Solutions 6: Scalar Yukawa Theory

In the interaction picture, the field operators in the scalar Yukawa theory, for real scalar field ϕ of mass m and complex scalar field ψ with mass M, take the form

$$\phi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}}\, e^{-ipx} + a^\dagger_{\boldsymbol{p}}\, e^{+ipx}\right), \qquad p_0 = \omega_{\boldsymbol{p}} \geq 0\,. \tag{1}$$

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}} \left(b_{\boldsymbol{p}}\, e^{-ipx} + c^\dagger_{\boldsymbol{p}}\, e^{ipx}\right), \qquad p_0 = \Omega_{\boldsymbol{p}} \geq 0\,. \tag{2}$$

where the dispersion relations are

$$\omega_{\boldsymbol{p}} = +\left|\sqrt{\boldsymbol{p}^2 + m^2}\right| \geq 0\,, \tag{3}$$

$$\Omega_{\boldsymbol{p}} = +\left|\sqrt{\boldsymbol{p}^2 + M^2}\right| \geq 0\,. \tag{4}$$

and the annihilation and creation operators obey their usual commutation relations

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q})\,, \qquad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{5}$$

The b and c annihilation and creation operators obey similar commutation relations while the different types of annihilation and creation operator always commute e.g. $\left[a_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = 0$. The fact that different types of annihilation and creation operators commute corresponds to the idea that they represent distinct types of particle or modes (different degrees of freedom)[1].

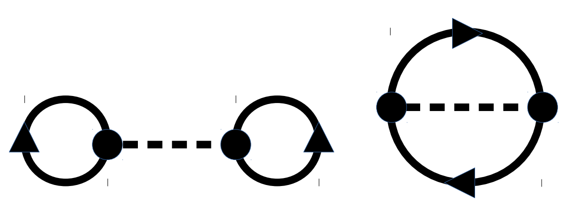
1. Vacuum Diagrams in Scalar Yukawa Theory

Let $Z = \langle 0|S|\, 0\rangle = \sum_n Z_n$ where Z_n is the order $O(g^n)$ term.

(i) At $O(g^0)$ the normalisation of the free vacuum is just $Z_0 = \langle 0|1|\, 0\rangle = 1$.

There are no $O(g^1)$ terms, $Z_1 = 0$.

There are two vacuum Feynman diagrams contributing to the normalisation of the free vacuum $Z = \langle 0|S|\, 0\rangle$ at $O(g^2)$:-



(6)

6a 6b

(ii) As noted above $Z_0 = 1$ and $Z_1 = 0$.

Diagram 6a gives us

$$Z_{2a} = \frac{1}{2}(-ig) \int d^4x_1\, (-ig) \int d^4x_2 \Delta_M(x_1 - x_1)\Delta_M(x_2 - x_2)\Delta_m(x_1 - x_2) \tag{7}$$

[1]Thus when particles and anti-particles can be distinguished, as in the complex scalar field ψ (or electrons/positrons) then we must have a different type of operator for particles and antiparticles. Conversely when particles and antiparticles are indistinguishable (as in the real scalar field ϕ or a photon). For electrons and photons, their spin means they have additional modes so in fact we need four distinct annihilation and creation operators for spin up/down electrons/positrons, while photons need two for the two transverse modes of propagation.

where the order of the arguments in the propagators doesn't matter as $\Delta_F(x-y) = \Delta_F(y-x)$.

$$\begin{aligned} Z_{2a} &= \frac{-g^2}{2} \int d^4x_1 \left(\int d^4x_2 \Delta_M(0)\Delta_M(0)\Delta_m(x_1-x_2) \right) & (8) \\ &= \frac{-g^2}{2} \left(\int d^4x_1 \right) \Delta_M(0)\Delta_M(0) \left(\int d^4x \Delta_m(x) \right) & (9) \\ &= \frac{-g^2}{2} \Omega \Delta_M(0)\Delta_M(0) \int d^4x \Delta_m(x) \,. & (10) \end{aligned}$$

To get from (7) to (10) you first note that $\Delta_M(x_1-x_1) = \Delta_M(0)$ which is a constant, and likewise for the $\Delta_M(x_2-x_2)$. Next change variables for the inner integral of (7) from x_2 to $x = x_1 - x_2$. The inner integral is now over x but the integrand is independent of x_1 allowing you can do the outer integral over x_1. Since there is no x_1 dependence left in the integrand, the integral over x_1 just gives the space-time volume factor $\Omega = \int d^4x_1$ (formally infinite).

In the same way we see that diagram 6b gives us

$$\begin{aligned} Z_{2b} &= \frac{1}{2}(-ig) \int d^4x_1 \, (-ig) \int d^4x_2 \Delta_M(x_1-x_2)\Delta_M(x_2-x_1)\Delta_m(x_1-x_2) & (11) \\ &= \frac{-g^2}{2} \int d^4x_1 \, \left(d^4x \Delta_M(x)\Delta_M(x)\Delta_m(x)\right) & (12) \\ &= \frac{-g^2}{2} \Omega \left(\int \, d^4x \Delta_M(x)\Delta_M(x)\Delta_m(x) \right) . & (13) \end{aligned}$$

(iii) If you are happy with the coordinate space rules and the results in you can arrive at in (10) and (13) then to get to the momentum space answer you just substitute the Fourier transform $\Delta(x) = (2\pi)^{-4} \int d^4k \, \exp(\pm ikx)\Delta(k)$.

Alternatively you can use the momentum space rules directly. For instance for diagram 6a we get

$$\begin{aligned} Z_{2a} &= \frac{1}{2} \int \bar{d}^4k_1 \Delta_M(k_1) \int \bar{d}^4k_3 \Delta_m(k_3) \int \bar{d}^4k_2 \Delta_M(k_2) \\ &\qquad \times(-ig)\bar{\delta}^4(k_1-k_1+k_3) \, (-ig)\bar{\delta}^4(k_2-k_2-k_3) & (14) \end{aligned}$$

The delta function of momenta at each vertex does not fix the momentum flowing round the loop as we have $\bar{\delta}^4(k_1-k_1+k_3) = \bar{\delta}^4(k_3)$. So then

$$\begin{aligned} Z_{2a} &= \frac{-g^2}{2} \left(\int \bar{d}^4k_1 \Delta_M(k_1) \right) \left(\int \bar{d}^4k_2 \Delta_M(k_2) \right) \int \bar{d}^4k_3 \Delta_m(k_3) \, \bar{\delta}^4(k_3) \, \bar{\delta}^4(k_3) & (15) \\ &= \frac{-g^2}{2} \left(\int \bar{d}^4k \Delta_M(k) \right)^2 \Delta_m(k_3=0) \, \bar{\delta}^4(k_3=0) & (16) \\ &= \frac{-g^2}{2} \left(\int \bar{d}^4k \Delta_M(k) \right)^2 \frac{i}{-m^2} \, \Omega & (17) \end{aligned}$$

The delta functions from the vertices only depend on the momentum k_3 flowing down the middle propagator but we get two of these $\delta(k_3)$. One kills the integral associated with the middle edge, $\int d^4k_3$. The second delta function gives the Ω factor as

$$\bar{\delta}^4(q=0) = \lim_{q \to 0} \int d^4x \, e^{iqx} = \int d^4x = \Omega \qquad (18)$$

if you allow me to be a little loose with regulation of this infinite quantities.

So to summarise the first diagram 6a gives us

$$Z_{2a} \quad = \quad \Omega\frac{-g^2}{2}\left(\int đ^4k\Delta_M(k)\right)^2\frac{i}{-m^2} \tag{19}$$

This diverges for large k (high energy or 'ultraviolet' limit) as $(\int d^4k\, k^{-2})^2 \sim \Lambda^4$ where $\Lambda \to \infty$ is some high energy cutoff scale.

In the same way diagram 6b gives us

$$Z_{2b} \quad = \quad \Omega\frac{-g^2}{2}\left(\int đ^4k_1 đ^4k_2\ \Delta_M(k_1)\Delta_M(k_2)\Delta_m(k_1-k_2)\right). \tag{20}$$

Counting the dimensions of the integration measures ($O(k^8)$) and comparing against the denominators ($O(k^6)$) we see this term only diverges as Λ^2.

2. Vacuum Expectation Value for ϕ in Scalar Yukawa Theory

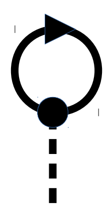
The **vacuum expectation value** or **vev** of the field ϕ is the 1-point Green function

$$v = G(x) = \langle 0|\mathrm{T}\hat{\phi}(x)S|\,0\rangle\,. \tag{21}$$

Note that because of space-time translation invariance we know that the vev will be the same wherever in space and time we measure it. That is $G(x)$ must be independent of the coordinate x. That is why the vev is usually denoted by a constant v with no space time arguments.
In this question we are looking at the perturbative expansion so we define $v = \langle 0|\mathrm{T}\hat{\phi}(x)S|\,0\rangle = \sum_n v_n$ where v_n is the term with all g^n contributions.

(i) We have at lowest order $S = 1$ and so $v_0 = \langle 0|\phi(x)|\,0\rangle = 0$. You find this by inserting the usual free field expression for the interaction picture field ϕ we have two terms, one with a single annihilation operator and the other with a single creation operator. These operators will be annihilate the bra vacuum and the ket vacuum respectively, leaving zero overall.

There is one Feynman diagram describing the $O(g^1)$ contribution to the ϕ vev v, that is it is the simplest **tadpole** diagram



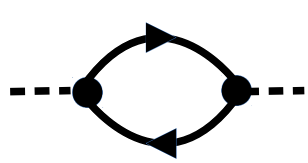
(22)

There are no order $O(g^2)$ diagrams.

To find the order $O(g^3)$ diagrams, one way to proceed is to adorn the $O(g^1)$ tadpole with vacuum or self-energy insertions. This gives

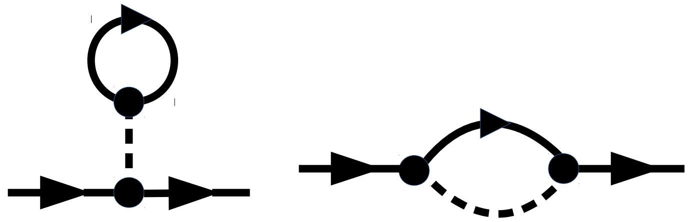
(a) Two diagrams where the tadpole of (22) is accompanied by one of the two vacuum diagrams of (6a) and (6b).

(b) A diagram where on the ϕ line we have a ϕ field self-energy insertion, i.e. we replace the single ϕ propagator by



(23)

(c) A diagram where on the ψ line in the loop we have one of the two ψ field self-energy insertions, i.e. the ψ line now looks like one of the following:-



(24)

24a 24b

Did you draw these out fully? In fact the approach suggested above gives **two** not three distinct $O(g^3)$ diagrams. The diagram from (b), when inserted in the lowest order diagram (22), gives the same $O(g^3)$ diagram as inserting the first diagram of Eqn. 23 (the tadpole self-energy contribution to a ψ propagator) into the lowest order diagram for v, (22). This shows that playing with diagrams is far from trivial even though its better than doing Wick's theorem[2].

The real answer at $O(g^3)$ is that there are just two distinct one-component ϕ tadpole diagrams contributing to $v = \langle 0|\mathrm{T}\hat{\phi}(x)S|\,0\rangle$, along with the two diagrams describe in (a) above with two components. You just sum over these two distinct graphs when calculating $v = \langle 0|\mathrm{T}\hat{\phi}(x)S|\,0\rangle$. If you arrive at the same graph in two different ways, great. Include the diagram just once and the symmetry factor takes care of any double counting. The real problem is if you miss a diagram.

(ii) The $O(g)$ contribution to the vev (vacuum expectation value) in coordinate space is

$$v_1 = \int d^4x\, (-ig)\Delta_m(y-x)\Delta_M(x-x) \tag{25}$$

$$= (-ig)\left(\int d^4x\ \Delta_m(x)\right)\Delta_M(x=0) \tag{26}$$

(iii) The momentum calculation in the previous part will be calculating the Fourier transform of the 1-point Green function $v = G(x)$, we can exploit the space-time translation invariance, the fact the vev is a constant, to see that the momentum space diagrams must give us something of the form

$$G(p) = \int d^4x\, \mathrm{e}^{-ipx}\, G(x) = \int d^4x\, \mathrm{e}^{-ipx}\, v = (2\pi)^4\delta^4(p)v\,. \tag{27}$$

Like all Green functions in momentum space, $G(p_1,\ldots,p_n)$, for our $G(p)$ there will be an overall momentum-conserving delta function coming from space-time translation invariance (in fact one per component), so we already expected on those grounds that $G(p)$ is of the form

$$G(p) = (2\pi)^4\delta^4(p)\tilde{G}(p)\,. \tag{28}$$

Here $\tilde{G}$ is just defined from this expression and must be the vev v. You can see this overall delta function in our results below for the perturbative expansion in momentum space[3].

[2]If you know about three-point 1PI diagrams and how they represent higher order corrections to the ordinary vertex, you could also have tried replacing the ordinary vertex in the $O(g^1)$ diagram with such 3-point 1PI diagrams. That though gives you the same diagram as when the second diagram in Eqn. 23 is inserted into the lowest order diagram (22).

[3]There is a subtle point that just because there is a symmetry in the theory as a whole, that does not mean you see this symmetry in each individual diagram, or even in the sum of all diagrams at the same order in the perturbation series. So here it is a good question to ask if $G_1(p)$, the first order contribution to the 1-point Green function. In fact space-time symmetry is true order by order. As a general point, any useful expansion, in any theory not just QFT, should respect all symmetries, something you can exploit to produce the Schwinger-Dyson equations of QFT.

In momentum space, using the Feynman rules we should find that the $O(g)$ contribution is[4]

$$v_1(p) = \int d^4x\, \mathrm{e}^{-ipx} v_1 = (2\pi)^4\delta^4(p) v_1 = (-ig)\Delta_m(p=0) \int đ^4k \Delta_M(k)(2\pi)^4\delta^4(p+k-k) \quad (29)$$

$$= (-ig)\Delta_m(p=0)(2\pi)^4\delta^4(p) \int đ^4k \Delta_M(k) \quad (30)$$

$$= -\frac{g}{m^2}(2\pi)^4\delta^4(p) \left(\int đ^4k \Delta_M(k) \right) \quad (31)$$

Note first that there is a single overall energy momentum conserving delta function. Next note that the integral clearly diverges as the square of some high energy scale Λ. That is we can with a little bit of work show that $\int^\Lambda dK\, K^3/K^2 \sim \Lambda^2$ for large momenta region of the integral. Using the space-time invariance properties of the first line (29) shows that

$$v_1 = -\frac{g}{m^2} \left(\int đ^4k \Delta_M(k) \right) . \quad (32)$$

Note on Vacuum Expectation Values

QFT is almost always done under the assumption that the vev of a field is zero. For this reason tadpole diagrams are rarely calculated explicitly and so they are often not included in discussions. Put another way, we almost always choose to work with a field $\eta(x)$ defined such that $\eta(x) = \phi(x) - \langle 0|\mathrm{T}\phi(x)S|\,0\rangle$ so that the (vacuum) expectation value of $\eta(x)$ is by definition zero. This corresponds to our assumptions that $\hat{a}_{\boldsymbol{k}}|\,0\rangle = 0$ and that there are no terms linear in fields in the classical Lagrangian. This is at the centre of discussions in the Unifcation course about symmetry breaking, superconductivity and superfluidity, and the Higgs mechanism.

3. Symmetry factors in Scalar Yukawa Theory

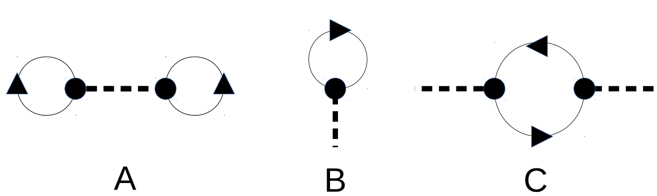


Figure 1: Diagrams representing contributions to three different quantities in scalar Yukawa theory of (38).

(i) **(A)** Diagram A in figure 1 is a **vacuum diagram** — it has no external legs. It represents the difference between the free vacuum and the full vacuum in the presence of interactions.

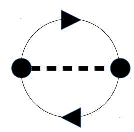
(B) The **tadpole** diagram, B of 1, has one external leg. It represents a contribution to $v = \langle 0|\phi(x)|\,0\rangle$ — the vev (vacuum expectation value) of the field ϕ. This has been assumed to be zero in this course as a quick calculation will show. However when we have symmetry breaking (i.e. when we are in a supefconducting/superfluid phase) this vev will be non-zero.

[4]The Green function in coordinate space is $G(x) = \langle 0|\phi(x)|\,0\rangle = v$ here as by translation symmetry you can deduce this is a constant. To first order in g we denote this as simply the constant v_1. In momentum space the corresponding Green function to first order in g is denoted $G(p) = v_1(p)$.

In perturbation theory, we would generally work with redefined fields to ensure all the fields have zero expectation value i.e. we are looking at "small" fluctuations around the true lowest energy state. This would be encoded by finding that a different state other than $|\,0\rangle$ (or $|\,\Omega\rangle$ if interactions are included) represents the true vacuum.

(C) Diagram C of 1 represents quantum corrections to the propagator for the real ϕ field — it contains a **self-energy** insertion. It will tell us if the effective mass changes with energy scale. That is if we are working at energy scales P then we will find that the ϕ propagator is roughly of the form $i/(P^2 - (M(P^2))^2)$ but that the effective mass $M(P)$ depends on P with only at the physical mass scale is this mass parameter exactly equal to the physical mass, i.e. we define the physical mass to be $M(P^2 = m^2_{\text{phys}}) = m^2_{\text{phys}}$. This $M(P^2)$ is called the "running mass" and reflects the fact that quantum fluctuations, the interactions with virtual particles, changes the propagation of energy and momentum in a complicated way but one we can calculate.

(ii) The other vacuum diagram at $O(g^2)$ is A2



A2 (33)

(iii) The symmetry factors for the diagrams in figure 1 are most easily found by simple counting of the contractions. You can try using the various descriptions of a rule given in books but I always find these confusing.

(A) The vacuum diagram in figure 1 has a symmetry factor of 2.

Each of the two tadpoles[5] can not be changed as the two legs are different (one with arrow in, one with arrow out). However we can switch the two vertices around. However I can never see this without doing the Wick expansion.

The other $O(g^2)$ vacuum diagram, A2 of (33), has a the same symmetry factor as again, switching the two vertices changes nothing.

This diagram comes from contributions of the form

$$V_2 = \frac{(-ig)^2}{2!} \int d^4x_1 d^4x_2 \, \langle 0| T\left(\phi_1 \psi_1^\dagger \psi_1 \phi_2 \psi_2^\dagger \psi_2\right) |\,0\rangle \tag{34}$$

where $\phi_i = \phi(x_i)$ etc. For our purposes we need only track the numerical factors such as the 2! in the denominator coming from the expansion of the exponential in the S matrix to second order. This is also why we may as well work in terms of the coordinate space representation not the momentum representation though the latter we would use for the actual calculation.

The first diagram, A of figure 1, comes from the case where we contract each $\psi_i^\dagger$ with the ψ_i of the same coordinate, giving us a factor of $\Delta_M(x = 0)$. There is only one way to do this. Likewise the last contraction always has to be between ϕ_1 and ϕ_2 giving $\Delta_m(x_1 - x_2)$ and there is only one way to do this. Thus we have no other numerical terms. Normally we would not include the $1/n!$ factor coming from the expansion of the S matrix, the factor $1/2!$ here,

[5] In diagram A in figure 1, each ψ propagator loop is part of a subdiagram with a single leg coming out, i.e. each can be considered to be a tadpole subdiagram.

when calculating Feynman diagrams. here there is nothing to cancel it so we must supplement our usual contributions with a 1/2 so the symmetry factor is $\mathcal{S} = 2$ in this case.

The second vacuum diagram, A2 of (33), comes from the case where we contract each $\psi_i^\dagger$ with the ψ of the opposite coordinate but again there is still only one way to do this. There is again no choice for the contraction between ϕ_1 and ϕ_2. Thus we again have no other numerical terms so only the factor of 2! in the denominator is left so the symmetry factor is again $\mathcal{S} = 2$ in this case.

(B) The **tadpole** diagram of figure 1 has a symmetry factor of 1. This diagram comes from the contribution at $O(g^1)$ to the vev (vacuum expectation value) of ϕ, $v(x_0) = \langle 0|\phi(x_0)|\,0\rangle$, of the form

$$v_1 \quad = \quad \frac{(-ig)}{1!} \int d^4x_1 \,\langle 0|T\left(\phi_0\phi_1\psi_1^\dagger\psi_1\right)|\,0\rangle \tag{35}$$

where $\phi_i = \phi(x_i)$ etc. There is only one way to contract the ϕ fields with each other) and only one way to contract the $\psi_a^\dagger$ with the ψ_1 so there are no numerical factors here.

(C) The diagram with a self-energy insertion shown in diagram (C) has a symmetry factor of 1. This diagram comes from the contribution at $O(g^2)$ to the full propagator of ϕ, $v(x_0) = \langle 0|\phi(x_0)\phi(x_3)|\,0\rangle$, of the form

$$\Pi_2(x_0 - x_3) \quad = \quad \frac{(-ig)^2}{2!} \int d^4x_1 d^4x_2 \,\langle 0|T\left(\phi_0\phi_1\psi_1^\dagger\psi_1\phi_2\psi_2^\dagger\psi_2\phi_3\right)|\,0\rangle \tag{36}$$

where $\phi_i = \phi(x_i)$ etc. To get this diagram the ϕ_0 can be contracted with a ϕ from either of the internal vertices, i.e. with ϕ_1 or ϕ_2. This gives a factor of 2, the usual permutation of the internal vertices which generally cancels the $1/n!$ factor coming from the expansion of the S matrix. Note the $\phi_0\phi_3$ contraction means we have a disconnected vacuum diagram and that is not C. So wlog we chose one of these two options, say that we have $\phi_0\phi_1$ and $\phi_2\phi_3$.

There are two options for the ψ contractions. If we contract each $\psi_i^\dagger$ with the ψ_i of the same coordinate, we get a different diagram. That is we get a pair of disconnected vev contributions, each part of the form shown in figure B. Again that is a different diagram from the one we have.

So we only need consider the contraction between each $\psi_i^\dagger$ with the ψ of the opposite coordinate. There is only one way to do this, so no more numerical factors are produced.

Note that if you are trying to find the symmetry factors by finding the number of permutations of internal lines which leave the diagram invariant, then you have to realise that the internal line in diagram C are not identical as the arrows are in opposite directions. Arrows can make a difference. If we had a diagram with no lines on the internal propagators (so we have some sort of $(g/2)\phi\eta\eta$ interaction where both ϕ and η are real fields) then the symmetry factor would be two not one.

(iv) The function under the integrals must be a function of $(x_1 - x_2)$ by Lorentz invariance. Indeed here is is simply $\Delta_m(x_1 - x_2)$. We can do the x_2 integral first and change variables to $x' = x_2 - x_1$ to see the result in independent of the last x_1 integration. Thus that generates a space-time volume factor of VT.

A similar argument works for the tadpoles. We can change the x_1 integration variable to be $x' = x_1 - x_0$ and then the result is clearly independent of x_0.

4. ψ^2 Scattering in Scalar Yukawa Theory

We are considering the the case of $\psi\psi \to \psi\psi$ scattering with incoming ψ particles of three-momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ while the outgoing ψ particles have three-momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. The matrix element is $\mathcal{M}$ where

$$\mathcal{M} = \langle f|S|\,i\rangle = \langle q_1, q_2|S|\,p_1, p_2\rangle = \sum_{n=0}^{\infty} \mathcal{M}_n\,, \qquad \mathcal{M} \sim O(g^n)\,. \tag{37}$$

(i) Interaction Hamiltonian for this theory is

$$H_{\text{int}} = g \int d^3\boldsymbol{x}\; \psi^\dagger(x)\psi(x)\phi(x) \tag{38}$$

where the coupling constant g is real and a measure of the interaction strength. The Lagrangian is just of the general form $L = \Pi\dot{\phi} - H$ so as the interaction term has no derivatives it simply appears as a $-H_{\text{int}}$ term in the Lagrangian. The moving to the density just removes the $\int d^3\boldsymbol{x}$ so the quadratic terms will be the usual ones for real ϕ and complex ψ fields (note the different factors of 1/2) with a $-g\psi^\dagger(x)\psi(x)\phi(x)$ from the non-linear, cubic, interaction term. So the scalar Yukawa theory has the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu\psi^\dagger)(\partial^\mu\psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x) \tag{39}$$

(ii) Here we are considering

$$\mathcal{M} = \langle f|S|\,i\rangle = \langle q_1, q_2|S|\,p_1, p_2\rangle = A\langle 0|b(\boldsymbol{q}_1)b(\boldsymbol{q}_2)Sb^\dagger(\boldsymbol{p}_1)b^\dagger(\boldsymbol{p}_2)|\,0\rangle \tag{40}$$

ignoring, for now, possible corrections to the vacuum coming from interactions (this will be considered later in the course). The factor $A = (16\Omega(p_1)\Omega(p_2)\Omega(q_1)\Omega(q_2))^{1/2}$ comes from the normalisation of operators and states used here. To lowest order in g we have $S = 1$ so that we want to consider

$$\begin{aligned}
A^{-1}.\mathcal{M}_0 &= \langle 0|b(\boldsymbol{q}_1)b(\boldsymbol{q}_2)b^\dagger(\boldsymbol{p}_1)b^\dagger(\boldsymbol{p}_2)|\,0\rangle && (41)\\
&= \langle 0|b(\boldsymbol{q}_1)\left(b^\dagger(\boldsymbol{p}_1)b(\boldsymbol{q}_2) + \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_2)\right)b^\dagger(\boldsymbol{p}_2)|\,0\rangle && (42)\\
&= \langle 0|b(\boldsymbol{q}_1)b^\dagger(\boldsymbol{p}_1)b(\boldsymbol{q}_2)b^\dagger(\boldsymbol{p}_2)|\,0\rangle + \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_2)\langle 0|b(\boldsymbol{q}_1)b^\dagger(\boldsymbol{p}_2)|\,0\rangle && (43)\\
&= \langle 0|\left(b^\dagger(\boldsymbol{p}_1)b(\boldsymbol{q}_1) + \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_1)\right)b(\boldsymbol{q}_2)b^\dagger(\boldsymbol{p}_2)|\,0\rangle \\
&\qquad + \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_2)\langle 0|\left(b^\dagger(\boldsymbol{p}_2)b(\boldsymbol{q}_1) + \delta^3(\boldsymbol{p}_2 - \boldsymbol{q}_1)\right)|\,0\rangle && (44)\\
&= \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_1)\langle 0|b(\boldsymbol{q}_2)b^\dagger(\boldsymbol{p}_2)|\,0\rangle + \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_2)\langle 0|b(\boldsymbol{q}_1)b^\dagger(\boldsymbol{p}_2)|\,0\rangle && (45)\\
&= \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_1)\delta^3(\boldsymbol{p}_2 - \boldsymbol{q}_2) + \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_2)\delta^3(\boldsymbol{p}_2 - \boldsymbol{q}_1) && (46)
\end{aligned}$$

(iii) If we have $\mathcal{M}_n$ for n is odd then we will have an odd number of ϕ terms coming from $(H_{\text{int}})^n$. From Wick's theorem we know that the only non-zero terms will be ones where all fields are contracted with another, terms with a normal ordered factor present are zero. The only contractions involving ϕ which are non-zero are contractions between two ϕ fields so we need an even number of ϕ fields to get a non-zero results. There are no ϕ terms in the initial and final states, so they must all come from the S matrix. So odd orders in n have odd numbers of ϕ factors and so must be zero.

In terms of diagrams for every power of g we get another vertex. These vertices have only one ϕ leg coming out. There are no external ϕ legs as the initial and final states are only built from ψ fields. So an odd number of vertices, n odd, means that we can not connect our ϕ edges in pairs as needed.

(iv) Substituting in the form (2) for the field ψ, you find that

$$\begin{aligned}
\left(\int d^3\boldsymbol{y}\, \exp\{-ipy\} 2\Omega(\boldsymbol{p})\right) \hat{\psi}(y)|\,0\rangle &= \int \frac{đ^3\boldsymbol{k}}{\sqrt{2\Omega(\boldsymbol{k})}} 2\Omega(\boldsymbol{p}) \int d^3\boldsymbol{y}\, e^{-i(p-k)y} && (47)\\
&= \int \frac{đ^3\boldsymbol{k}}{\sqrt{2\Omega(\boldsymbol{k})}} 2\Omega(\boldsymbol{p}) \deltā^3(\boldsymbol{p}-\boldsymbol{k})\, e^{-i(\Omega(p)-\Omega(k))t} \hat{b}^\dagger_{\boldsymbol{k}}|\,0\rangle && (48)\\
&= \sqrt{2\Omega(\boldsymbol{p})}\hat{b}^\dagger_{\boldsymbol{k}}|\,0\rangle = \sqrt{2\Omega(\boldsymbol{p})}|\,\psi(\boldsymbol{p})\rangle = |\,\psi(p)\rangle && (49)
\end{aligned}$$

where $t = y^0$. The $|\,\psi(p)\rangle$ state is the one ψ particle state with the appropriate normalisation for relativistic calculations while $|\,\psi(\boldsymbol{p})\rangle = \hat{b}^\dagger_{\boldsymbol{k}}|\,0\rangle$ has the standard normalisation usually encountered when first looking at QHO.

We can repeat the process for the second ψ particle but now we need to introduce labels on the momenta of incoming particles, p_1 and p_2.

$$\begin{aligned}
|\,\psi(p_1),\psi(p_2)\rangle &= \sqrt{2\omega(\boldsymbol{p}_1)}\sqrt{2\omega(\boldsymbol{p}_2)}|\,\psi(\boldsymbol{p}_1),\psi(\boldsymbol{p}_2)\rangle && (50)\\
&= \left(\int d^3\boldsymbol{y}_1\, \exp\{-ip_1y_2\} 2\Omega(\boldsymbol{p}_1)\right)\left(\int d^3\boldsymbol{y}_2\, \exp\{-ip_2y_2\} 2\Omega(\boldsymbol{p}_2)\right) \hat{\psi}(y_1)\hat{\psi}(y_2)|\,0\rangle && (51)\\
&= \prod_{i=1,2}\left(\int d^3\boldsymbol{y}_i\, \exp\{-ip_iy_i\} 2\Omega(\boldsymbol{p}_i)\right) \hat{\psi}(y_1)\hat{\psi}(y_2)|\,0\rangle && (52)
\end{aligned}$$

Taking the hermitian conjugate will give us the final state in this case, provided we also switch the labels for momenta and coordinates appropriately. That is we have

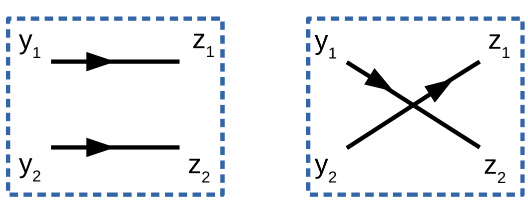
$$\langle\psi(q_1),\psi(q_2)| = \prod_{f=1,2}\left(\int d^3\boldsymbol{z}_f\, \exp\{+iq_fz_f\} 2\omega(\boldsymbol{q}_f)\right)\langle 0|\hat{\psi}(z_1)\hat{\psi}(z_2) \qquad (53)$$

From our expression (37), we then have that the relationship between the matrix element $\mathcal{M}$ and the relevant Green function for this $\psi\psi \to \psi\psi$ scattering process in Scalar Yukawa theory is just

$$\begin{aligned}
\mathcal{M}(\phi \to \bar{\psi}\psi) &= \prod_{f=1,2}\left(\int d^3\boldsymbol{z}_f\, \exp\{+iq_fz_f\} 2\omega(\boldsymbol{q}_f)\right)\prod_{i=1,2}\left(\int d^3\boldsymbol{y}_i\, \exp\{-ip_iy_i\} 2\Omega(\boldsymbol{p}_i)\right)\\
&\qquad \times G(z_1,z_2,y_1,y_2) && (54)\\
G(z_1,z_2,y_1,y_2) &= \langle 0|\mathrm{T}\psi(z_1)\psi(z_2)\psi(y_1)\psi(y_2)S|\,0\rangle && (55)
\end{aligned}$$

where the order you write the operators in the vacuum expectation value is irrelevant as that is fixed by the time ordering.

(v) The two Feynman diagrams for G which contribute to $\mathcal{M}_0$ are (no vertices in either)



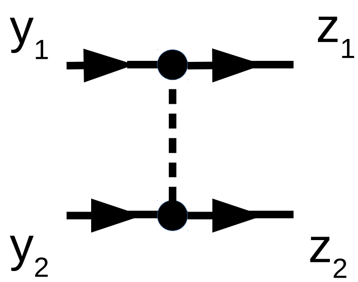
(56)

(vi) In terms of Feynman diagrams, the lack of $O(g^1)$ diagrams contributing to $\psi\psi \to \psi\psi$ scattering is encoded in the rules we have for joining vertices by propagators. If we had a diagram at $O(g^1)$

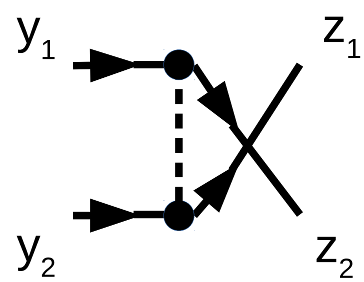
it means we have one vertex. That means it has one ϕ leg. However, this can not end anywhere. There are no other vertices allowed at this order so it can not end at another vertex. There are no ϕ in the initial or final states which could be linked to this propagator. We can not construct a legal diagram at this order if we follow the rules for diagrams in this theory.

(vii) The Feynman diagrams which contribute to the $\psi\psi \to \psi\psi$ scattering matrix element $\mathcal{M}_2$ at $O(g^2)$ are shown in the following equations 57, 58 and 59. Further similar diagrams are indicated in the accompanying text.

Diagrams representing first non-trivial contribution ($O(g^2)$) in the perturbative expansion for the $\psi\psi \to \psi\psi$ scattering process in scalar Yukawa theory of (39) are shown in equation (57).



57a



57b

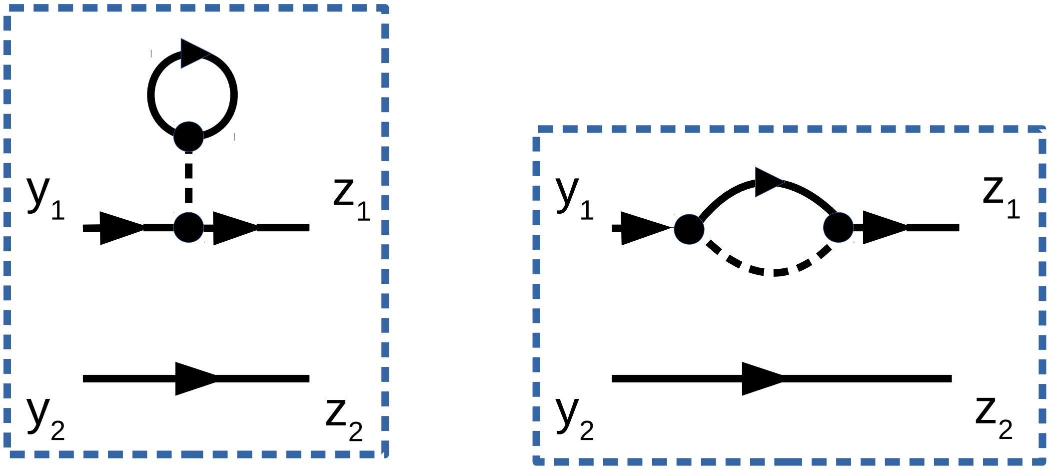
(57)

For these diagrams (57), the symmetry factors are $\mathcal{S} = 1$, no symmetries under exchange of internal lines. Remember that the arrows on the lines make a difference when assessing if exchanging internal lines leaves the diagram invariant.

To find the number of loops in diagrams (57)a and (57)b, using this formula is overkill as indeed it is for most examples we have in this course. You can see, literally, there are no loops in either (57)a or (57)b.

If you want to practice using the formula then in both diagrams of (57), the number of internal edges is $I = 1$, there are two vertices so $V = 2$, and each has just one component so $C = 1$. Using $L = I - V + C$ we find there are no loop momenta confirming our visual identification.

The diagrams in (58) also contribute at $O(g^2)$ to the $\psi\psi \to \psi\psi$ scattering process in scalar Yukawa theory of (39) but consist of self-energy contributions to the ψ propagator so do not change the non-trivial interactions. There are also the same diagrams but with q_1 and q_2 switched round and with the self-energy contribution on the p_2 leg not on the p_1 leg. This gives us 8 distinct diagrams in total of this self-energy type.



(58)

58a 58b

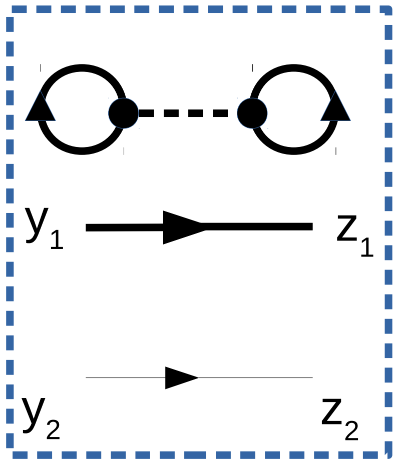
The symmetry factors in the diagrams of (58) are $\mathcal{S} = 1$, no symmetries under exchange of internal lines. Note that the first diagram, 58a, is a tadpole contribution. Such tadpole contributions are considered in the question on vacuum expectation values of ϕ.

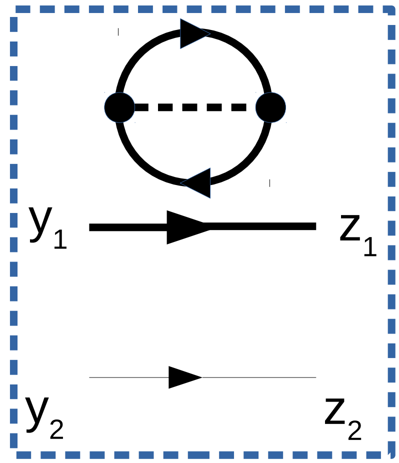
To find the number of loops is trivial here by observation; the diagrams of (58) each have one loop, $L = 1$. However, if you want to use the formula, then consider each component of each diagram separately. So start with the non-trivial component of (58) (the top part with y_1 and z_1 external legs). There we see the number of internal edges is $I = 2$, there are two vertices so $V = 2$ and there is just one component in that subdiagram, $C = 1$. Using $L = I - V + C$ we find there is one loop momenta.

It is important to not two things here. First the contribution to the formula $L = I - V + C$ from each separate component is linear, so we can apply this formula to each component separately and it is easier to do so. Second this formula $L = I - V + C$ **fails** when applied to the diagram of a single line, at least given the rules stated in this course[6]. For a single line we have one component and no internals lines or vertices yet clearly no loop momenta. This is why best to focus on the vacuum subdiagram which, like any diagram with at least one internal vertex, follows the $L = I - V + C$ rule. Better still, just spot the loops by inspection or at least as a common sense check.

[6]It is possible to change the way we count the elements of a diagram but the only problem with our framework is for this simple example. I prefer to just treat this case, where an external line is not connected to an internal vertex, as an exception.

These last type of diagrams at $O(g^2)$ are





(59)

59a 59b

These are not usually included when calculating contributions at $O(g^2)$ to the $\psi\psi \to \psi\psi$ scattering process in scalar Yukawa theory of (39). The disconnected parts are vacuum diagrams which capture the difference between the vacuum in free theory ($|\,0\rangle$) and in the fully interacting theory ($|\,\Omega\rangle$). These contributions are cancelled when the normalisation factor ($Z = \langle 0|S|\,0\rangle$) is included which is used to express the execration value in the vacuum of the full interacting theory ($|\,\Omega\rangle$) in terms of free vacuum expectation values $\langle 0|T(\text{fields})|\,0\rangle$. There are two further diagrams which are identical except the q_1 and q_2 are swapped making a total of 4 distinct diagrams with vacuum contributions.

The vacuum diagrams in (59) have symmetry factors $\mathcal{S} = 2$ coming from the interchange of the vertices as there are no external legs pinning these down. I had to use Wick's theorem to see this.

As before, the linearity of the $L = I - V + C = 2$ formula means we can just focus on the non-trivial vacuum components, ignoring the single line components which are again the only exceptions to this rule. Each of the vacuum subdiagrams in (59) has two loop momenta since the number of internal edges is $I = 3$, there are two vertices so $V = 2$ and there is just one component in that subdiagram, $C = 1$. Using $L = I - V + C = 2$ we find there are two loop moments, certainly obvious in the diagram 59a, perhaps less so in 59b.

I would again note that using this formula is overcomplicated for 59a as I would contend there are two very obvious loops in the diagram. I would concede that without some experience is harder to know that there are two independent loops in the vacuum subdiagram of 59b so knowingthe formula might help here but this example is about as difficult as we are likely to get in this course.

(viii) We have for the first diagram in 57

$$G(z_1, z_2, y_1, y_2) = \\ -g^2 \int d^4x_1 \int d^4x_2 \; \Delta_M(y_1 - x_1)\Delta_M(x_1 - z_1)\Delta_m(x_1 - x_2)\Delta_M(y_2 - x_2)\Delta_M(x_2 - z_2) \quad (60)$$

The symmetry factor is 1 in each case. The second diagram is the same expression except that z_1 (labelled as q_1 in the diagram 57b) switched is switched with with z_2.

We have for the first diagram with self-energy contributions, shown in 58a, that

$$G_{2A}(z_1, z_2, y_1, y_2) = \\ -g^2 \int d^4x_1 \int d^4x_2 \; \Delta_M(y_1 - x_1)\Delta_M(x_1 - z_1)\Delta_m(x_1 - x_2)\Delta_M(x_2 - x_2)\Delta_M(y_2 - z_2) \quad (61)$$

where again the symmetry factor is 1. There is another term where the self-energy contribution is on the other leg, so basically where we switch the 1 and 2 labels on y_i and z_f (not on the x's) to give a total of two terms. These two terms each have a partner where just y_1 is switched with y_2, leaving us with four terms of the same form but with various permutations of the coordinates (momenta) of the external lines, i.e. of the final and initial states.

For the second diagram with a self-energy contribution in 58 we have that

$$G_{2A}(z_1, z_2, y_1, y_2) = \\ -g^2 \int d^4x_1 \int d^4x_2 \; \Delta_M(y_1 - x_1)\Delta_m(x_1 - x_2)\Delta_M(x_1 - x_2)\Delta_M(x_2 - z_1)\Delta_M(y_2 - z_2) \quad (62)$$

where again the symmetry factor is 1. There is another term where the self-energy contribution is on the other leg, so basically where we switch the 1 and 2 labels on y_i and z_f (not on the x's) to give a total of two terms. If we switch just the labels on the two external legs alone, we get two more contributions. This leaves us with four terms of the same form but with various permutations of the external leg coordinates, y_1, y_2, z_1 and z_2 (likewise with external momenta if you work in momentum space).

For the diagrams with a vacuum contribution we always pick up the same contribution as given above for the $O(g^0)$, multiplied by a vacuum diagram contribution. The vacuum contributions are written out in equations (7) and (11) below. The vacuum diagram in the both cases has a symmetry factor is 2 (see the next question to see how I found that for 59a).

(ix) The type of diagram is mentioned in the text above for each case. The tadpole contributions are considered in the question on vacuum expectation values of ϕ. The question on vacuum diagrams gives more detail on those diagrams.

(x) The diagrams are exactly the same.

However the diagrams now represent a different expression for a different object. The expression is now written in terms of momentum dependent propagators $\Delta_M(p)$ and $\Delta_m(p)$. This expression is a contribution in the perturbative expansion of $G(p_1, p_2, q_1, q_2)$ — the Fourier transform of the coordinate space Green function $G(z_1, z_2, y_1, y_2)$. In the earlier parts of this question using the same diagrams but applying the coordinate space Feynman rules we were finding perturbative contributions to this coordinate space Green function $G(z_1, z_2, y_1, y_2)$.

Another way to see this is to take the contributions to the coordinate Green function $G(z_1, z_2, y_1, y_2)$ represented by the expression given by the coordinate space Feynman rules for the same diagrams. Now take the Fourier transform of that expression which gives us $G(p_1, p_2, q_1, q_2)$ but written in terms of coordinate space propagators $\Delta(x)$. If you then substitute for these coordinate propagators using their expression in terms of a Fourier transform of a momentum propagator, $\Delta(x) = \int \delta^4 k \Delta(k)$, you will find you can do all the coordinate integrations associated with the vertices. What you will end up with is just the same answer as we got using momentum space rules directly on the same set of Feynman diagrams.

5. $\phi \to \psi\bar{\psi}$ Decay in Scalar Yukawa Theory

Consider the decay of a ϕ particle of Mass m into a ψ-$\bar{\psi}$ pair (each of mass M) in the scalar Yukawa theory with interactions defined by (38).

(i) The matrix element we wish to have is

$$\begin{aligned}
\mathcal{M}(\phi \to \bar{\psi}\psi) &= \langle \psi(q_1), \bar{\psi}(q_2)|S|\,\phi(p)\rangle && (63)\\
\langle \psi(q_1), \bar{\psi}(q_2)| &= \langle 0|\hat{b}(q_1)\hat{c}(q_2)\sqrt{4\Omega(q_1)\Omega(q_2)} && (64)\\
|\,\phi(p)\rangle &= \sqrt{2\omega(p)}\hat{a}^\dagger(p)|\,0\rangle && (65)
\end{aligned}$$

where $\hat{a}(p) = \hat{a}_p$ etc is used to make the subscripts on the momentum arguments clear.

(ii) Substituting in the form (1) for the field ϕ, you find that

$$\begin{aligned}
\left(\int d^3\boldsymbol{y}\,\exp\{-ipy\}2\omega(\boldsymbol{p})\right)\hat{\phi}(y)|\,0\rangle &= \int \frac{d\!\!{}^-{}^3\boldsymbol{k}}{\sqrt{2\omega(\boldsymbol{k})}} 2\omega(\boldsymbol{p}) \int d^3\boldsymbol{y}\, e^{-i(p-k)y} && (66)\\
&= \int \frac{d\!\!{}^-{}^3\boldsymbol{k}}{\sqrt{2\omega(\boldsymbol{k})}} 2\omega(\boldsymbol{p})\delta\!\!{}^-{}^3(\boldsymbol{p}-\boldsymbol{k})\, e^{-i(\omega(p)-\omega(k))t}\hat{a}^\dagger_{\boldsymbol{k}}|\,0\rangle && (67)\\
&= \sqrt{2\omega(\boldsymbol{p})}|\,\phi(\boldsymbol{p})\rangle = |\,\phi(p)\rangle && (68)
\end{aligned}$$

where $t = y^0$. The $|\,p\rangle$ state is the one ϕ particle state with the appropriate normalisation for relativistic calculations while $|\,\phi(\boldsymbol{p})\rangle = \hat{a}^\dagger_{\boldsymbol{k}}|\,0\rangle$ has the standard normalisation usually encountered when first looking at QHO.

Taking the hermitian conjugate shows us that (switching to the convention in the lectures of using z and q for final state coordinates and momentum, write this out to check if you want)

$$\langle 0|\hat{\phi}(z)\left(\int d^3\boldsymbol{z}\,\exp\{+iqz\}2\omega(\boldsymbol{q})\right) = \langle \phi(q)|. \quad (69)$$

Changing to the complex field case will work exactly the same way so we deduce (or write it out to check this) that

$$\begin{aligned}
\langle 0|\hat{\psi}(z)\left(\int d^3\boldsymbol{z}\,\exp\{+iqz\}2\omega(\boldsymbol{q})\right) &= \langle \psi(q)|. && (70)\\
\langle 0|\hat{\psi}^\dagger(z)\left(\int d^3\boldsymbol{z}\,\exp\{+iqz\}2\omega(\boldsymbol{q})\right) &= \langle \bar{\psi}(q)|. && (71)
\end{aligned}$$

Since the two particles in the final state are distinct there is no problem applying the each of the one-particle ψ and $\psi^\dagger$ examples together (all the relevant operators commute). From our expression (65), we then have that the relationship between the matrix element $\mathcal{M}$ and the relevant Green function for this $\phi \to \psi\bar{\psi}$ decay process in Scalar Yukawa theory is just

$$\begin{aligned}
\mathcal{M}(\phi \to \bar{\psi}\psi) &= \left(\int d^3\boldsymbol{z}_1\,\exp\{+iq_1z_1\}2\omega(\boldsymbol{q}_1)\right)\left(\int d^3\boldsymbol{z}_2\,\exp\{+iq_2z_2\}2\omega(\boldsymbol{q}_2)\right)\\
&\qquad \left(\int d^3\boldsymbol{y}\,\exp\{-ipy\}2\omega(\boldsymbol{p})\right)G(z_1,z_2,y) && (72)\\
G(z_1,z_2,y) &= \langle 0|\mathrm{T}\psi(z_1)\psi^\dagger(z_2)\phi(y)S|\,0\rangle && (73)
\end{aligned}$$

where the order you write the operators in the vacuum expectation value is irrelevant as that is fixed by the time ordering.

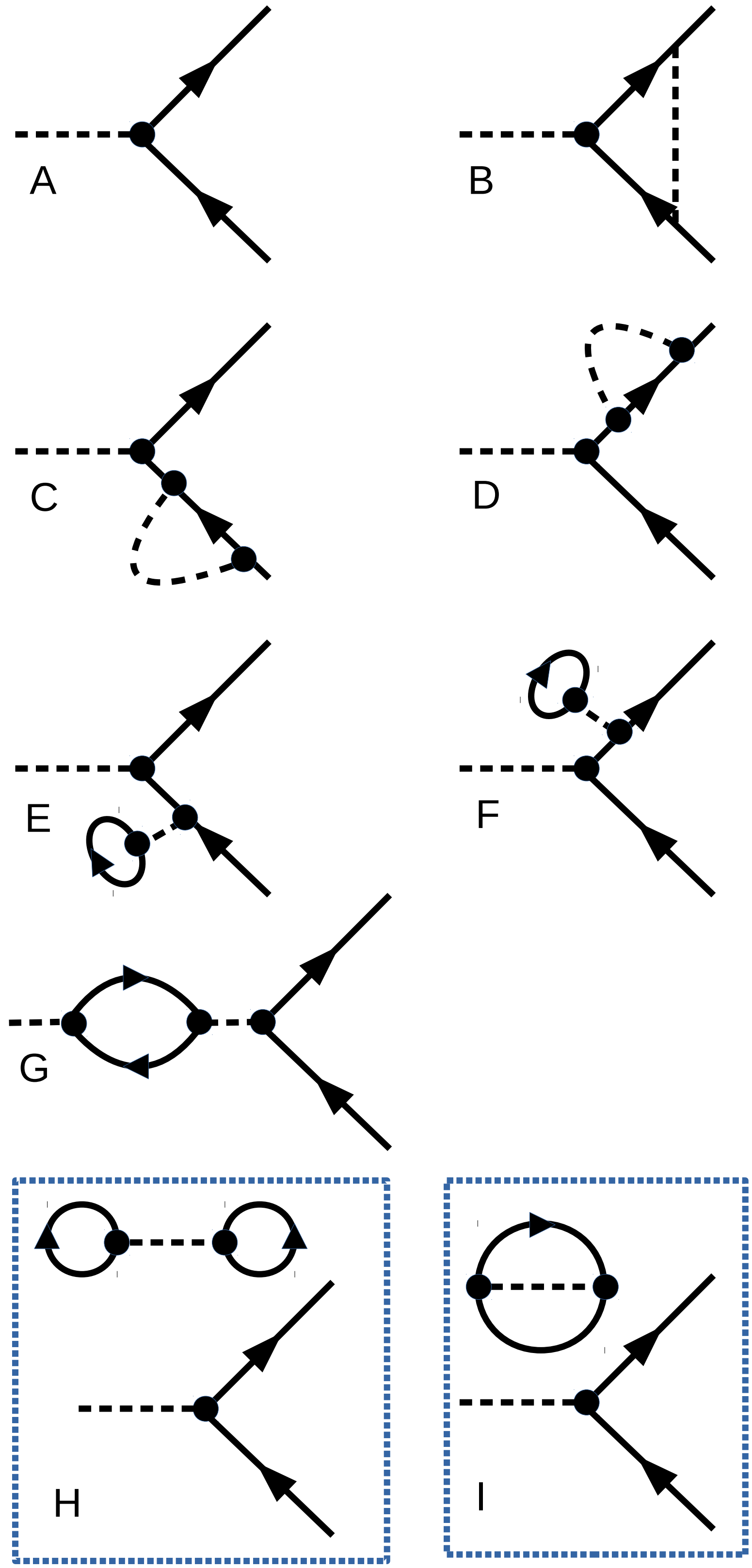


Figure 2: The Feynman diagrams for the full Green function describing the decay $\phi \to \phi\bar{\psi}$ in Scalar Yukawa theory showing all contributions up to $O(g^3)$.

(iii) The Feynman diagrams for the full Green function which correspond to contributions to $\mathcal{M}$ up to $O(g^3)$ for the decay of the ϕ particle.

The symmetry factor and the number of loop momenta for each diagram is as follows

Diagram	Symmetry factor $\mathcal{S}$	Loop momenta L	Type
A	1	0	Γ_1
B	1	1	Γ_3
C	1	1	Σ_ψ
D	1	1	Σ_ψ
E	1	1	Σ_ψ and tadpole
F	1	1	Σ_ψ and tadpole
G	1	1	Σ_ϕ
H	2	2	Vacuum
I	2	2	Vacuum
Other			More diagrams with tadpoles

(74)

(iv) The type of each diagram is given in (74).

The contributions labelled Γ_n are three-point 1PI diagram of order $O(g^n)$. These are the core contributions which really capture the QFT effects of the interaction and they describe how quantum fluctuations alter the effective strength of the interaction[7]. A 1PI diagram not be cut into two parts by cutting any one internal line. This is not true for the remaining diagrams and the effects described by the remaining diagrams will all be absorbed into other quantities once you have learnt the tricks to deal with such corrections.

The contributions labelled *vacuum* have *vacuum diagrams*, that is parts of the whole diagram are disconnected from the external legs. These disconnected parts represent the virtual fluctuations in the vacuum of a fully interacting theory. They are eliminated when we use the vacuum of the full interacting theory ($|\,\Omega\rangle$) not the free vacuum state, $|\,0\rangle$, we are currently using.

The contributions labelled Σ_ϕ come from self-energy corrections to the ϕ field propagator, while the Σ_ψ is a ψ field self-energy correction. By cutting these diagrams in the right place you would have two parts, one the core diagram (A) representing the decay process, and the second would be a contribution to a two-point Green function so represent corrections to the free field propagators. Such propagator corrections can be summed up and absorbed into the effective mass and propagator (wavefunction) normalisation so again these are not really addressing the decay process directly.

Any diagram with a *tadpole* has a part connected to the rest of the diagram by a single edge (here a ϕ field propagator). By themselves these tadpole diagrams give the vev (vacuum expectation value) of the ϕ field, that is $\langle 0|\mathrm{T}\phi(x)S|\,0\rangle$. In this QFT course we are implicitly assuming such vev are zero for all fields e.g. the vev depends on the normal ordered product of a single field which we are assuming is zero[8]. You can then deduce that diagrams with tadpoles will sum with other

[7] The proper way to tackle this issue is to use a renormalisation group approach which which be encountered in more advanced discussions.

[8] To be more precise the relation that $\hat{a}_k|\,0\rangle = 0$ assumes that $|\,0\rangle$ is the free non-interacting vacuum we want. In fact as problem sheet 1 indicated there are other (infinitely many) possible free vacua. It turns out that when we have symmetry breaking (see the Unification course), we need to work with a different free vacuum. We need to rearrange our fields if we are

tadpole diagrams to give zero because the vev is zero. If we have a problem where the dynamics of the theory produce a non-zero vev for a scalar field, then we have symmetry breaking (the Higgs mechanism) which requires further work outside this course.

In fact there are more tadpole diagrams in the general Green function. For instance the incoming decaying ϕ line can connect to one of these tadpoles. This would represent a ϕ particle being absorbed by the vacuum. The ψ and $\bar{\psi}$ lines can then connect to each other. This is just an $O(g)$ diagram so you can then throw in self-energy or vacuum corrections to get many more $O(g^3)$ terms built on this type of $O(g)$ diagram. There are several reasons I ignored this type without thinking but formally it will appear in an expression for the Green function. First the tadpoles are zero as $\langle 0|\phi(x)|\,0\rangle = 0$. Secondly the way the ψ propagator matches the final state particles in this case already means this diagram can not contribute to the matrix element for ϕ decay, though it is formally present in the generic Green function. Finally these will be disconnected diagrams and these can always be reinterpreted as corrections to other quantities. Just as the vacuum diagram contributions can be dropped by demanding that we work with the full interacting vacuum, any disconnected diagram will represent "trivial" corrections to other aspects, here the ψ propagator or the ϕ vacuum expectation value.

(v) You may work in terms of coordinates or in terms of momenta. The shape of the diagrams for Green functions are the same in both cases. To get from the coordinate diagrams given above, a representation of $G(z_1, z_2, y_1, y_2)$, just replace the labels on the external legs, so $y_i \to p_i$ and $z_f \to p_f$ in the conventions used here.

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to work with Wick's theorem in the way we have done in this course. What we need is to make sure the vacuum expectation value of odd numbers of fields is zero. That implies we must work with fields with a zero vev. So the first step is to work with a new scalar field $\eta(x) = \phi(x) - \langle 0|\mathrm{T}\phi(x)S|\,0\rangle$ so that $\langle 0|\mathrm{T}\eta(x)S|\,0\rangle = 0$ by definition. We can then use the work of this course without further problems.