\psi, t\rangle_1 + H_{0,s} e^{-iH_{0,s} t} |\psi, t\rangle_1 = H_S \exp\{-iH_{0,s} t\} |\psi, t\rangle_1
\]  
(34)
\[
\Rightarrow \quad ie^{-iH_{0,s}t} \frac{d}{dt} |\psi, t\rangle_1 = -H_{0,s} \exp\{-iH_{0,s} t\} |\psi, t\rangle_1 + H_S \exp\{-iH_{0,s} t\} |\psi, t\rangle_1
\]  
(35)
\[
= (H_S - H_{0,s}) \exp\{-iH_{0,s} t\} |\psi, t\rangle_1
\]  
(36)
\[
= H_{\text{int},s} \exp\{-iH_{0,s} t\} |\psi, t\rangle_1
\]  
(37)
\[
\Rightarrow \quad \frac{d}{dt} |\psi, t\rangle_1 = -i \exp\{+iH_{0,s}t\} H_{\text{int},s} \exp\{-iH_{0,s} t\} |\psi, t\rangle_1
\]  
(38)
\[
\Rightarrow \quad \frac{d}{dt} |\psi, t\rangle_1 = -i H_{\text{int},1}(t) |\psi, t\rangle_1
\]  
(39)

where we have used the definition of an interaction picture operator in (17) to produce the \( H_{\text{int},1}(t) \) factor. Thus we have
\[
i \frac{d}{dt} |\psi, t\rangle_1 = H_{\text{int},1}(t) |\psi, t\rangle_1 .
\]  
(40)

Substituting in the definition of \( U \) from (18) then gives us
\[
i \frac{d}{dt} U(t, t_0) = H_{\text{int},1}(t) U(t, t_0) .
\]  
(41)
From (40) we have for infinitesimal $\epsilon$ (not necessarily positive just yet) that

\[
\frac{i}{\epsilon} (|\psi, t + \epsilon\rangle_1 - |\psi, t\rangle_1) \approx H_{\text{int},1}(t) |\psi, t\rangle_1
\]

\[
\Rightarrow \frac{i}{\epsilon} |\psi, t + \epsilon\rangle_1 \approx \frac{i}{\epsilon} |\psi, t\rangle_1 + e\epsilon H_{\text{int},1}(t) |\psi, t\rangle_1
\]

\[
\Rightarrow |\psi, t + \epsilon\rangle_1 \approx |\psi, t\rangle_1 - i\epsilon H_{\text{int},1}(t) |\psi, t\rangle_1 - (1 - i\epsilon H_{\text{int},1}(t)) |\psi, t\rangle_1
\]

Note that $H_{\text{int},1}(t)$ is an operator so it is vital that we are careful with the order. Now we iterate to find that

\[
|\psi, t + 2\epsilon\rangle_1 \approx \exp\{-i\epsilon H_{\text{int},1}(t + \epsilon)\} \exp\{-i\epsilon H_{\text{int},1}(t)\} |\psi, t\rangle_1
\]

and then

\[
|\psi, t + N\epsilon\rangle_1 \approx \exp\{-i\epsilon H_{\text{int},1}(t + (N - 1)\epsilon)\} \exp\{-i\epsilon H_{\text{int},1}(t + (N - 2)\epsilon)\} \ldots \exp\{-i\epsilon H_{\text{int},1}(t + \epsilon)\} \exp\{-i\epsilon H_{\text{int},1}(t)\} |\psi, t\rangle_1.
\]

Again note we are being very careful with operator ordering. This may be written in a short hand notation as a product

\[
|\psi, t + N\epsilon\rangle_1 \approx T \left( \prod_{n=0}^{(N-1)} \exp\{-i\epsilon H_{\text{int},1}(t + n\epsilon)\} \right) |\psi, t\rangle_1.
\]

as the time-ordering operator $T$ encode the operator ordering we need provided that $\epsilon > 0$. This notation only works in this case which will be useful only if $t_2 > t_1$ in our $U(t_2, t_1)$ of (18).

Now we need the identity proved in the next part that the time ordering operator takes care of all the operator ordering issues so that we can write $T(e^A e^B) = T(e^{(A+B)})$. We find that

\[
|\psi, t + N\epsilon\rangle_1 \approx T \left( \exp\{-i \sum_{n=0}^{(N-1)} \epsilon H_{\text{int},1}(t + n\epsilon)\} \right) |\psi, t\rangle_1.
\]

Taking the limit $\epsilon \to 0^+$ with $N = (t_2 - t)/\epsilon$, we arrive at the key result

\[
|\psi, t_2\rangle_1 = T \left( \exp\{-i \int_t^{t_2} dt' H_{\text{int},1}(t')\} \right) |\psi, t_1\rangle.
\]

Looking at the definition of $U$ in (18) we see that we have

\[
U(t_2, t_1) = T \left( \exp\{-i \int_{t_1}^{t_2} dt' H_{\text{int},1}(t')\} \right).
\]

(v) The Baker-Campbell-Hausdorff identity (BCH) is

\[
\exp\{\hat{A}\} \exp\{\hat{B}\} = \exp\{\hat{A} + \hat{B} + \frac{1}{2} [\hat{A}, \hat{B}] + \frac{1}{12} [\hat{A}, [\hat{A}, \hat{B}]] - \frac{1}{12} [\hat{B}, [\hat{A}, \hat{B}]] + \ldots\}
\]

The additional terms in the \ldots represent terms containing all possible combinations of $\hat{A}$ and $\hat{B}$ operators in all possible multiple commutators, multiplied by a known c-number.

**Second order proof:**
Consider two operators, $A$ and $B$. The expansion of $e^Ae^B$ to second order is

$$e^Ae^B = (1 + A + \frac{1}{2}A^2 + \ldots)(1 + B + \frac{1}{2}B^2 + \ldots)$$

$$= 1 + A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 + \ldots$$

(54)

Then compare this to the expansion of $e^{A+B}$

$$\exp\{A + B\} = 1 + (A + B) + \frac{1}{2}(A + B)^2 + \ldots$$

$$= 1 + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \ldots$$

(56)

We note that the order of the terms is important if, as for many operators, $A$ and $B$ do not commute. The difference between these expressions is the commutator which is the first non-trivial term in the BCH expression (53). That is

$$\exp\{A + B + \frac{1}{2}[A,B]\} = 1 + (A + B) + \frac{1}{2}(A + B)^2 + \frac{1}{2}[A,B] + \ldots$$

$$= 1 + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \frac{1}{2}[A,B] + \ldots$$

(58)

agrees with (55) to second order as the BCH formula (53) says it should. However we also note that a time-ordering operator changes (57) to

$$\exp\{A + B\} = 1 + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \ldots$$

(57)

as this overrides any operator ordering in the expression. Note that only the $AB$ term has any ordering issue at this order. Likewise

$$\exp\{A + B\} = 1 + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \ldots$$

(60)

So now there is complete agreement between (64) and (57) and we have that up to second order

$$\exp\{A + B\} = \exp\{Ae^B\} = 1 + A + B + \frac{1}{2}A^2 + T(AB) + \frac{1}{2}B^2 + \ldots$$

(65)

**All orders proof:**

Essentially the issue of operator ordering is encoded in the commutators in the BCH expression of (53). However we see that

$$T([A,B]) = T(AB) - T(BA)$$

(66)

but because the $T$ specifies the ordering $T(AB) = T(BA)$ and so $T([A,B]) = 0$. This will guarantee the commutator corrections in the BCH expression give zero e.g. when we expand out the exponential in (53), so we arrive at the conclusion that

$$T\left(\exp\{\hat{A}\} \exp\{\hat{B}\}\right) = T\left(\exp\{\hat{A} + \hat{B}\}\right) .$$

(67)
(vi) As we have written out the derivation of the form of \( U(t_2, t_1) \) in detail for the case \( t_2 > t_1 \), we can just pick this up from where we used the time ordering. So we can start from the expression in (48) which is valid for any \( t_2 \) and \( t_1 \). It is the next line, (49), which we must change for the case \( t_2 < t_1 \). So suppose that \( \epsilon < 0 \) then we have in terms of an infinitesimal \( \eta = -\epsilon > 0 \) that we must write

\[
|\psi, t - N\eta\rangle_1 \approx \mathcal{T} \left( \prod_{n=0}^{(N-1)} \exp\{+i\eta H_{\text{int},1}(t - n\eta)\} \right) |\psi, t\rangle_1 .
\]

Here \( \mathcal{T} \) is the anti-time-ordering operator, that is \( \mathcal{T}(AB) = \theta(t_a - t_b)BA + \theta(t_b - t_a)AB \) where \( t_a \) is the time associated with operator \( A \) and \( t_b \) is the time associated with operator \( B \). So \( \mathcal{T} \) puts the operator in order of their time with operators at the earliest times on the left.

Another way to see this is to realise that \( U(t_1, t_2)U(t_2, t_1) = 1 \) from (23). So if \( t_2 > t_1 \) we have the largest times in the middle of the expression so we must have \( U(t_1, t_2) \) ordered in the opposite way from \( U(t_1, t_2) \) (and also with a minus sign difference in the exponential) in order for cancellation to occur.

3. Contractions

(i) For bosonic fields, a contraction is defined and denoted as

\[
\overline{\phi_1 \phi_2} = \Delta_{12} = \mathcal{T}[\phi_1 \phi_2] - N[\phi_1 \phi_2] \tag{69}
\]

where \( \phi_1 = \phi_1(x_1) \) and \( \phi_2 = \phi_2(x_2) \) are any two bosonic fields. Here \( N[...] \) is general normal ordering where, for a given split of fields \( \phi_i = \phi_i^+ + \phi_i^- \), \( \phi_i^\pm \) are moved to the right of all \( \phi_i^- \), switching terms as few times as possible.

\[
N[\phi_1 \phi_2] = N[(\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)] = \left( \phi_1^+ \phi_2^- + \phi_1^- \phi_2^+ + \phi_1^- \phi_2^- + \phi_1^+ \phi_2^- \right) . \tag{70}
\]

Time ordering moves fields with latest times to the left.

\[
\mathcal{T}[\phi_1 \phi_2] = \mathcal{T}[(\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)] = \theta(t_1 - t_2) \left( \phi_1^+ \phi_2^- + \phi_1^- \phi_2^+ + \phi_1^- \phi_2^- + \phi_1^+ \phi_2^- \right) + \theta(t_2 - t_1) \left( \phi_2^+ \phi_1^- + \phi_2^- \phi_1^+ + \phi_2^- \phi_1^- + \phi_2^+ \phi_1^- \right) . \tag{71}
\]

Substituting (70) and (72) into (69) gives

\[
\overline{\phi_1 \phi_2} = \Delta_{12} = \theta(t_1 - t_2) \left[ [\phi_1^+ \phi_2^-] + [\phi_1^- \phi_2^+] \right] + \theta(t_2 - t_1) \left( [\phi_2^+ \phi_1^-] + [\phi_2^- \phi_1^+] \right) . \tag{73}
\]

The same argument works if we consider \( \overline{\phi_2 \phi_1} \) so we deduce that

\[
\overline{\phi_2 \phi_1} = \Delta_{21} = \theta(t_2 - t_1) \left[ [\phi_2^+ \phi_1^-] + [\phi_2^- \phi_1^+] \right] + \theta(t_1 - t_2) \left( [\phi_1^+ \phi_2^-] + [\phi_1^- \phi_2^+] \right) . \tag{74}
\]

Comparing the two expressions (73) and (74) we see we have a symmetric contraction \( \overline{\phi_1 \phi_2} = \overline{\phi_2 \phi_1} \) only if the term with a commutator of two plus parts cancels the commutator with two minus parts, i.e. for splits of the fields where

\[
[\phi_2^+ \phi_1^+] + [\phi_2^- \phi_1^-] = 0 \tag{75}
\]

for all times.

\(^1\)Fermionic fields have some extra signs in these definitions.
(ii) For the remainder of this question we will consider the standard definition of normal ordering (denoted with : : : ) where annihilation (creation) operators are put to the right (left) so that $\langle 0 | (\text{any fields}) | 0 \rangle = 0$.

What this means is that $[\phi_i^+, \phi_j^+] = [\phi_i^-, \phi_j^-] = 0$ for any fields $\phi_i$ because $\phi_i^+$ ($\phi_i^-$) contains only creation (annihilation) operators and these always commute. Thus for standard normal ordering we have that

$$\overline{\phi_1 \phi_2} = \Delta_{12} = T[\phi_1 \phi_2] - : \phi_1 \phi_2 := \theta(t_1 - t_2) [\phi_1^+, \phi_2^+] + \theta(t_2 - t_1) [\phi_2^+, \phi_1^-]$$  \hfill (76)

To show that the standard definition of normal ordering gives a symmetric normal ordering, i.e. that

$$\phi_1(x) \phi_2 :=: \phi_2(x) \phi_1 :,$$  \hfill (77)

we just need to show that $[\phi_2^+, \phi_1^+] + [\phi_2^-, \phi_1^-] = 0$ of (75) according to the analysis above. Since the + parts are always pure annihilation operators they will always commute, where they are for the same field or different fields. Likewise for the minus parts. So we see each term in (75) is zero and the sum is therefore zero. Hence we deduce that the standard definition of normal ordering leaves us with a simple symmetric normal ordering.

(iii) For the scalar Yukawa theory of (??), the fields in the interaction picture expressions are given by

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} (\hat{a}_p e^{-ipx} + \hat{a}^+_p e^{ipx}), \quad p_0 = \omega(p) = \sqrt{p^2 + m^2},$$  \hfill (78)

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\Omega(p)}} (\hat{b}_p e^{-ipx} + \hat{b}^+_p e^{ipx}), \quad p_0 = \Omega(p) = \sqrt{p^2 + M^2},$$  \hfill (79)

where the annihilation and creation operators obey the usual commutation relations

$$[\hat{a}_p, \hat{a}^+_q] = (2\pi)^3 \delta^3(p - q), \quad [\hat{a}_p, \hat{a}_q] = [\hat{a}^+_p, \hat{a}^+_q] = 0.$$  \hfill (80)

For the standard split used in QFT we find that

$$\hat{\phi}^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p e^{-ipx}, \quad p_0 = \omega_p,$$  \hfill (81)

$$\hat{\phi}^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a^+_p e^{ipx}, \quad p_0 = \omega_p.$$  \hfill (82)

For the complex fields we have to be careful with the notation, distinguish the hermitian conjugate operator (denoted with a dagger $\dagger$) from the plus symbol (+) used to indicate the annihilation operator parts. The split for the field $\psi$ in terms of $\Omega_p$ of (??) is

$$\hat{\psi}^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_p}} b_p e^{-ipx}, \quad p_0 = \Omega_p,$$  \hfill (83)

$$\hat{\psi}^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_p}} c^+_p e^{ipx}, \quad p_0 = \Omega_p.$$  \hfill (84)

The split for the hermitian conjugate field $\psi^\dagger$ is (now the notation is getting a little clumsy)

$$(\psi^\dagger)^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_p}} c_p e^{-ipx}, \quad p_0 = \Omega_p,$$  \hfill (85)

$$(\psi^\dagger)^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_p}} b^+_p e^{ipx}, \quad p_0 = \Omega_p.$$  \hfill (86)
Note that the hermitian conjugate of the ‘positive’ part of $\psi^\dagger$, that is $((\psi^\dagger)^+)^\dagger$, is not $\psi^\dagger$.

Now from (76) we find the following results

(a)

$$\overline{\phi(x)\phi(y)} = \theta(x_0 - y_0) [\phi^+(x), \phi^-(y)] + \theta(y_0 - x_0) [\phi^+(y), \phi^-(x)]$$  \hspace{1cm} (87)

Now use (81) and (82) to find

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \frac{1}{\sqrt{2\omega_q}} e^{-ipx + iqx} [a_p, a_q^\dagger]$$  \hspace{1cm} (88)

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-i\omega_p(x_0 - y_0) + ip(x - y)}.$$  \hspace{1cm} (89)

Now we need to use the standard tricks used for the energy integrations in the complex plane used for two-point functions. Here we need to see that

$$\int_{-\infty}^{+\infty} dz \, e^{-izt} \frac{e^{-it\omega}}{z - \omega + i\epsilon} = -2\pi i \theta(t) e^{-i\omega t}$$  \hspace{1cm} (90)

where $\epsilon$ is the usual infinitesimal but positive real parameter and $\omega$ is real and positive. To prove this you need to see that you can only close the contour with the upper semicircle at infinity (so with positive imaginary part) when $t < 0$ and then this part picks up no pole inside the contour and is zero. For $t > 0$ you have to close the contour in the lower half plane which then picks up the pole at $z = \omega - i\epsilon$ though you are going around the pole in the negative sense giving a factor of $-2\pi i$.

Using this on the $t = (x_0 - y_0)$ dependent terms in (89) we can rewrite our expression as an integral over $z = p_0$ as follows

$$\theta(x_0 - y_0) [\phi^+(x), \phi^-(y)] = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dz \, e^{-izt} \frac{1}{z - \omega + i\epsilon} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip(x - y)}$$  \hspace{1cm} (91)

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(x - y)}$$  \hspace{1cm} (92)

where we have also changed variables from $p$ to $-p$ exploiting the fact that $\omega_p$ is independent of this change.

For the second term of (87) is identical except we have $x$ and $y$ switched round.

$$\overline{\phi(x)\phi(y)} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(x - y)} + \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(y - x)}$$  \hspace{1cm} (93)

If we change variables in the second term, switching $p^\mu$ to $-p^\mu$, then we find

$$\overline{\phi(x)\phi(y)} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(x - y)} + \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(y - x)}$$  \hspace{1cm} (94)

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \left( \frac{i}{p_0 - \omega + i\epsilon} + \frac{i}{p_0 - \omega + i\epsilon} \right) e^{-ip(x - y)}$$  \hspace{1cm} (95)

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{2\omega_p} \left( \frac{-2\omega + 2i\epsilon}{p_0 - \omega + i\epsilon} \right) e^{-ip(x - y)}$$  \hspace{1cm} (96)
In the numerator \( \omega \geq m > 0 \) is assumed so we can always drop the infinitesimal here. This leaves us with

\[
\phi(x)\phi(y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{2\omega_p} \left( \frac{-2\omega}{-(p_0)^2 + (\omega - i\epsilon)^2} \right) e^{-ip(x-y)} \tag{97}
\]

\[
= \int \frac{d^4p}{(2\pi)^4} \left( \frac{i}{(p_0)^2 - \omega^2 - 2i\epsilon\omega - \epsilon'^2} \right) e^{-ip(x-y)} \tag{98}
\]

Now the \( \epsilon'^2 \) term is negligible while the \( 2\epsilon\omega = \epsilon' \) is infinitesimal, real and positive (not negligible at the pole of course). However we can relabel this \( \epsilon' \) as \( \epsilon \) (it has the usual properties) giving us

\[
\phi(x)\phi(y) = \int \frac{d^4p}{(2\pi)^4} \left( \frac{i}{(p_0)^2 - \omega^2 + i\epsilon} \right) e^{-ip(x-y)} = \Delta_m(x - y) \tag{99}
\]

**Note:** I will use the same \( \Delta_m \) notation for propagators in both position and momenta with the arguments indicating which is meant, i.e. it is clear which form is meant for \( \Delta_m(x - y) \) and \( \Delta_m(p - q) \).

(b) Here

\[
0 = \phi(x)\psi(y) \tag{100}
\]

as \( \psi \) and \( \phi \) fields have different types of annihilation and creation operators (i.e. they represent different types of particle). Thus all the commutators of any parts of these fields always commute.

(c) The previous argument applies here too

\[
0 = \phi(x)\psi(y) \tag{101}
\]

(d) Here we have from (76) that

\[
\psi(x)\psi(y) = \theta(t) \left[ \psi^+(x), \psi^-(y) \right] + \theta(-t) \left[ \psi^+(y), \psi^-(x) \right] \tag{102}
\]

Again by inspection we see that from (83) and (84) that \( \psi^+ \) and \( \psi^- \) contain different types of annihilation and creation operators, \( b \)'s in \( \psi^+ \) and \( c \)'s in \( \psi^- \).

(e) The argument for the previous parts works here too.

\[
0 = \psi^\dagger\psi^\dagger \tag{103}
\]

(f) We start from

\[
\psi(x)\psi(y) = \theta(x_0 - y_0) \left[ \psi^+(x), (\phi^\dagger)^-(y) \right] + \theta(y_0 - x_0) \left[ (\psi^\dagger)^+(y), \psi^-(x) \right] \tag{104}
\]

Now use (83), and (86) we find that

\[
\left[ \psi^+(x), (\phi^\dagger)^-(y) \right] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_p} \sqrt{2\Omega_q}} e^{-ipx+iqy} \left[ b_p, b^\dagger_q \right] \tag{105}
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\Omega_p} e^{-i\Omega_p(x_0 - y_0) + ip(x-y)}. \tag{106}
\]
For the second term in (104), using (84) and (85) we get

\[
\left((\psi^\dagger)^+(y), \psi^-(x)\right) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_p}} \frac{1}{\sqrt{2\Omega_q}} e^{-ipx+iqy} \left[c_p, c_q^\dagger\right] e^{-i\Omega_p(x_0-y_0)+ip(x-y)}. \tag{107}
\]

These two forms are identical to those found for the real scalar field contraction of (87) which gave us the two terms (89) and (93) except \(\omega\) is replaced by \(\Omega\). So we can immediately deduce that

\[
\psi(x)\psi^\dagger(y) = \Delta_M(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2-M^2+i\epsilon} \tag{109}
\]

**Note:** I will use the same \(\Delta_M\) notation for propagators in both position and momenta with the arguments indicating which is meant, i.e. it is clear which form is meant for \(\Delta_M(x-y)\) and \(\Delta_M(p-q)\).

(g) We just have to note our result above in (77) to see that with the standard normal ordering definition the contractions are symmetric so that

\[
\psi^\dagger(x)\psi(y) = \Delta_M(y-x) \tag{110}
\]

which is exactly the same as (104) except the space-time arguments are reversed.

The contraction is symmetric in our standard choice of normal ordering for vacuum expectation values. This means the order of the field and its hermitian conjugate in the definition of the contraction is irrelevant provided we label the coordinates appropriately. Equation (110) is correctly labelled with the \(y\) and \(x\) swapped compared to (109) above.

However for complex (and real) scalar fields the propagator is in fact completely symmetric \(\Delta(x-y) = \Delta(y-x)\). You can show this explicitly by switching each component of the four \(k_\mu = -k'_\mu\) integration variables in turn (remembering to set the range of integration appropriately).

### 4. Wick’s theorem for four bosonic fields

(i) Let \(\phi_i = \phi_1(x_1), \Delta_{12} = \phi_1\phi_2, N_{12} = N(\phi_1\phi_2), N_{1234} = N(\phi_1\phi_2\phi_3\phi_4)\), etc., then for four scalar (or indeed bosonic) fields Wick’s theorem states that

\[
T_{1234} = T(\phi_1\phi_2\phi_3\phi_4) = N(\phi_1\phi_2\phi_3\phi_4)
\]

\[
\quad + N(\phi_1\phi_2\phi_3\phi_4) + N(\phi_1\phi_2\phi_3\phi_4) + N(\phi_1\phi_2\phi_3\phi_4)
\]

\[
\quad + N(\phi_1\phi_2\phi_3\phi_4) + N(\phi_1\phi_2\phi_3\phi_4) + N(\phi_1\phi_2\phi_3\phi_4)
\]

\[
\quad + N(\phi_1\phi_2\phi_3\phi_4) + N(\phi_1\phi_2\phi_3\phi_4) + N(\phi_1\phi_2\phi_3\phi_4)
\]  \tag{111}

\[
= N_{1234} + \Delta_{12}N_{334} + \Delta_{13}N_{24} + \Delta_{14}N_{23}
\]

\[
+ \Delta_{23}N_{14} + \Delta_{24}N_{13} + \Delta_{34}N_{12}
\]

\[
+ \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}
\] \tag{112}
(ii) Whenever normal ordering has been defined such that expectation values of any normal product are zero, i.e. \( \langle N(\text{fields}) \rangle = 0 \). Then when we take expectation values of a time ordered product, and if we use Wick’s theorem any term containing a normal ordered operator will be zero so only terms which are products of contractions \( \Delta_{ij} = \langle T(\phi_i \phi_j) \rangle \) will survive. In this case we will just pick up the last term

\[
G_{1234} = \langle T(\phi_1 \phi_2 \phi_3 \phi_4) \rangle = \Delta_{12} \Delta_{34} + \Delta_{13} \Delta_{24} + \Delta_{14} \Delta_{23}
\]  
(113)

(iii) For the usual vacuum expectation values we want to define our normal ordering in terms of the usual split into annihilation parts for \( \phi^+ \) and creation parts in \( \phi^- \) as given in (81) and (82). We can substitute this in to our expression for a vacuum expectation value of the time-ordered product of four fields with \( t_1 > t_2 > t_3 > t_4 \)

\[
G_{1234} = \langle 0| T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) |0\rangle = \langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |0\rangle
\]  
(114)

\[
= \left(\prod_{i=1,2,3,4} \int \frac{d^3p_i}{(2\pi)^3 \sqrt{2\omega_i}} \right) \times \langle 0|(a_1 e^{-ip_1 x_1} + a_1^\dagger e^{+ip_1 x_1})(a_2 e^{-ip_2 x_2} + a_2^\dagger e^{+ip_2 x_2}) \rangle \times (a_3 e^{-ip_3 x_3} + a_3^\dagger e^{+ip_3 x_3})(a_4 e^{-ip_4 x_4} + a_4^\dagger e^{+ip_4 x_4}) |0\rangle
\]  
(115)

where \( a_i = a_{p_i} \). Now for a single operator \( a_p \) the only non-zero answers must come from terms with two annihilation operators \( a_j \) and two creation operators \( a_j^\dagger \). Otherwise we will have in initial and final states with different numbers of quanta and hence the overlap will be zero.

So writing \( A_j = a_j e^{-ip_j x_j} \), and exploiting that \( a_j |0\rangle = 0 \) and \( \langle 0|a_j^\dagger = 0 \), then we have using the commutation relations (??) that

\[
\langle 0|A_1(A_2 + A_2^\dagger)(A_3 + A_3^\dagger)A_4^\dagger|0\rangle =
\]  
(116)

\[
= \langle 0|A_1 A_2 A_3^\dagger A_4^\dagger |0\rangle + \langle 0|A_1^\dagger A_2^\dagger A_3 A_4^\dagger |0\rangle
\]  
(117)

\[
= \langle 0|A_1 \left( A_3^\dagger A_2 + \delta^3(p_2 - p_3)e^{-ip_2(x_2-x_3)} \right) A_4^\dagger |0\rangle + \langle 0| \left( A_2 A_1 + \delta^3(p_1 - p_2)e^{-ip_1(x_1-x_2)} \right) A_3 A_4^\dagger |0\rangle
\]  
(118)

\[
= \langle 0| \left( A_3^\dagger A_1 + \delta^3(p_1 - p_3)e^{-ip_1(x_1-x_3)} \right) A_2 A_4^\dagger |0\rangle + \delta^3(p_2 - p_3)e^{-ip_2(x_2-x_3)}\langle 0|A_1 A_4^\dagger |0\rangle
\]  
(119)

An inspection for the form of the propagator shows that for \( t_1 > t_2 \) we have that

\[
\Delta_{12} = D(x_1 - x_2) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega} e^{-ip(x_1-x_2)}.
\]  
(121)

and likewise for other combinations. This makes it clear what we have in (120) are just the terms found above in (113), so indeed we have that

\[
G_{1234} = \Delta_{12} \Delta_{34} + \Delta_{13} \Delta_{24} + \Delta_{14} \Delta_{23}
\]  
(122)
§5. Normal Ordering for Thermal Expectation Values

The expectation values here, $\langle \ldots \rangle$, are now thermal expectation values where

$$\langle \hat{O} \rangle = \frac{1}{Z} \text{Tr}\{e^{-\beta \hat{H}} \hat{O}\}, \quad Z = \text{Tr}\{e^{-\beta \hat{H}}\},$$

(123)

and $\beta = 1/(KT)$ is the inverse temperature. Here $\text{Tr}\{\ldots\}$ indicates a sum over all states in any basis, i.e. $\text{Tr}\{\hat{O}\} \equiv \sum_n \langle n | \hat{O} | n \rangle$.

(i) We have a single quantum harmonic oscillator with the usual annihilation and creation operators $\hat{a}^\dagger$ and $\hat{a}$ and a Hamiltonian $\hat{H} = \omega \hat{a}^\dagger \hat{a}$. The states are the usual normalised Fock space energy/number eigenstates

$$| n \rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n | 0 \rangle.$$  

(124)

This gives that

$$Z = \text{Tr}\{e^{-\beta \hat{H}}\} = \sum_{n=0}^{\infty} \langle n \rangle \exp(-\beta \omega \hat{a}^\dagger \hat{a}) | n \rangle$$

(125)

$$= \sum_{n=0}^{\infty} \langle n \rangle \sum_{m=0}^{\infty} (-\beta \omega \hat{a}^\dagger \hat{a})^m | n \rangle$$

(126)

$$= \sum_{n=0}^{\infty} \langle n \rangle \sum_{m=0}^{\infty} (-\beta \omega)^m | n \rangle$$

(127)

$$= \sum_{n=0}^{\infty} \langle n | e^{-\beta \omega n} | n \rangle$$

(128)

$$= \sum_{n=0}^{\infty} (e^{-\beta \omega})^n$$

(129)

$$= \frac{1}{1 - e^{-\beta \omega}}$$

(130)

Note we used the binomial expansion of $(1 - x)^{-1}$ and the normalisation of the states.

We can calculate $\langle \hat{a}^\dagger \hat{a} \rangle$ directly as above or use the usual trick with partition functions and recognise that

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{1}{\omega} \langle \hat{H} \rangle = \frac{1}{\omega Z} - \frac{d}{d\beta} \text{Tr}\{e^{-\beta \hat{H}}\} = -\frac{1}{\omega Z} \frac{dZ}{d\beta}$$

(131)

$$= -\frac{1}{\omega Z} \frac{d}{d\beta} \frac{1}{1 - e^{-\beta \omega}}$$

(132)

$$= -\frac{1}{\omega Z} \frac{-\omega e^{-\beta \omega}}{(1 - e^{-\beta \omega})^2}$$

(133)

$$= \frac{e^{-\beta \omega}}{(1 - e^{-\beta \omega})}$$

(134)

$$= \frac{1}{(e^{\beta \omega} - 1)} = n(\omega)$$

(135)

This is the Bose-Einstein distribution as should have been expected.

Using the commutator $[\hat{a}, \hat{a}^\dagger] = 1$ we have that

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle \hat{a}^\dagger \hat{a} + 1 \rangle = 1 + \frac{1}{(e^{\beta \omega} - 1)} = \frac{e^{\beta \omega}}{(e^{\beta \omega} - 1)} = 1 + n(\omega) = \frac{1}{(1 - e^{-\beta \omega})}$$

(136)
We can see that \( \langle \hat{a} \hat{a} \rangle \) and \( \langle \hat{a}^\dagger \hat{a}^\dagger \rangle \) both zero because all the terms in the sum under the trace are of the form \( \langle n|\hat{a}\hat{a}|n \rangle \propto \langle n|n-2 \rangle = 0 \) and \( \langle n|\hat{a}^\dagger \hat{a}^\dagger |n \rangle \propto \langle n|n+2 \rangle = 0 \).

(ii) Here we will use the notation : . . . : to indicate the ‘usual’ ‘traditional’ normal ordering as used in QFT based around vacuum expectation values — zero temperature QFT. This normal ordering : . . . : is defined such that annihilation (creation) operators are moved to the right (left) of creation (annihilation) operators\(^2\).

Clearly
\[
\langle 0| : \hat{a} \hat{a}^\dagger : |0 \rangle = \langle 0|\hat{a}^\dagger \hat{a}|0 \rangle = 0
\]
but from above we have that
\[
\langle : \hat{a} \hat{a}^\dagger : \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = \frac{1}{(e^{\beta \omega} - 1)} \neq 0.
\]


### 6. Normal Ordering for Thermal Field Theory

(i) Consider a single real scalar field \( \phi(x) \), defined as usual as
\[
\hat{\phi}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} (\hat{a}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx}), \quad p_0 = \omega(p) = \sqrt{p^2 + m^2},
\]
where \( \hat{a}_p \) and \( \hat{a}_p^\dagger \) are the usual annihilation and creation operators obeying standard commutation relations. The Hamiltonian is then just \( \hat{H} = \int d^3 k \hat{a}_k^\dagger \hat{a}_k \). Note we can ignore any (infinite) constants because this represents a shift in the zero of energy. Equilibrium statistical mechanics does not depend on the zero of energy e.g. you can include rest mass energy or not and it does not make any difference to the statistical mechanics.

The **thermal Wightman function** is given as follows:-
\[
D(x-y) = \langle \phi(x)\phi(y) \rangle = \frac{1}{Z} \text{Tr} \{ e^{-\beta \hat{H}} \phi(x)\phi(y) \}.
\]

\[
= \frac{1}{Z} \text{Tr} \left\{ e^{-\beta \hat{H}} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega(q)}} (\hat{a}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx})(\hat{a}_q e^{-iqy} + \hat{a}_q^\dagger e^{iqy}) \right\}
\]

We can generalise the result that \( \langle \hat{a} \hat{a} \rangle \) and \( \langle \hat{a}^\dagger \hat{a}^\dagger \rangle \) are both zero to see that \( \langle \hat{a}_p \hat{a}_q \rangle \) and \( \langle \hat{a}_p^\dagger \hat{a}_q^\dagger \rangle \) are always zero as the bra and kets will have different numbers of \( p \) and \( q \) states even if \( p = q \). For

\(^2\)To be more precise, as you might want to be in an exam if asked explicitly for a definition, we should also note that within the set of annihilation operators, their relative order is maintained. Likewise for the creation operators.
instance if \( \mathbf{p} \neq \mathbf{q} \) then

\[
\langle \hat{a}_p \hat{a}_q \rangle = \frac{1}{Z} \text{Tr} \left\{ e^{-\beta \hat{H}} \hat{a}_p \hat{a}_q \right\} = \frac{1}{Z} \left( \sum_{n_p=0}^{\infty} \langle n_p | e^{-\beta \hat{a}_p \hat{a}_p} | n_p \rangle \right) \left( \sum_{n_q=0}^{\infty} \langle n_q | e^{-\beta \hat{a}_q \hat{a}_q} | n_q \rangle \right) \times \prod_{r \neq p, q} \left( \sum_{n_r=0}^{\infty} \langle n_r | e^{-\beta \hat{a}_r \hat{a}_r} | n_r \rangle \right)
\]

(142)

The \( e^{-\beta \hat{a}_p \hat{a}_p} \) terms do not alter particle number of the states so the terms with \( \hat{a}_p \) and \( \hat{a}_q \) always involve overlaps between \( \langle n | n-1 \rangle = 0 \). If \( \mathbf{p} = \mathbf{q} \) then we will have one term of \( \langle n | n-2 \rangle = 0 \).

The same applies to all the other cases for thermal expectation values of pairs of annihilation and creation operators when \( \mathbf{p} \neq \mathbf{q} \).

In fact we can see the only cases where the thermal expectation values of a pair of annihilation and creation operators is non-zero is when then are a number operator, i.e. the two cases

\[
\langle \hat{a}_p^\dagger \hat{a}_q \rangle = n(\omega_p)(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) ,
\]

(145)

\[
\langle \hat{a}_p \hat{a}_q^\dagger \rangle = (1 + n(\omega_q))(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q})
\]

(146)

where we have used (136) and (135) and written the answer in terms of the Bose-Einstein distribution defined as \( n(\omega_p) = (e^{\beta \omega} - 1)^{-1} \). Note the normalisation for these continuous momentum space \( \mathbf{p} \) states and operators is now

\[
\langle n_p | n_q \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \delta_{n_p, n_q}.
\]

(147)

and so forth. It is sometimes easier to do this calculation in discrete momentum space and take the continuum limit at the end but we would have to change the notation and definitions used in this course.

So we can reduce this thermal Wightman function to

\[
D_\beta(x - y) = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\mathbf{p})2\omega(\mathbf{q})}} \langle \hat{a}_p \hat{a}_q^\dagger \rangle e^{-ip|x-y|} + \langle \hat{a}_p^\dagger \hat{a}_q \rangle e^{ip|x-y|}
\]

(148)

\[
D_\beta(x - y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\mathbf{p})} \left[ n(\omega_p)(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) e^{-ip|x-y|} + (1 + n(\omega_q))(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) e^{ip|x-y|} \right]
\]

(149)

Note that if we take \( \beta \rightarrow \infty \) we get the usual Wightman function we encountered in zero temperature QFT, i.e.

\[
D(x - y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\mathbf{p})} e^{ip(x-y)}.
\]

(151)
(ii) We split our field using a general linear split of the annihilation and creation parts as follows

\[ \hat{\phi}^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} ((1 - f_p)\hat{a}_p e^{-ipx} + g_p \hat{a}_p^\dagger e^{+ipx}), \]

\[ \hat{\phi}^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} (f_p \hat{a}_p e^{-ipx} + (1 - g_p) \hat{a}_p^\dagger e^{+ipx}), \]

where \( p_0 = \omega(p) = \sqrt{p^2 + m^2} \) as usual. Here \( f_p = f(p) \) and \( g_p = g(p) \) are two functions to be determined.

We have from the definition of normal ordering that if \( \phi_1 = \phi(x) \) and \( \phi_2 = \phi(y) \) the as in (70) we have that

\[ N(\phi(x)\phi(y)) = (\phi_1^+ \phi_2^+ + \phi_2^- \phi_1^+ + \phi_1^- \phi_2^+ + \phi_1^- \phi_2^-) \]

For instance the first term is

\[ \langle \phi_1^+ \phi_2^+ \rangle = \left\langle \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} ((1 - f_p)\hat{a}_p e^{-ipx} + g_p \hat{a}_p^\dagger e^{+ipx}) \right. \]

\[ \times \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega(q)}} ((1 - f_q)\hat{a}_q e^{-iqy} + g_q \hat{a}_q^\dagger e^{+iqy}) \right\rangle \]

\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega(q)}} \]

\[ \times \left( (1 - f_p)g_p(\hat{a}_p \hat{a}_q^\dagger e^{+iqy - ipx} + g_p(1 - f_q)(\hat{a}_p^\dagger \hat{a}_q)e^{-iqy + ipx} \right) \]

\[ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} \left( (1 - f_p)g_p(1 + n(\omega_p))e^{+ip(y-x)} + g_p(1 - f_p)n(\omega_p)e^{-ip(y-x)} \right) \]

The remaining terms are

\[ \langle \phi_2^- \phi_1^+ \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} \left( f_p g_p(1 + n(\omega_p))e^{-ip(y-x)} + (1 - g_p)(1 - f_p)n(\omega_p)e^{+ip(y-x)} \right) \]

\[ \langle \phi_1^- \phi_2^+ \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} \left( f_p g_p(1 + n(\omega_p))e^{+ip(y-x)} + (1 - g_p)(1 - f_p)n(\omega_p)e^{-ip(y-x)} \right) \]

\[ \langle \phi_1^- \phi_2^- \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(p)}} \left( f_p(1 - g_p)(1 + n(\omega_p))e^{+ip(y-x)} + g_p(1 - f_p)n(\omega_p)e^{-ip(y-x)} \right) \]
(iii) We now demand that \( \langle N(\phi(x)\phi(y)) \rangle = 0 \). This gives us the following

\[
0 = \int \frac{d^3p}{(2\pi)^3 2\omega(p)} \left[ (1 - f_p)g_p(1 + n(\omega_p))e^{ip(y-x)} + g_p(1 - f_p)n(\omega_p)e^{-ip(y-x)} \right]
\]

\[
+ \left( f_p g_p(1 + n(\omega_p))e^{-ip(y-x)} + (1 - g_p)(1 - f_p)n(\omega_p)e^{ip(y-x)} \right)
\]

\[
+ \left( f_p g_p(1 + n(\omega_p))e^{ip(y-x)} + (1 - g_p)(1 - f_p)n(\omega_p)e^{-ip(y-x)} \right)
\]

\[
+ \left( f_p(1 - g_p)(1 + n(\omega_p))e^{ip(y-x)} + g_p(1 - f_p)n(\omega_p)e^{-ip(y-x)} \right)
\]

\[= \int \frac{d^3p}{(2\pi)^3 2\omega(p)} \left[ e^{ip(y-x)} ((1 - f_p)g_p(1 + n(\omega_p)) + (1 - g_p)(1 - f_p)n(\omega_p))
+ f_p g_p(1 + n(\omega_p)) + f_p(1 - g_p)(1 + n(\omega_p)) \right]
\]

\[+ e^{-ip(y-x)} (g_p(1 - f_p)n(\omega_p) + f_p g_p(1 + n(\omega_p))
+ (1 - g_p)(1 - f_p)n(\omega_p) + g_p(1 - f_p)n(\omega_p)) \]

(161)

So we require that

\[ f_p g_p = -n(\omega_p), \quad (1 - f_p)(1 - g_p) = 1 + n(\omega_p). \]

(162)

These have two solutions

\[ f_p = -n + s\sqrt{n(n+1)}, \quad g_p = -n - s\sqrt{n(n+1)}, \quad s = \pm 1. \]

(163)

(iv) Perturbation theory works formally in thermal field theory exactly as before.

If we choose a field split such that the thermal expectation value of a pair of normal ordered fields is zero then from the definition of the contraction we still have that

\[ \langle T\phi(x)\phi(y) \rangle = \langle \overline{\phi(x)\phi(y)} \rangle \]

(164)

Exactly as before, this leads to (see (73))

\[ \overline{\phi(x)\phi(y)} = \theta(t_1 - t_2) \left[ \phi_1^+ \phi_2^- \right] + \theta(t_2 - t_1) \left[ \left[ \phi_1^+ , \phi_2^+ \right] + \left[ \phi_2^+ , \phi_1^- \right] + \left[ \phi_1^- , \phi_2^- \right] \right]. \]

(165)

where \( \phi_1 = \phi(x) \) and \( \phi_2 = \phi(y) \) here.

Only when we substitute in the specific form for the split to be used in thermal theory do we find the new form for the propagator to be used in perturbation theory of thermal field theory.

We could calculate the contraction directly as that will give us our thermal propagator for the Feynman rules as the contraction has the same form as in (69) and (73). By choosing the split such that the thermal expectation value of normal ordered products is zero we will find the correct propagator to use in perturbation theory.
However it is easier just to find the expectation value of the time-ordered product of two interaction picture fields directly. This is simply given in term of the thermal Wightman function (150)

\[
\Delta_\beta(x - y) = \langle T\phi(x)\phi(y) \rangle
\]

\[
= \theta(x_0 - y_0)D_\beta(x - y) + \theta(y_0 - x_0)D_\beta(y - x)
\]

\[
= \Delta_0(x - y) + \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(p)} n(\omega_p) \left( e^{-ipx + ipy} + e^{+ip(x-y)} \right)
\]

\[
= \Delta_0(x - y) + \int \frac{d^4p}{(2\pi)^4} n(|p_0|)(2\pi)^4\delta^4(p^2 - m^2)e^{-ip(x-y)}
\]

where \(\Delta_0(x - y)\) is our usual Feynman propagator i.e. for zero temperature field theory. Using our previous results for \(\Delta_0(x - y)\) we have that

\[
\Delta_\beta(x - y) = \int \frac{d^4p}{(2\pi)^4} \left( \frac{i}{p^2 - m^2 + i\epsilon} + n(|p_0|)(2\pi)^4\delta^4(p^2 - m^2) \right) e^{-ip(x-y)}
\]

\[
\Delta_\beta(p) = \frac{i}{p^2 - m^2 + i\epsilon} + n(|p_0|)(2\pi)^4\delta^4(p^2 - m^2).
\]

So we see that the thermal correction to our usual propagator only comes from physical on-shell particles weighted by the Bose-Einstein factor. These correspond to the effects of propagating through a heat bath at inverse temperature \(\beta\) of real physical particles (i.e. no virtual energy fluctuations, their energies are always equal to \(\omega_p\)).