The Standard Model

Prior to consideration of neutrino masses, the Standard Model Lagrangian is determined by 
$\text{SU}(3) \times \text{SU}(2) \times U(1)$, Yang-Mills gauge invariance, the requirements of renormalizability ($\Rightarrow$ only Lagrangian terms of dimensions $\leq 4$ in $D=4$ spacetime) and the following choice of group representations (SU(3), SU(2))_Y charge:
- SU(3) x SU(2) x U(1) gauge fields: $E^a_\mu (B, \xi), W^\mu_\alpha (1, 1/6, B_\nu, 1/2)$, $\gamma_5$
- leptons:
  - left (1, 2, 1/2) $L_\alpha = (\psi^\alpha)$ m=1, 2, 3 generations
  - right (1, 1, 1) $E_{\alpha \mu}$ m=1 "electron", m=2 "muon", m=3 "tau"
- quarks:
  - left (3, 2, 1/6) $Q_{\alpha \mu} = (A_{\alpha \mu})$ A=1, 2, 3 colour SU(3) index
  - right (3, 1, 1/3) $Q_{\tau \mu}$
  - (3, 1, -1/3) $d_{\alpha \mu}$
- Higgs doublet (1, 2, 1/2) $\Phi_\alpha = (\phi^+, \phi^0)$

With these choices, the most general gauge invariant and renormalizable Lagrangian is
\begin{align*}
L_m &= L_{\text{gauge}} + L_{\text{spinor}} + L_{\text{Higgs}} + L_{\text{Lagrangian}} + L_{\text{fermion}}
\end{align*}

where
\begin{align*}
L_{\text{gauge}} &= -\frac{1}{4} G_{\mu \nu}^g \varepsilon^{\mu \nu} - \frac{1}{4} W_{\mu \nu}^a \varepsilon^{\mu \nu} - \frac{1}{2} B_{\mu \nu} B^{\mu \nu} \\
L_{\text{spinor}} &= -\frac{i}{2} \bar{L}_m \gamma_\mu \gamma^\mu L_m - i \bar{\epsilon}_{\alpha \mu} \gamma_{\alpha \mu} - i \bar{\epsilon}_{\alpha \mu} \gamma_{\alpha \mu} + i \bar{\epsilon}_{\alpha \mu} \gamma_{\alpha \mu}
\end{align*}

with the covariant derivatives are
\begin{align*}
D_{L_m} &= \partial_{L_m} + \left( -\frac{i}{2} g_2 W^\mu_\tau + \frac{i}{6} g_3 B_\mu \right) L_m; D_{\bar{\epsilon}_{\alpha \mu}} = \partial_{\bar{\epsilon}_{\alpha \mu}} + i g_1 B_\mu \bar{\epsilon}_{\alpha \mu}
\end{align*}
\[ L_{\text{Higgs}} = -\partial_{\mu} \Phi^* D^\mu \Phi - \lambda \left( \Phi^* \Phi - \frac{\mu^2}{2A} \right)^2 \quad \Phi = (\Phi^+, \Phi^0) \]
\[ D_\mu \Phi = \partial_\mu \Phi + \left( \frac{i}{2} g_1 W^\mu \Phi - \frac{i}{2} g_2 B^\mu \Phi \right) \Phi \]

- Note that for the direct-product gauge group \( SU(3) \times SU(2)_L \times U(1)_Y \), the three simple factor groups can each have their own separate coupling constants: \( g_3, g_2, g_1 \).

\[ L_{\text{ Yukawa}} = -i \left( \bar{l}_{mn} \tilde{e}_{mn} \phi \right) + \bar{h}_{mn} \bar{Q}_{mn} \tilde{d}_{mn} \phi + \bar{u}_{mn} \bar{Q}_{mn} \tilde{u}_{mn} \phi \phi + \text{h.c.} \]

- Note how all the gauge-group contractions make invariants:
  \[ \bar{e}_{mn} \phi = \bar{e}_{mn} \phi \phi \quad \text{hyperccharge} \quad \frac{1}{2} - 1 + \frac{1}{2} = 0 \]
  \[ \bar{Q}_{mn} \tilde{d}_{mn} \phi = \bar{Q}_{mn} \tilde{d}_{mn} \phi \phi \quad \text{hyperccharge} \quad -\frac{1}{6} - \frac{1}{3} + \frac{1}{2} = 0 \]
  \[ \bar{Q}_{mn} \tilde{u}_{mn} \phi \phi \quad \text{hyperccharge} \quad -\frac{1}{6} + \frac{2}{3} - \frac{1}{2} = 0 \]

where \( \phi^* = e_{ab} \phi^b = (\phi^0, \phi^+) \) transforms in the \( (1, \bar{2}, -\frac{1}{2}) \)

- Complex conjugation of complex representations like \( \phi_a \), \( \tilde{d}_{\alpha} \)
  produce conjugate representations denoted with raised indices:
  \( (\phi_a)^* = \phi^\alpha \), \( (\tilde{d}_{\alpha})^* = (\tilde{d}_{\alpha})^\alpha \) where the Lorentz charge-conjugate is \( \lambda_c = \lambda^T \), \( \bar{\gamma} = \bar{\gamma}^\gamma \), \( \gamma_c = \gamma^c \) = conjugation matrix

- The prime notation on \( \bar{u}_{mn}, \bar{Q}_{mn}, \bar{d}_{mn}, \bar{e}_{mn} \)
  anticipate redefinitions in the family (generation) labels
  in which will be made to diagonalize the quark mass matrix. These redefinitions will also have
  an important effect on the quark couplings to the charged \( W^\pm \) gauge fields.
The remaining part $\Theta$ of the Standard Model Lagrangian has no effect at the classical level, but nonetheless has an important set of rules to play in the quantum theory.

$$\Theta = -\frac{e^2}{32\pi^2} G^\mu \nabla G - \frac{g^2}{32\pi^2} W^\mu \nabla W - \frac{g_s}{32\pi^2} B^\mu \nabla B$$

in which the field strengths with Higgs $G^\mu$, $W^\mu$, $B^\mu$ are Hodge duals of the original field strengths:

$$G^\mu = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} G^{\nu \rho \sigma}, \quad W^\mu = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} W^{\nu \rho \sigma}, \quad B^\mu = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} B^{\nu \rho \sigma}$$

Note that the Levi-Civita tensor $\epsilon_{\mu \nu \rho \sigma}$ is a numerically invariant tensor only under $SO(3,1)$, but not under $O(3,1)$.

Under $O(3,1)$ transformations $\Lambda^\nu$ with $\det(\Lambda) = 1$, $\epsilon_{\mu \nu \rho \sigma}$ flips by a factor of $(-1)$. Hence $\Theta$ is odd under transformations such as parity, $(x^1, x^2, x^3, t) \rightarrow (x^1, x^2, x^3, -t)$.

Non-invariance under parity is also a standart feature of the Standard Model in the fermion sector because of the key role of chiral fields:

$$P + L^\mu = L^\mu, \quad P \epsilon_{\mu \nu \rho \sigma} = \epsilon_{\mu \nu \rho \sigma}; \quad P + \left( \frac{1}{2} (1 \pm i) \delta \right), \quad \delta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\$$

The reason why $\Theta$ does not affect the theory at the classical level is that each of the three terms is a total divergence. One quick way to see this is to consider the variation of such a term and use the Yang-Mills version of the Palatini identity:

$$\langle \delta S^G \rangle = \langle \delta \bar{G}^0 \rangle - \langle \delta \bar{G}^i \rangle, \text{ in which } \bar{G}^i \text{ is the covariant derivative in the adjoint representation. Then}$$

$$\langle S(\bar{G}^0 \nabla G) \rangle = 2 \langle \bar{G}^0 \nabla \bar{G}^0 \rangle = 4 \langle \bar{G}^0 \bar{G}^0 \nabla G \rangle = 4 \left( \langle \bar{G}^0 \bar{G}^0 \nabla G \rangle \right)$$

but $\nabla G = 0$, so $\langle S(\bar{G}^0 \nabla G) \rangle = 4 \langle \bar{G}^0 \bar{G}^0 \nabla G \rangle$.\)
The total divergence property also holds for the unvaried terms. One has, e.g.,
\[ \nabla_\mu v^\nu = \partial_\mu v^\nu ; \quad K^\mu = 2 \nabla_\mu \rho \nu \left( \epsilon^{\alpha \beta \gamma \delta} \partial_\alpha v_\beta \rho + \frac{1}{3} \epsilon^{\alpha \beta \gamma \delta} \epsilon_\rho \sigma \nu \delta \right) \]

\( K^\mu \) is the Chern-Simons current.

Bosonic masses and the mixing angle

Under SU(2)\(_L\)xU(1)\(_Y\), the Higgs doublet transforms as
\[ \phi^a \rightarrow \exp \left( \frac{i}{2} \sigma \cdot a \right) \phi^b \quad \text{SU(2)}_L \]
\[ \phi^a \rightarrow \exp \left( \frac{1}{2} \rho \right) \phi^a \quad \text{U(1)}_Y \]

For the potential \( V = \lambda \left( \phi^\dagger \phi - \frac{\mu^2}{2} \right)^2 \), the vacuum expectation value \( \phi_0 \) may be taken to be
\[ \phi_0 = \left( \begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} \nu \end{array} \right) \quad \text{with} \quad \nu = \frac{\mu}{\sqrt{\lambda}}, \quad \nu \text{ real.} \]

The stabilizing subgroup \( H \) for this broken symmetry system is identified by making infinitesimal transformation around \( \phi_0 \):
\[ \delta \phi = \frac{i}{2} \left( \mathbf{5} \otimes \sigma + \rho \right) \phi_0 = \frac{i}{2} \left( \begin{array}{c} 0 \\ \mathbf{5}_1 \end{array} \right) \left( \begin{array}{c} \mathbf{5}_1 + \mathbf{5}_2 - \mathbf{5}_3 + \rho \\ \mathbf{5}_1 - \mathbf{5}_2 + \mathbf{5}_3 + \rho \end{array} \right) \left( \begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} \nu \end{array} \right) \]

Requiring \( \delta H \phi = 0 \), one finds \( \mathbf{5}_1 = \mathbf{5}_2 = 0 \) and \( \mathbf{5}_3 = \rho \), so the unbroken stability subgroup \( H = \text{U}(1)_{\text{em}} \) is generated by \( T^3 + Y \).

Expand \( \phi \) about \( \phi_0 \) using an exponential parametrization:
\[ \phi = \exp \left( \frac{i}{\sqrt{2}} (\mathbf{3} \otimes \mathbf{1}) k^i \right) \left( \begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} (\mathbf{1} + H(x)) \end{array} \right) \]

where the \( k^i \) are the broken generator combinations \( \frac{1}{2} \mathbf{1} \), \( \frac{1}{2} \mathbf{2} \), and \( \frac{1}{2} \mathbf{2} - \mathbf{1} \), and the \( H(x) \) are the would-be Goldstone bosons. Eliminate the \( H(x) \) by going to unitary gauge, where \( H(x) = 0 \), in which case \( \phi_{\text{unitary}} = \left( \begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} (\mathbf{1} + H(x)) \end{array} \right) \).
In order to work out the boson field masses, we need the covariant derivative $D_\mu \phi$ written in unitary gauge:

$$D_\mu \phi = \frac{1}{i} \left[ Q_\mu \phi - \frac{1}{2 \hbar} \left( g_1 W_3^2 + g_2 B_3 \right) (g_2 W_\mu^3 + g_1 B_\mu) \right] (0).$$

Use this to expand the kinetic term for $\phi$:

$$D_\mu \phi^* D^\mu \phi = -\frac{1}{4} \partial_\mu H \partial^\mu H - \frac{1}{8} (\partial_\mu H)^2 (W_\mu^1 + W_\mu^3) \left( W_\mu^1 + i W_\mu^3 \right) - \frac{1}{8} (\partial_\mu H)^2 (g_2 W_\mu^3 + g_1 B_\mu) (g_2 W_\mu^3 + g_1 B_\mu)$$

while the potential term $-V$ becomes

$$-V = \frac{1}{4} \left[ (\partial_\mu H)^2 - \mu^2 \right] = \left( \lambda v^2 H^2 + \lambda^2 H^3 + \frac{1}{4} \lambda^4 H^4 \right).$$

We can now read off the bosonic field masses.

Comparing the expansion of $-V$ up to second order with a standard mass term $\frac{1}{2} m^2 H^2$ to find $m^2 = 2 \lambda v^2 = 2 \mu^2$.

For the spin-one particles, one has from the gauge employ $m_{\phi_1}^2$:

$$-\frac{1}{8} g_2 v^2 \left( W_\mu^1 - i W_\mu^3 \right) \left( W_\mu^1 + i W_\mu^3 \right) - \frac{1}{8} v^2 (g_2 W_\mu^3 + g_1 B_\mu) (g_2 W_\mu^3 + g_1 B_\mu)$$

so the fields $W_{\mu}^1$ and $W_{\mu}^3$ appear nicely in the diagonal combination $-\frac{1}{8} g_2 v^2 \left( W_\mu^1 W_\mu^1 - W_\mu^3 W_\mu^3 \right)$.

Comparing to a standard spin-one mass term $-\frac{1}{2} M^2 W_\mu^1 W_\mu^1 - \frac{1}{2} M^2 W_\mu^1 W_\mu^3$, one identifies $M_1^2 = M_2^2 = g_2^2 v^2 / 4$. It is no accident that $W_\mu^1$ and $W_\mu^3$ have the same mass, because they transform into each other under the unbroken stability subgroup $H = \mathrm{U}(1)_{\mathrm{em}}$ which is generated by $T^3 + i Y$.

Setting $g_1 = g_2 = 0$ and $g_3 = g_1$, find $\delta (W_\mu^1) = \delta (W_\mu^3) = \delta (W_\mu^1) = \delta (W_\mu^3)$. So $W_\mu^1$ and $W_\mu^3$ are charged under $H_{\mathrm{em}}$. Combine them into a conjugated pair of complex vector fields $W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \pm i W_\mu^3)$. These transform as $W_\mu^\pm \rightarrow e^{\pm i \theta} W_\mu^\pm$ under $H_{\mathrm{em}}$.

Compare the mass term to $-M^2_{\pm} W_\mu^\pm W_\mu^\pm$ for a complex vector to obtain $M_{\pm} = M_1 = M_2 = g_2 v^2 / 2$. 

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**Handwritten Notes:**

- Handwritten notes are present on the page. The text is not clearly legible, but it seems to be related to the mathematical content of the document.
- There are mathematical expressions and equations present, indicating a focus on physics, likely quantum field theory or a related field.
- The text appears to be discussing the derivation of boson field masses, involving derivatives, potentials, and mass terms.

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**Additional Information:**

- The page number is 3, suggesting this is part of a larger document or textbook.
- The handwritten notes include symbols and mathematical notations that are typical in physics and mathematics, such as $\phi$, $D_\mu$, $H$, $\lambda$, and $v^2$.
The remaining vector fields $W^3$ and $B^u$ appear only in the combination $-g_1 B^u + g_2 W^3$ in the mass term. So define a normalized combination

$$Z_u = \frac{-g_1 B^u + g_2 W^3}{\sqrt{g_1^2 + g_2^2}}$$

which can also be rewritten $Z_u = W^3 \cos \theta_w - B^u \sin \theta_w$ where $\cos \theta_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}$, $\sin \theta_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}$.

determine the weak mixing angle (or Weinberg angle) $\theta_w$.

In terms of $Z_u$, the kinetic term has the standard form $-\frac{1}{4} (\partial_{\mu} Z - \partial^\mu Z) (\partial_{\nu} Z^\nu - \partial^\nu Z^\nu)$ while the mass term $-\frac{1}{2} M^2 Z_u Z^\mu \bar{Z}_u Z^\mu$ is $-\frac{1}{8} v^2 (g_1^2 + g_2^2) Z_u Z^\mu$. So one identifies $M^2 = \frac{1}{2} \sqrt{g_1^2 + g_2^2}$.

The remaining vector field combination, orthogonal to $Z_u$, is $A^u = W^3 \sin \theta_w + B^u \cos \theta_w = g_1 W^3 - g_2 B^u$.

N.B. From the coefficient of $A^u$ in $B^u$, identify the change $e = g_1 \cos \theta_w = g_2 \sin \theta_w$.

The $A^u$ field combination has no mass term and corresponds to the unbroken $W^3$ em gauge symmetry. The vector field $A^u$ is accordingly identified as the photon field.

Current values:

$$m_h = 125.09 \pm 0.21 \text{ (stat)} \pm 0.11 \text{ (syst)} \text{ GeV/c}^2$$
$$m_W = 80.385 \pm 0.015 \text{ GeV/c}^2$$
$$M^2 = 91.0876 \pm 0.0021 \text{ GeV/c}^2$$

$\sin^2 \theta_w = 0.23120 \pm 0.00015$ at $91.2 \text{ GeV/c}$ momentum transfer (so $\theta_w \sim 30^\circ$). $\theta_w$, however, "runs" as a function of momentum transfer—a key prediction of electroweak theory at the quantum level.

General Structure of Non-abelian Symmetry Breaking.

T.J.N. Kibble, Symmetry breaking in non-abelian gauge theories
Phys. Rev. 155 (1967) 1554 - 1561
Custodial $SU(2)$ symmetry

The Standard Model is formulated in the simplest way that fits with elementary particle physics data. In particular, it presumes just one Higgs doublet for the symmetry-breaking mechanism. There are many ways in which this could be generalized (e.g., by including more Higgs scalars), and it is appropriate to keep track of which features are required by the symmetry-breaking pattern $SU(2)_L \times SU(2)_R \rightarrow SU(2)$ and which features may be accidental.

Custodial symmetry is such an accidental feature. It is a rigid symmetry of the non-gauge theory obtained upon setting $g_2 = 0$, under which the Goldstone bosons transform as an $SU(2)$ (equivalently $SO(3)$) triplet. This also implies for the Standard Model gauge theory, that the vector mass matrix sub-block for $W^+_L$, $W^-_L$, and $Z$ leaves unbroken a rigid $SU(2)$ (equiv. $SO(3)$) under which the $W^{+,-,\pm}$ transform as a triplet.

Consider the non-abelian Higgs effect in general for a system with original local gauge symmetry $G$ which breaks down to $H \subset G$. This system has Yang-Mills gauge fields $A_{\alpha i}^\tau$, $i=1,\ldots,\dim(G)$ and a set of $n$ real scalars $\phi_{a}$, $a=1,\ldots,n$. Even for symmetry breaking involving complex Higgs fields, one may consider them in real form by separating real and imaginary parts. (So for the Standard Model, the complex Higgs doublet may be considered as a real quartet $\phi, i\phi$.)

The Yang-Mills + Higgs sector then has an action

$$I_{\text{YMH}} = \int \mathcal{D}v \int \mathcal{D}v_- \frac{i}{2} \left( \mathcal{D}^\tau - i g_A T^\tau \right) \phi_{a} \left( \mathcal{D}_{\alpha} - i g A_{\alpha i} T^i \right) \phi_a \left( \mathcal{D}_{\alpha}^\tau - i g_A T^\tau \right) \phi_{a} \left( \mathcal{D}^\tau - i g_A T^\tau \right)$$

where the $g_A$ coupling constants are the same for all $T^i$ generators within any simple subgroup of $G$, but may differ between different subgroup factors, e.g., between the $SU(3)$, $SU(2)$, $SU(1)$ factors.
The potential \( V(\phi) \) in a renormalizable theory will be a polynomial with up to quartic terms in \( \phi_a \). Let it be minimized for \( \phi_a = \phi_a^0 \), constant in \( x^\mu \) and invariant under the stability subgroup \( HCG \) (aka the "little group").

Divide the \( \text{dim}(G) \) generators \( T^i \), into \( e_i, i = 1, \ldots, \text{dim}(H) \) little group generators annihilating the Higgs vacuum, \( e_i \phi = 0 \), plus \( (\text{dim}(G) - \text{dim}(H)) \) broken generators \( k^i \) that do not annihilate the Higgs vacuum, \( k^i \phi = 0 \). The \( k^i \phi \) are independent and span a \( (\text{dim}(G) - \text{dim}(H)) \) dimensional subspace of the original \( n \)-dimensional representation target space of the \( \phi_a \) fields. In the absence of gauge coupling, there would be \( (\text{dim}(G) - \text{dim}(H)) \) massless Goldstone boson scalar fields corresponding to the \( k^i \) broken generators.

Parametrize \( \phi \) by \( \phi = \exp \left( i \frac{\phi_a}{\sqrt{2}} k^i \right) (\bar{\phi} + \eta_a) \)

where \( \ell \) repeated is Einstein summation and \( \nu^2 = \phi_a \phi_a^0 \).

The Higgs fields \( \eta_a(x) \) belong to the \( n - (\text{dim}(G) - \text{dim}(H)) \) complement to the \( k^i \phi \) subspace of the original \( n \)-dimensional representation space. In general, to \( \eta_a(x) \) Higgs fields will be massive (having accidental vanishing) with a \( (\text{mass})^2 \) matrix \( (D^\nu \phi_a D^{\nu} \phi^a) = \phi_{\mu} \phi_{\mu} \).

One eliminates the world-be Goldstone fields by going to unitary gauge where \( \phi_a(x) = 0 \). In unitary gauge, the action depends only on the \( \eta_a(x) \) Higgs fields and the vector fields \( A^i \), of which \( (\text{dim}(G) - \text{dim}(H)) \) are now massive. The term in the Lagrangian responsible for the vector field masses is \( \frac{1}{2} (g_{ij} T^i \phi, g_{ij} T^j \phi) A^i A^j \) where \((V, w)\) is the ordinary inner product in the real \( n \)-dimensional scalar representation space. Hence, the vector-field \( (\text{mass})^2 \) matrix is

\[
(M^2)^{ij} = -g_{ij} g_{kl} (T^i \phi, T^l \phi) = g_{ij} g_{kl} (\phi_{\mu} \phi_{\mu})^{(T^i T^l \phi)}
\]

in which \( i \) and \( j \) are not summed; \( g_{ij} \) is \( \delta_{ij} \) for all \( \ell \) in a simple factor.
The second form of $(M^2)^i$ follows since the $T_i$ are Hermitian for a compact gauge group $G$. The matrix $(M^2)^i$ is real and symmetric. Restricting $(M^2)^i$ to the broken $\kappa \Phi = 0$, one finds a positive definite $(\text{mass})^2$ submatrix for $((\text{dim} G) - \text{dim}(H))$ vector fields corresponding to the broken symmetries.

Now consider $SU(2)_L \times U(1)_Y$ symmetry breaking to $U(1)_em$. In the Standard Model with a single Higgs doublet $\Phi D$, formulate the complex field system in terms of four real fields as above. Write $\Phi D = \frac{i}{\sqrt{2}} (\Phi_3 - i \Phi_4)$ and consider the gauge-field couplings in $D_\mu \Phi D$.

$$ - i g_2 W_\mu^1 \left( \begin{array}{c} \Phi_1 + i \Phi_2 \\ \Phi_3 + i \Phi_4 \end{array} \right) - i g_2 W_\mu^2 \left( \begin{array}{c} \Phi_2 - i \Phi_1 \\ - \Phi_4 + i \Phi_3 \end{array} \right)$$

$$ - i g_2 W_\mu^3 \left( \begin{array}{c} \Phi_3 + i \Phi_4 \\ - \Phi_1 - i \Phi_2 \end{array} \right) - i g_1 B_\mu \left( \begin{array}{c} \Phi_3 + i \Phi_4 \\ \Phi_1 + i \Phi_2 \end{array} \right) = - i g_1 A_\mu^i T^i \Phi D$$

Rewrite this now in terms of the real quartet $\phi_{\text{(4R)}}^i = \left( \begin{array}{c} \phi_{\text{(4R)}}^1 \\ \phi_{\text{(4R)}}^2 \\ \phi_{\text{(4R)}}^3 \\ \phi_{\text{(4R)}}^4 \end{array} \right)$. Then for the four contributions to $- i g_1 A_\mu^i T^i \Phi D_{\text{(4R)}}$, one has

$$ g_2 W_\mu^1 \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \phi_{\text{(4R)}}^1$$

$$ g_2 W_\mu^2 \left( \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \phi_{\text{(4R)}}^2$$

$$ g_2 W_\mu^3 \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \phi_{\text{(4R)}}^3$$

Then, recalling that one writes the covariant derivative gauge field terms as $- i g_1 A_\mu^i T^i \Phi D$ with $T_i$ Hermitian, one needs to insert $1 = (-i) i$ to identify

$$ T^1_{\text{(4R)}} = \frac{1}{2} \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$ T^2_{\text{(4R)}} = \frac{1}{2} \left( \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$ T^3_{\text{(4R)}} = \frac{1}{2} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$ T^4_{\text{(4R)}} = T^2_{\text{(4R)}} = \frac{1}{2} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

All purely imaginary and antisymmetric.
One can now obtain the vector field (mass)\(^2\) matrix from \(- D_\mu \Phi (x) D^\mu \Phi (x) = - \frac{1}{2} D_\mu \Phi (x) D^\mu \Phi (x)\) by setting \(\Phi (x)\) to its vacuum value \(\Phi (0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\). 

One obtains 
\[
(M^2)_{ij} = \frac{1}{4} \begin{pmatrix}
9 g^2 v^2 & 0 & 0 \\
0 & g^2 v^2 & 0 \\
0 & 0 & g^2 v^2 - g_2^2 v^2
\end{pmatrix}
\]

noting that 
\[
(T_1)^2 = (T_2)^2 = (T_3)^2 = \frac{1}{4} T^2 = (T_4)^2 \quad \text{and} \quad T_3 T_4 = T_4 T_3 = \frac{1}{4} \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Now ask whether \((M^2)_{ij}\) has a more special structure than one might expect just from a requirement that \(SU(2) \times U(1)\) break spontaneously to \(U(1)_{\text{em}}\) generated by \(T_3 + \gamma\). As we have seen, requiring \(T_3 + \gamma\) to be unbroken implies \((M^2)_{11} = (M^2)_{22}\). This can also be seen directly using \((T_3 + \gamma) \Phi = 0\), so one requires \(0 = g_2 \Phi, [T_3, [T_3, T_3]] \Phi\) (recall: \(i\) and \(j\) are not summed here). Using \([T_1, T_i] = i T_3 + \text{etc.}\) and \([Y, T_i] = 0\), one gets for \(i = 1, j = 2\)
\[
g^2 (\Phi, [T_3, T_1 T_3] \Phi) = ig^2 (\Phi, T_1 T_3 \Phi) - (\Phi, T_1 T_3 \Phi) = 0
\]

So \((M^2)_{11} = (M^2)_{22}\). Similarly, taking \(i = j = 1\), one finds
\[
g^2 (\Phi, T_1 T_3 \Phi) = ig^2 (\Phi, T_1 T_3 \Phi) = 0 \quad \text{so (M^2)_{12} (M^2)_{21} = 0.}
\]

From \(i = 1, j = 3\) one finds \(g^2 (\Phi, T_3 T_1 \Phi) = 0 \quad \text{so (M^2)_{13} = 0.}\)

while from \(i = 2, j = 3\) one finds \((M^2)_{13} = 0.\) From \(g_2 (\Phi, [T_3, T_3] \Phi) = 0\)

for \(i = 2, 3\) one obtains \((M^2)_{14} = (M^2)_{24} = 0.\) So the upper right-hand block of \((M^2)_{ij}\) must vanish, and similarly for the lower left-hand block.

The remaining lower right-hand block is symmetric, but is not otherwise constrained by the above argument, since \([T_3, T_3] = [T_3, T_4] = 0.\) However, this block is constrained by the requirement that \((M^2)_{ij}\) have a zero eigenvalue for the unbroken \(U(1)_{\text{em}}.\) This requires the lower right-hand blocks determinants vanish.
Taken together with the other block structure, one accordingly finds that \((M^2)_{ij}^{0}\) can thus be written
\[
(M^2)_{ij}^{0} = \frac{1}{4}
\begin{pmatrix}
 g_{\mu}^2 & 0 & 0 & 0 \\
 0 & g_{\mu}^2 & 0 & 0 \\
 0 & 0 & g_{\nu}^2 - g_{\mu}^2 u^2 & 0 \\
 0 & 0 & 0 & g_{\nu}^2 (g_{\mu}^2 u^2)
\end{pmatrix}
\]
which is similar to the structure of \((M^2)_{\text{SM}}\) except that in \((M^2)_{\text{SM}}\) one doesn't necessarily have \(u = v\).

The zero eigenvector of the lower right-hand block is, up to normalization, \((g_1, g_2)\), or, normalizing, \((\sin \theta_w, \cos \theta_w)\) with
\[
\sin \theta_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \quad \cos \theta_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} \quad \text{as before.}
\]

The non-zero eigenvalue of the lower right-hand block is given by
\[
M_z^2 = u^2 \left( g_{\mu}^2 - (g_{\mu}^2 + g_{\nu}^2) u^2 \right) = u^2 (g_{\mu}^2 + g_{\nu}^2) = (M^2)_{33}^0 (1 + \tan^2 \theta_w)
\]
since \(\tan \theta_w = \frac{g_1}{g_2}\) . So \(M_z^2 = (M^2)_{33}^0 \sec^2 \theta_w\), i.e. \(M_3^2 = \cos^2 \theta_w M_\mu^2\).

Letting \((M^2)^{11} = M_\mu^2 = M_W^2\) and \((M^2)^{33} = M_3^2\), the special property of the Standard Model vector mass pattern is \(M_3 = M_W\), or \(u = v\). This specific structure is a consequence of having used just one \(SU(2)\) scalar field doublet to break \(SU(2)_L \times U(1)_Y\) to \(U(1)_{\text{em}}\).

The Standard Model potential \(V(\phi^+ \phi)\) can be written in terms of \(\phi(4R)\) as \(V(\phi^1 \phi^2, \phi^2 \phi^3, \phi^3 \phi^4)\). As required by the \(SU(2)_L \times U(1)_Y\) gauge symmetry, this is locally invariant under this 4-parameter gauge group. But it is also invariant under an \(SO(4) = SO(3) \times SO(3)\) rigid symmetry.

Symmetry breaking with a vacuum \(\phi(4R) = (\frac{\pi}{2}, 0, 0, 0)\) breaks this \(SO(4)\) rigid symmetry down to \(SO(3) = SU(2)/Z_2\).

This unbroken symmetry of the Higgs sector is the custodial \(SU(2)\).
The Standard Model relation \( M_W = M_2 \cos^2 \theta_W \) is thus a consequence of the \( SU(2) \) electroweak custodial symmetry, following from the simple Higgs spontaneous symmetry breaking mechanism with a single \( SU(2) \) scalar doublet \( \phi_0 \). More complicated models of the Higgs mechanism can give deviations from this, even at the tree level. Even in the simple Standard Model, custodial symmetry is broken by couplings of \( \phi_0 \) to \( B_u^m \) and the Yukawa couplings. Accordingly, radiative corrections can be expected to alter the \( M_Z \) to \( M_W \) mass relation. It is useful to define

\[
\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} \quad \text{(Current experimental value: } \rho = 1.0004 \text{)}
\]

Denoting the tree-level value of \( \rho \) as \( \rho_0 \), one has \( \rho_0 = 1 \) in the Standard Model.

Consider new couplings to currents \( J_{u,2,3} \) and \( J_{y+} \):

\[
g_1(W_u^+ J_{u,2}^+ + W_3^+ J_{y,3}^+) = g_2 W_3^+ J_{3,3}^+ = g_1 B_u^{2,4}
\]

Identify the electromagnetic charge \( e = g_1 \), \( C = g_2 S \) (with shorthand \( C = \cos \theta_W \), \( S = \sin \theta_W \)). Then the \( W_u \) and \( B_u \) couplings can be rewritten

\[
g_2 W_3 = J_3^m - g_1 B_u J_4^m = \frac{e}{\cos \theta_W} \left( A_u + Z_u \right) J_3^m - \frac{e}{\sin \theta_W} \left( A_u - Z_u \right) J_4^m
\]

Then writing \( J_{Em} = J_3^m + J_4^m \), one has the couplings

\[
e A_u J_{Em} = Z \left( \frac{e}{\cos \theta_W} J_3^m - \frac{e}{\sin \theta_W} J_4^m \right)
\]

as is appropriate for electromagnetic coupling to \( A_u \).

After symmetry breaking, \( W_3^+ \) and \( W_2^+ \) become massive with mass \( M_W \), and \( Z_u \) with mass \( M_Z \).

Ignore higher (cubic, quartic) self couplings and consider integrating out these massive fields to obtain current-current effective theory interactions.
For a real vector field $\mathbf{A}$ with mass $m$ coupled to a conserved $(\tilde{\mathcal{J}}^\mu)$ current $\mathcal{J}^\mu$ with charge coupling $q$, one has the action

$$\int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} \mathbf{A}_\mu \mathbf{A}^\mu - q \mathcal{J}^\mu \right), \quad F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu.$$ 

giving field equations $\partial_\nu F^{\mu\nu} = q \mathcal{J}^\mu$, which in momentum space becomes

$$(k \cdot k_\mathbf{A} - (k^2 M^2) \mathcal{J}^\mu) V^\mu(k) = q \mathcal{J}^\mu.$$ 

Solve this:

$$V^\mu = -q \left( \frac{\mathcal{J}^\mu + k^\mu k^\nu \mathcal{J}_\nu}{k^2 + M^2} \right) = -q \left( \frac{\mathcal{J}^\mu + k^\mu k^\nu \mathcal{J}_\nu}{k^2 + M^2} \right).$$

For low momenta $|k| \ll M$, the factor $\frac{1}{k^2 + M^2} \approx \frac{1}{M^2}$ and for a conserved current $\mathcal{J}^\mu$, $k^\mu \mathcal{J}_\nu = 0$, so the propagator

$$-\frac{\mathcal{J}^\mu + k^\mu k^\nu \mathcal{J}_\nu}{k^2 + M^2}$$

becomes just $-\frac{\mathcal{J}^\mu}{M^2}$ and accordingly one has $V^\mu \approx -q \frac{\mathcal{J}^\mu}{M^2}$.

At low momentum transfers, the terms in the effective Lagrangian arising from $W$ and $Z$ vector field exchanges are

$$-\frac{g}{\sqrt{2}} (W_{\mu \nu} J^\mu + W_{\nu \mu} J^\nu) + Z_{\mu} \left( \frac{e}{\sqrt{2}} J^\mu - \frac{e \sqrt{2}}{c} J^\mu_{\text{em}} \right) - \frac{e}{\sqrt{2}} (Z_{\mu} - \frac{e \sqrt{2}}{c} J^\mu_{\text{em}}) g_{\mu \nu} \left( J^\nu + s^2 J^\nu_{\text{em}} \right).$$

The conventional definition of the Fermi constant is

$$g_F = \frac{\sqrt{2} e^2 / (8 \pi^2 M^4)}{2},$$

so the current-current interactions arising from $W$ and $Z$ exchange can be written

$$\frac{8}{\sqrt{2}} g_F \left[ J^\mu_{\text{em}} + i J^\mu_{\text{em}} \right]^2 + \phi \left( J^\mu_{\text{em}} + s^2 J^\mu_{\text{em}} \right)^2.$$ 

So the custodial symmetry implying $p_0 = 1$ is a symmetry relating the strength of the weak ($p_2$) part of the neutral-current interactions to the strength of the charged-current ($p_2^\mp$) interactions. The currents $J^\mu_{\text{em}}$ and the neutral current ($J^\mu_{\text{em}} + s^2 J^\mu_{\text{em}}$) are quadratic in fermions, so this gives the structure of the current-current 4-Fermi effective interactions:

For the leptons, one has a chiral charged current

$$\lambda_{\mu\nu}(x) = \frac{1}{2} \bar{e}_\alpha \gamma_\mu (1 + \gamma_5) e_\nu + \frac{1}{2} \bar{\mu}_\alpha \gamma_\mu (1 + \gamma_5) \mu_\nu + \frac{1}{2} \bar{\tau}_\alpha \gamma_\mu (1 + \gamma_5) \tau_\nu, \quad \text{electron} \quad \mu \quad \tau \quad \text{e, mu, tau}.$$
A bit of history: in December 1956-January 1957, George Sudarshan, a student of Robert Marshak, was studying non-parity-violating beta decays and found the known experiments on angular correlations to be inconsistent. By the time of the Rochester-Savannah conference in Spring 1957, he had come to the conclusion that the interaction had to have a chiral V-A structure (vector-minus-axial vector). He asked to present this at the conference, but was not allowed, since he was only a graduate student. P.T. Matthews (a founder, together with Salam of the Imperial Theory Group) was asked to report on the V-A theory briefly, but forgot to do so.

Marshak was going to be at the RAND Corporation in Los Angeles and offered Sudarshan and another student one-month salaries during the summer to work in LA. However, as an alien, Sudarshan could not enter RAND, so they met outside in restaurants. Gel'fand was also at RAND at that time and Marshak invited him to such a lunch. Sudarshan gave a full presentation on the V-A theory. The data at that time were internally inconsistent; he also pointed out which experiments were likely to be mistaken.

Marshak asked Sudarshan to write up the work, which he did. Instead of publishing it immediately, however, Marshak decided to wait until a Sept. 1957 conference in Padua and Venice. A later paper was sent to the Physical Review, but in the meantime, Feynman and Gel'fand published a paper in the Physical Review asserting the V-A structure of the weak interactions. They thanked Sudarshan for "important discussions." Most often, reference is made only to the Feynman and Gel'fand paper. Feynman, however, acknowledged Sudarshan's discovery of V-A in later writings.
Yukawa and gauge field couplings of fermions

In addition to the masses of \( W^\pm \) and \( Z \), spontaneous symmetry breaking also generates masses for the leptons and quarks via the Yukawa interactions. In the unitary gauge, one has \( \phi = \left( \begin{array}{c} 0 \\ \frac{i}{2} (U + H \bar{U}) \end{array} \right) \) (recall \( \phi_0 = e^{i \beta} \phi^{*} \)), so upon inserting these into the SM Yukawa terms one has

\[
\text{Yukawa} = -\frac{i}{2} (U + H) \left( f_{mn} e_{Lm} e_{Rn} + h_{mn} d_{Lm} d_{Rn} + k_{mn} \bar{l}_{Lm} \bar{l}_{Rn} \right) \text{etc.}
\]

Fermion masses arise from the \( \frac{U}{2} \) terms, while the other terms involving \( H(x) \) are Higgs - (lepton)\(^2\) and Higgs - (quark)\(^2\) couplings.

The mass terms for the fermions involve mixtures of the three \( m = 1, 2, 3 \) generations for general \( f_{mn}, h_{mn} \) and \( k_{mn} \). However, one can diagonalize the structure of the mass terms by making non-symmetry \( U(3) \) unitary field redefinition transformations independently for all the leptons and quark flavours:

\[
E_{Lm} \rightarrow U_{mn} E_{Ln}, \quad E_{Rm} \rightarrow U^{*}_{mn} E_{Rn}, \quad W_{Lm} = U_{mn} W_{Ln}, \quad W_{Rm} = U^{*}_{mn} W_{Rn}, \quad d_{Lm} = U_{mn} d_{Ln}, \quad \bar{d}_{Rm} = U^{*}_{mn} \bar{d}_{Rn}.
\]

It is at this point that we define the unprimed quarks:

\[
E_{Lm} \rightarrow E_{Lm}, \quad W_{Lm} = W_{Lm}, \quad d_{Lm} = d_{Lm}, \quad \text{etc.}
\]

Note that all kinetic terms are consequently left unchanged by these transformations:

\[
-i \bar{E}_{Lm} \gamma^\mu E_{Lm} \rightarrow -i \bar{E}_{Lm} U_{mn} \gamma^\mu U^{*}_{mn} E_{Ln} = -i \bar{E}_{Lm} \gamma^\mu E_{Ln}, \quad \text{etc.}
\]

A note on nomenclature: for the quarks, there are six flavours, comprising the up and down quarks of each of the three generations, or families. At this point, after SU(2)\(_L\) x U(1)\(_Y\) symmetry breaking, the up and down quarks of each generation are treated separately.
The effects of these field-rede finition transformations on the Yukawa coupling matrices are biunitary transformations
\[ f_{mn} \rightarrow U_{mn}^* f_{mn} U_{mn}, \quad h_{mn} \rightarrow U_{mn}^* h_{mn} U_{mn}, \quad k_{mn} \rightarrow U_{mn}^* k_{mn} U_{mn}. \]
Note that although prior to symmetry breaking the Yukawa interactions are independent, \( f_{mn}, h_{mn}, k_{mn} \) are all involved in doublets (capitalized fields), the unitary transformations for each doublet must make the unitary transformation matrices of \( f, h \) and \( k \).

With the unitary transformation matrices acting on the first and second indices of \( f_{mn}, h_{mn} \) and \( k_{mn} \) independently, the biunitary transformations can be used to make \( f_{mn}, h_{mn} \) and \( k_{mn} \) diagonal and with positive real values:
\[ f_{mn} = \text{diag} \left( f_1, f_2, f_3 \right), \quad h_{mn} = \text{diag} \left( h_1, h_2, h_3 \right), \quad k_{mn} = \text{diag} \left( k_1, k_2, k_3 \right) \]
where the \( f_m, h_m \) and \( k_m \) are all real and positive.

The mass terms are then
\[ -\frac{i}{\sqrt{2}} \left( f_{mn} E_{rm} \overline{E_{rm}} + h_{mn} \overline{h_{mn}} + k_{mn} \overline{k_{mn}} \right) \]
where the indices \( m \) correspond to the same values as their corresponding counterparts, but do not trigger Einstein summation.

Then the lepton and quark masses are:
\[ m_e = \frac{V}{\sqrt{2}} f_n, \quad m_d = \frac{V}{\sqrt{2}} h_n, \quad m_u = \frac{V}{\sqrt{2}} k_n. \]

Note that since right-handed neutrinos have not been introduced into the original Standard Model, no mass terms arise, so far, for the neutrinos.

The absence of right-handed neutrinos, and consequently the neutrino mass terms has another consequence. One may freely transform the left-handed neutrino fields \( \nu_l \) with the same \( U(3) \) transformation as for the left-handed \( \nu_e \). Then the whole \( SU(2)_L \) doublet \( \nu_l \) transforms into \( U_{mn} \nu_{mn} \), and consequently the gauge-coupled kinetic term
\[ -\frac{1}{2} \overline{\nu_{mn}} \nu_{mn} \]
under this \( U(3) \) redefinition. Similarly, the full gauge-coupled right-handed kinetic term
\[ -\frac{1}{2} \overline{\nu_{mn}} \nu_{mn} \]
is unchanged under \( U(3) \).
CKM Structure

Now consider what happens to the quark field gauge couplings. In order to diagonalize the quark mass matrix, one needs to make independent $U(3)$ transformations $U^u_{mn}$ and $U^d_{mn}$, so there is no uniform transformation of the left-handed quark doublet $Q_{LM}$. The pure derivative kinetic terms $-i \bar{Q}_{LM} d_{LM}$ and $-i \bar{Q}_{LM} d^*_{LM}$ are naturally diagonal in the $u_{LM}$ and $d_{LM}$ quarks, and consequently are themselves left unchanged. The same is the case for the other terms that are naturally diagonal in $u_{LM}$ and $d^*_{LM}$. Since the $T^3$ and $Y$ generators do not mix the top and bottom components of SU(2)$_L$ doublets, this applies to the quark gauge couplings to $W^3$ and $B^0$ (i.e. to $A_0$ and $Z_0$). So the neutral gauge couplings are unchanged.

The gauge-coupling terms involving the charged gauge fields $W^\pm$ are not all invariant, however. The charged currents' interaction terms for the quarks are

$$\mathcal{L}_{cc} = -\frac{g_2}{\sqrt{2}} \left[ W^+_{LM} \bar{u}_{LM} \gamma^\mu d_{LM} + W^-_{LM} \bar{d}_{LM} \gamma^\mu u_{LM} \right],$$

and these transform into

$$\mathcal{L}_{cc} = -\frac{g_2}{\sqrt{2}} \left[ W^+_{LM} \bar{V}_{mn} u_{nL} \gamma^\mu d_{LM} + W^-_{LM} \bar{V}_{mn} d_{nL} \gamma^\mu u_{LM} \right],$$

where $V_{mn} = (U^d_{LM} (U^u_{LM})^\dagger)_{mn}$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Note that $V^TV = 1$, so $V_{mn}$ is itself a unitary matrix.

[Note that the couplings of $W^\pm$ to the leptons, i.e.

$$-\frac{g_2}{\sqrt{2}} \left[ W^+_{LM} \bar{V}_{mn} \gamma^\mu e_{LM} + W^-_{LM} \bar{V}_{mn} e_{LM} \gamma^\mu \bar{e}_{LM} \right]$$

are invariant under $U^{\pm(3)}$, as noted above, once one transforms $V_{mn}$ with the same $U^{\pm(3)}$ matrix as $e_{LM}$.]

A priori, the CKM matrix, which is a $U(3)$ matrix, would depend on $3^2 = 9$ real parameters. However, not all components of $V_{mn}$ are of equal significance; some can be eliminated by putting $V_{mn}$ into one of another special form using further non-symmetry transformations.
In order to simplify $V_{\text{min}}$, one can make pure phase transformations of the 6 quark flavours, acting in the same way on $L$ and $R$ chiralities, i.e. like $U_{\text{km}} \rightarrow e^{i \alpha_{\text{km}}} U_{\text{km}}$, $U_{\text{em}} \rightarrow e^{i \alpha_{\text{em}}} U_{\text{em}}$, etc. Such pure phase transformations leave the pure derivative $D$ kinetic terms and also mass terms such as $y_{\text{trij}} U_{\text{ij}}$ invariant. Note, however, that only non-symmetry transformations can be used in this way. There is one phase symmetry of the whole $S_N$ Lagrangian which has no effect on $V_{\text{min}}$.

Overall transformations of all six quark flavours by the same phase. Consequently, the number of parameters that can be adjusted to put $V_{\text{min}}$ into a special form is $6 - 1 = 5$. The remaining, physically significant, number of components of $V_{\text{min}}$ is $9 - 5 = 4$.

There are a number of different conventional ways to parametrize the CKM matrix. It is appropriate first to count the number of parameters $V_{\text{min}}$ would have if it were purely real. In that case, it would be an orthogonal $S(3)$ matrix, determined by three parameters. In addition to these, there is one parameter determining the essentially complex nature of $V_{\text{min}}$, a phase. The three $S(3)$ parameters can be taken to be three angles $\Theta_{12}$, $\Theta_{23}$ and $\Theta_{13}$ corresponding to rotations in the $[\text{min}]$ flavour planes. This phase is often denoted $\delta$.

In the parametrization advocated by the Particle Data Group, one has

$$V = \begin{pmatrix}
1 & 0 & 0 \\
0 & C_{23} & S_{23} \\
0 & -S_{23} & C_{23}
\end{pmatrix} \begin{pmatrix}
C_{12} & 0 & S_{13} e^{i \delta} \\
0 & 1 & 0 \\
-S_{13} e^{-i \delta} & 0 & C_{13}
\end{pmatrix} \begin{pmatrix}
C_{12} & S_{12} & 0 \\
-S_{12} & C_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}$$

with $C_{\text{mn}} = \cos \Theta_{\text{mn}}$, $S_{\text{mn}} = \sin \Theta_{\text{mn}}$.
Numerical values for the angles and phase are
\[ \Theta_{12} = 13.02 ^\circ \pm 0.04 ^\circ, \quad \Theta_{23} = 2.36 ^\circ \pm 0.08 ^\circ, \quad \Theta_{13} = 0.20 ^\circ \pm 0.02 ^\circ \]
and \[ J = 69 ^\circ \pm 5 ^\circ \] [Particle Data Group 2012]. The CKM matrix \( V_{\text{CKM}} \) expresses the transformation between the original basis of fields in which generations (families) do not mix in the gauge boson interactions, known as flavour basis, and the basis in which all the fermionic mass terms are diagonal, known as mass basis.

All the rotation angles are relatively small; one has \[ \Theta_{13} \ll \Theta_{23} \ll \Theta_{12} \ll 1 \] when expressed in radians. So charged-current interactions that link fermions of different generations are highly suppressed \( \leftrightarrow\) \( V_{\text{CKM}} \) is close to being a unit matrix. The dominant amount of flavour mixing is given by \( \Theta_{12} \). Letting \( \lambda = s_{12} = \sin \Theta_{12} \approx 0.23 \),

another common parametrization of \( V_{\text{CKM}} \) is the Wolfenstein parametrization, which more clearly indicates the size of each matrix element. Through third order in \( \lambda \) it is

\[
V = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 (\rho - i \eta) \\
-\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\
A \lambda^3 (1 - \rho - i \eta) & -A \lambda^2 & 1
\end{pmatrix}
\]

where \( A \) and \( \beta + i \eta \) are \( \Theta(1) \).

The above discussion is for the currently understood case of \( N=3 \) generations. It is also instructive to consider what happens for general \( N \) generations of fermions. The CKM matrix is an \( N \times N \) unitary matrix, so depends a priori on \( N^2 \) real parameters. If it were real, it would be an \( O(N) \) matrix depending on \( \frac{1}{2} N(N-1) \) parameters. The difference between these two numbers, \( N^2 - \frac{1}{2} N(N-1) = \frac{1}{2} N(N+1) \) is the number of phase parameters determining the complex structure of \( V_{\text{CKM}} \).

However, one can, as in the \( N=3 \) case, make non-symmetry phase transformations of the \( 2N \) quark flavours. Among these, \( V_{\text{CKM}} \) is one combination that is a symmetry (overall \( \text{CP} \) field phase invariant).
Thus, from the $\frac{1}{2} N(N+1)$ a priori phase parameters in $V_{\text{mix}}$, one may use $(2N-1)$ parameters to fix a special form for $V_{\text{mix}}$ without disturbing the diagonal mass structure. The number of physically meaningful phases is then

$$P = [N^2 - \frac{1}{2} N(N-1)] - (2N-1) = \frac{1}{2} N(N-1)(N-2).$$

If there were only $N=2$ generations, which seemed to be the case before the discovery of the tau family, then one would have $P = 0$. In that case, the CKM matrix would be an $O(2)$ matrix

$$V_{\text{mix}} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \theta = \theta_{12} \quad \text{Cabibbo angle}$$

As one can see from the Wolfenstein parametrization of the $N=3$ case, the $2 \times 2$ submatrix of the $N=3$ CKM matrix $V_{\text{mix}}$ contains just this $N=2$ submatrix. Exponential values for the $\theta_{13}$ and $\theta_{13}$ angles give only very small corrections to this structure.

Another question that can be addressed is that of a possible fourth generation. The overall CKM matrix is unitary by construction. However, if there were a fourth generation, the restriction of the CKM matrix to the three-generation subsector would not be unitary. So testing the $N=3$ $V_{\text{mix}}$ for unitarity is a way to look indirectly for physics beyond the Standard Model. Unitarity requires $\sum_{p} V_{\text{mix}} V_{\text{mix}}^{*} = \delta_{p q}$ for any $p$ and $q$. For example $V_{\text{ud}} V_{\text{ud}}^{*} + V_{\text{cd}} V_{\text{cd}}^{*} + V_{\text{td}} V_{\text{td}}^{*} = 0$. The best measured of these three quantities is $V_{\text{ud}} V_{\text{cd}}^{*}$; dividing by it leads to

$$\frac{V_{\text{ud}} V_{\text{ud}}^{*}}{V_{\text{cd}} V_{\text{cd}}^{*}} + 1 = 0.$$ 

This expresses closure of the unitarity triangle:

$$\begin{bmatrix} V_{\text{ud}} V_{\text{ud}}^{*} \\ V_{\text{ud}} V_{\text{cd}}^{*} \\ V_{\text{cd}} V_{\text{cd}}^{*} \end{bmatrix} = 0$$

If $V_{\text{mix}}$ were real, the unitarity triangle would collapse to a line. For the non-rescaled triangle, the Jarlskog invariant is defined as

$$J = \text{Im} \left( V_{\text{ud}} V_{\text{cd}}^{*} V_{\text{td}} V_{\text{db}}^{*} \right) = (2.96 \pm 0.02) \times 10^{-5}.$$
Neutrino masses

When the Standard Model was originally formulated, there was no evidence for neutrino masses, so the SM was constructed with massless neutrinos. A key puzzle regarding neutrinos emerged, however, from the neutrino-counting experiment of Ray Davis in the Homestake Gold Mine, Lead, South Dakota (1970 to 1994). John Bahcall had calculated the expected rate of neutrino capture for neutrinos emitted by nuclear fusion in the Sun. The Homestake experiment consisted of 100,000 gallons of dry-cleaning fluid (perchloroethylene), which is rich in chlorine. 1478 meters underground. The neutrino capture reaction is $\nu_e + ^{37}\text{Cl} \rightarrow ^{37}\text{Ar} + e^-$. Every few weeks, Davis bubbled helium through the tank to collect the argon. Argon 37 is radioactive, with a half-life of 35 days. In this way, a few tens of $^{37}\text{Ar}$ could be detected, and the count of neutrinos captured made.

The puzzle posed by the Homestake experiment was that the detected rate of neutrino capture was only about 1/3 of that predicted by Bahcall. The experiment could, however, be calibrated by bringing radioactive, neutrino-producing sources down near the detector. Another experimental finding was the discovery by Masatoshi Koshiba at the Super Kamiokande experiment (Kamioka, Japan) of oscillations between the three generations of neutrinos in a well controlled beam from the KEK accelerator in Tsukuba. The Kamioka experiments also found a deficit in the flux of neutrinos from the Sun. The conversions $\nu_e \rightarrow \nu_x$ & $\nu_x \rightarrow \nu_e$ can explain this deficit, since the Homestake detector was sensitive only to $\nu_e$ electron neutrinos and the Kamioka detector predominantly.

The idea of neutrino oscillations had been put forward by Bruno Pontecorvo in 1957, not long after he emerged from sequestration by the KGB following his defection to Russia in 1950.
Pontecorvo's original proposal was for neutrino-antineutrino transitions. This has not been observed, but a 1968 paper by Pontecorvo and L. N. Grigor'ev ("Neutrino astronomy and lepton change," Physics Letters B28 (1969), 493) elaborated the key idea that neutrino mixing is a natural consequence of massive neutrinos. Since 2001, results from the Sudbury Neutrino Observatory provide clear evidence of neutrino-flavour change, with only about 35% of Solar neutrinos remaining as νe when they reach the Earth.

These observations give evidence for neutrino masses, in particular for differences in the (mass)² values for νe, νμ, and ντ, but they don't give any single value. The Δm² differences are now known to be on the order of 10⁻⁴ eV², with indications that one of the Δm² differences is 0.002 eV. Consequently, at least one neutrino species must have a mass that is at least the square root of this, i.e. 0.05 eV. Bounds on the sum of the masses for the known three neutrinos are obtained from cosmology, particularly galaxy surveys and analyses of cosmic microwave background data. These generally indicate that the sum of the masses is less than about 0.3 eV.

The upshot of the various experimental and observational results is that at least some neutrino species have masses on the order of 10⁻² eV. This is vastly different from the masses of other SM leptons, e.g. the electron mass of 0.511 MeV. So a big challenge is to find a mechanism that makes such a tiny scale occur naturally.

One way to obtain small masses for the known neutrino species is to consider the possibility of "sterile" right-handed singlets with respect to all SM gauge symmetries (i.e., (1,1,0), not currently observed. Denoting such right-handed spinor fields
There are two new types of terms that can occur in the Lagrangian. The first of these is a type of Yukawa coupling: \(-i \bar{\psi}_{Lm} \Gamma_{Rn} \phi + \text{h.c.}\). Check hypercharge invariance: \(+\frac{1}{2} - \frac{1}{2} = 0\). Taken alone, and assuming the \(\psi_{Rn}\) coefficients to be of the same order of magnitude as the SM \(\psi_{Lm}\), \(\Gamma_{Rn}\) and \(\bar{\psi}_{Lm}\) coefficients, these would produce neutrino mass terms after symmetry breaking that are far too large, perhaps comparable to the electron, muon, or tau masses \((m_e \sim 0.5 \text{ MeV}, m_\mu \sim 100 \text{ MeV}, m_\tau \sim 1.8 \text{ GeV})\).

Achieving small neutrino masses can be arranged in a somewhat paradoxical way by including the second type of new term in the Lagrangian, with an even larger mass value. For a sterile neutrino that is an SU(3) x SU(2) x U(1) singlet, one can construct bilinear Majorana mass terms:

\[-i M_{mn} \left( \bar{\psi}_{Rn} \Gamma^c \psi_{Lm} + \bar{\psi}_{Lm} \Gamma^c \psi_{Rn} \right), \text{ where } \Gamma^c_{\nu e} = (\bar{W}^\dagger)_{\nu e}\]

are the charge-conjugated fields. Such spinor bilinear mass terms are ruled out for all the other SM spinor fields because they would be incompatible with gauge invariance. Only for \((LeL)\) gauge singlets are they allowed. Instead of being small, the \(M_{mn}\) masses could be very large, perhaps on the order of \(10^{10} - 10^{12} \text{ GeV}\).

To see how this counter-intuitive system can work, first review some spinor field formalism. The correct "hermitian" group needed to characterize spinor fields is \(\mathrm{SL}(2, \mathbb{C})\), which bears a similar relationship to \(\mathrm{SO}(3,1)\) as the relationship of \(\mathrm{SU}(2)\) to \(\mathrm{SO}(3)\), i.e. \(\mathrm{SO}(3,1) \cong \mathrm{SL}(2, \mathbb{C}) / \mathbb{Z}_2\) \((\mathrm{SL}(2, \mathbb{C})\) is the "double cover" of \(\mathrm{SO}(3,1)\)). This relationship is made explicit in 2-component notation, using the Weyl representation for the Dirac \(\gamma\)-matrices:

\[
\gamma_\mu = \begin{pmatrix} 0 & (\delta^\mu)_\nu \\ (\delta^\mu)_\nu & 0 \end{pmatrix}, \quad \delta, \bar{\delta} = 1, 2 \, \text{ undotted} \\
\delta^\dagger, \delta = 1, 2 \, \text{ dotted}
\]
where the $\mathbf{2}$, $\mathbf{3}$ are the Vander Waerden matrices:

$$(\mathbf{2}_m)_{4\times4} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\mathbf{3}_m)_{4\times4} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\sigma_i = (0,1), \sigma_2 = (0,-i), \sigma_3 = (1,0)$ are the Pauli matrices.

In the 2-component notation, standard 4-component Dirac spinors are composed of two 2-component $\text{Sl}(2,\mathbb{C})$ spinors: $\Psi_{\text{Dirac}} = \begin{pmatrix} \lambda \chi \\ \chi^\dagger \sigma \end{pmatrix}$, $\chi = 1,2$ Note index positions!

The fundamental $\chi$ and $\chi^\dagger$ spinors are complex, giving rise to $2 \times 4 = 8$ real components, just like any complex 4-component Dirac spinor.

The undotted-index $\chi$ and the dotted-index $\chi^\dagger$ $\text{Sl}(2,\mathbb{C})$ spinors are independent, and one or the other may consistently be set to zero in $\Psi_{\text{Dirac}}$; the 4-component Dirac spinor representation is reducible. Setting $\chi^\dagger = 0$, one obtains a left-handed chiral spinor

$$\Psi_L = \begin{pmatrix} \lambda \chi \\ 0 \end{pmatrix}$$

while setting setting $\chi = 0$, one obtains a right-handed chiral spinor

$$\Psi_R = \begin{pmatrix} 0 \\ \chi^\dagger \end{pmatrix}$$

In the 4-component notation, one obtains left

by projection with $P_L = \frac{1}{2} (1 + \sigma)$ on a general Dirac spinor, where $\sigma = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note: $(P_L) = P_L^2$ in Weyl rep.

Taken together with their conjugate and anticonjugate forms, there are four basic $\text{Sl}(2,\mathbb{C})$ representations. For $A^\dagger \in \text{Sl}(2,\mathbb{C})$, one has the hermitian conjugate $A^\dagger \sigma_3$, giving transformation types $\chi \rightarrow A^\dagger \chi \sigma_3$; conjugated to $\chi \rightarrow \sigma_3 A^\dagger \chi$; and conjugated transformations $\chi \rightarrow \chi^\dagger (A^\dagger) \chi$, conjugated to $\chi \rightarrow (A^\dagger) \chi^\dagger \chi$. 

$A^\dagger$ $A^\dagger$
The relationship between \( SL(2, \mathbb{C}) \) and \( So(3,1) \) is not expressed precisely by the existence of a homomorphism 
\( \mu: SL(2, \mathbb{C}) \rightarrow So(3,1) \) given by 
\( \mu: A^\beta \rightarrow L^\alpha \)
where 
\( L^\alpha(A) = -\frac{1}{2} \text{Tr}(A^\alpha A \gamma) \). Note the rule that 2-component index contractions are only made between upper and lower indices. The index structures of \( A, A^\gamma, \gamma^\alpha = \gamma^\alpha \gamma^\nu \)
and \( \gamma_\nu \) are just as needed to make this covariant construction. This homomorphism map is clearly 2 to 1 since both the \( SL(2, \mathbb{C}) \) matrices \( A^\alpha \) and \( -A_\beta \) (both with determinant 1) map to the same \( L^\alpha \in So(3,1) \). So \( So(3,1) \cong SL(2, \mathbb{C})/\mathbb{Z}_2 \).

Now return to the right-handed neutrinos. In 2-component notation, they are 
\[ \nu^\text{Rm} = \begin{pmatrix} 0 \\ \nu^\text{Rm} \end{pmatrix} \]  
Another way to repackage covariantly such a spinor is as a 4-component Majorana spinor 
\[ \nu^\alpha = \nu^\text{Rm} \chi^\alpha \]  
in which \( \chi^\alpha = (\begin{pmatrix} 0 \\ 1 \end{pmatrix}) \) is a numerically invariant \( SL(2, \mathbb{C}) \) bispinor. [One also has \( \exp(\hat{\alpha}_R) \) and \( \exp(\hat{\alpha}_L) \) numerically invariant bispinors.]

Note: \( \exp(\hat{\alpha}_R) = \mu^\dagger \gamma_\alpha \mu \), \( \exp(\hat{\alpha}_L) = \mu \gamma^\dagger \mu \) and \( \chi^\alpha \equiv \exp(\hat{\alpha}_R) \chi^\alpha \exp(\hat{\alpha}_L) \) allow for \( SL(2, \mathbb{C}) \) covariant raising and lowering (with \( \exp(\hat{\alpha}_R) \) \( \exp(\hat{\alpha}_L) \) 2-component indices.)

In the Weyl representation, the 4x4 charge conjugation matrix is 
\[ C = \begin{pmatrix} i \exp(\hat{\alpha}_R) & 0 \\ 0 & i \exp(\hat{\alpha}_L) \end{pmatrix} \]  
so for \( \psi_{\text{m}} = \begin{pmatrix} \chi^\alpha \\ \bar{\chi}^\beta \end{pmatrix} \) one has the charge conjugate 
\[ \psi^c = \xi \bar{\gamma} \psi \]  
A Majorana spinor may thus be constructed from a right-handed spinor as 
\[ \psi^\text{maj} = \psi^R + \psi^L = \begin{pmatrix} \chi^\alpha \\ \bar{\chi}^\beta \end{pmatrix} \]  
Note: \( \psi^c + (\psi^c)_c \) also makes a Majorana spinor, \( \begin{pmatrix} \chi^\alpha \\ \bar{\chi}^\beta \end{pmatrix} \).

The Majorana mass term for the right-handed neutrinos when written in 2-component notation is
\[ -\frac{1}{2} M_{\nu} (\bar{\nu}^m (\nu^Rm)c + (\nu^Rm) \bar{\nu}^m) = -M_{\nu} (\bar{\nu}^m \nu^m \chi^\alpha + \bar{\nu}^m \bar{\chi}^\beta \chi^\alpha) \]
Majorana

This mass term can also be written in terms of the Majorana spinor as $-i M m \bar{\nu}_m \nu_m \nu_m$. Now consider what happens after $SU(2)_L \times U(1)_Y$ symmetry breaking and making the unitary gauge choice in which $\phi = (\frac{1}{\sqrt{2}} (\nu_1 \bar{\nu_2})$. The Yukawa terms $-i \sum_{m} M_{mn} \bar{\nu}_m \nu_n$ then yield for off-diagonal mass terms $-i \frac{1}{\sqrt{2}} M_{mn} \bar{\nu}_m \nu_n = -i \frac{1}{\sqrt{2}} M_{mn} \bar{\nu}_m \nu_n \bar{\nu}_n \nu_n$ and similarly for the Hermitian conjugate. Adding these to their conjugates, one has $-i \frac{1}{\sqrt{2}} \left( M_{mn} \bar{\nu}_m \nu_n + M_{mn} \bar{\nu}_n \nu_m \right)$.

To understand more clearly what happens in such a system with coefficients $M_{mn}$ and $M_m$, consider first a single generation, e.g. $m = 1$. Phase rotation of $\nu_1$ can be used to make $\nu_1 = \nu$ real and positive. For simplicity, repackage now both $\nu_1$ and $\nu\bar{\nu}$ spinors into Majorana spinors (always covariant for $\nu_1$, harmless after symmetry breaking and unitary gauge choice for $\nu\bar{\nu}$): let

$$N = \begin{pmatrix} \nu_1 \\ \nu_1 \bar{\nu} \end{pmatrix} \quad N = \begin{pmatrix} \nu \bar{\nu} \\ \nu \bar{\nu} \end{pmatrix}$$

and combine the mass terms $-i \left( M \bar{N} N + \frac{1}{\sqrt{2}} (\bar{N} N + N \bar{N}) \right)$, giving a mass matrix

$$N \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & M \end{pmatrix} N$$

The vanishing matrix element in the $nn$ position reflects the ordinary impossibility of having a mass term for $\nu$ spinors in the original standard model, as imposed by the starting $SU(2)_L \times U(1)_Y$ gauge invariance prior to symmetry breaking.

One finds the eigenvalues of the nondiagonal neutrino mass matrix in the standard way by solving: \[ \ldots \]
\[ -\lambda (M^2 - \nu^2 p^2 / 8) = 0, \text{ giving } \lambda = \frac{M^2}{2} \left( 1 \pm \sqrt{1 - \frac{2\nu^2}{2M^2}} \right). \]

Thus, if \( M \) is very large compared to \( \frac{\nu^2}{2M^2} \), one finds one very large eigenvalue \( \lambda \sim M \) corresponding to a (currently unknown) very heavy neutrino species and also a very small eigenvalue \( \lambda = \frac{\nu^2}{2M^2} \ll \frac{\nu^2}{2} \).

This mechanism thus naturally produces a very light mass value for the known types of neutrino, at the cost of proposing some very heavy unknown neutrino species.

The above mechanism for generating a very small neutrino mass by introducing a large mass \( M \) for an as-yet unknown sterile neutrino species is called the Seesaw mechanism. It has the virtue of preserving renormalizability, since all the extensions to the minimal Standard Model involve operators of dimensions \( d \leq 4 \). The fact that the Majorana mass \( M_{\nu} \) for the right-handed neutrino singlets \( \nu_{\text{Rm}} \) may be very large (perhaps \( 10^{10} \text{ GeV or so} \)) invites another kind of description in terms of an effective theory. If one wants to describe the effects of mass for the known light neutrinos, one can integrate out the heavy \( \nu_{\text{Rm}} \) fields. We saw earlier how integrating out the \( W \) and \( Z \) fields produces current-current interactions of dimension 6 in the effective 4-fermion theory. One may do something similar to integrate out the \( \nu_{\text{Rm}} \) fermion fields. Instead of constructing the \( \nu_{\text{Rm}} \) propagators and extracting the leading local parts of the interactions with light fields produced by \( \nu_{\text{Rm}} \) exchange, it is simplest to consider just the Majorana mass terms and Yukawa terms and eliminate the \( \nu_{\text{Rm}} \) using their lowest-order algebraic equations.

For simplicity of presentation, combine the \( \nu_{\text{Rm}} \) and their charge conjugates \( \nu_{\text{Rmc}} = (\nu_{\text{Rm}})^T \) into Majorana spinors

\[ N_m = \nu_{\text{Rm}} + \nu_{\text{Rmc}}, \text{ or, in 2-component form, } (\nu_{\text{Rm}} \nu_{\text{Rmc}}). \]
The Majorana mass and Yukawa interactions involving $N_m$ are then

$$ -i M_m N_m \tilde{N}_m = - i \bar{p}_{mn} N_n (L_m)^a \tilde{\phi}_a - i \bar{p}_{nn} \tilde{N}_n L_m e^{ab} \phi _b. $$

Note that since $L_m$ is right-handed ($P \cdot L_m = L_m$), the first Yukawa term involves only the right-handed projection of $N_n$ (i.e. $\tilde{N}_n$) and the second Yukawa term involves only the left-handed projection of $N_n$ (i.e. $N_n$). Taking just the above terms and deriving the algebraic equation for $N_m$, one has $N_m = - \frac{1}{2M_m} \left[ (L_m)^a \bar{p}_{mn} \tilde{\phi}_a + L_n e^{ab} \phi _b \right]$. Using this algebraic result together with its Dirac conjugate for $\tilde{N}_m$, one obtains the dimension $d = 5$ operators

$$ -i \left( \frac{\tilde{N}_m}{4M_m} \right) \left[ (L_m)^a \bar{p}_{mn} \tilde{\phi}_a \phi_d P_{km} + (L_n)^b L_k e^{ab} \phi _e \phi_d P_{km} \right] $$

involving two dimension $3/2$ spinor fields and two dimension $1$ Higgs fields.

This dimension 5 operator is non-renormalizable, similarly to the dimension 6 four-fermion operator obtained by integrating out the W and Z fields. One does not insist upon renormalizability in the context of an effective theory, and it is only taken to be valid up to some scale where other physical effects can take place. In this case, the validity of the above dimension 5 operator could not extend beyond the $M_m$ scales of the VEV Majorana masses. Consideration of an effective theory always presumes the existence of a satisfactory ultraviolet completion. For the neutrino masses, the ultraviolet completion could be a seesaw mechanism, as above, or it could be another type of much higher energy physics. Carrying out the integrations of the VEV to higher order involves expanding fermionic propagators $\left[ (P - iM)^{-1} = \frac{1}{P^2 + M^2} \right] = \frac{i}{P^2 + M^2} \left[ 1 + \frac{P^2 M^2}{P^2 + M^2} + \cdots \right]$, the second term of which generates dimension $5 + 6$ terms in the effective theory, and eventually higher. Phenomenological studies often motivate dimensions $5 + 6$. 
The PMNS matrix

When neutrino masses are included, one faces a similar issue upon diagonalizing their mass matrix $\delta m^2$ to that encountered with the quarks: a mixing matrix occurs distinguishing the diagonal mass basis from the flavour basis in which the $W^\pm$ couplings are $\text{min}$ diagonal. This is the PMNS matrix (Pontecorvo, Maki, Nakagawa & Sakata), the neutrino analogue of the CKM matrix for the quarks.

Consider the mass terms for the light neutrinos after integrating out the heavy species $N_\mu$ and $N_\tau$ and going to the unitary gauge where $\Phi = \begin{pmatrix} 0 & \delta \end{pmatrix}$, $\Phi = \begin{pmatrix} \delta & 0 \end{pmatrix}$:

$$\frac{1}{8 M_m} \left( \nu^\prime_{Lm} V_{Lc} \nu^\prime_{km} + \nu^\prime_{Lc} V_{Lk} \nu^\prime_{km} \right)$$

One sees that the resulting mass matrix $(\frac{1}{8 M_m}) \nu^\prime_{Lm} \nu^\prime_{km} = M_{\nu_{Lm}}$ is a complex, symmetric $3 \times 3$ matrix. Note already a difference with respect to the initial quark mass matrix prior to diagonalization, where the $t\nu_t$, $h\nu_t$ and $k\nu_t$ matrices were not in general symmetric. The symmetry of the $M_{\nu}$ mass matrix derives from the relation $-i \Phi \chi = -i \Phi \chi$ in the case $\Phi = \nu^\prime_L$, $\chi = \nu^\prime_L$, so $M_{\nu} \nu^\prime_L \nu^\prime_{Lc} = M_{\nu} \nu^\prime_{Lc} \nu^\prime_L$, and similarly for the conjugate term $-i M_{\nu} \nu^\prime_L \nu^\prime_{Lc}$.

Under $U(3)$ transformations of the left-handed neutrinos $\nu^\prime_L = \nu^\prime L e^* \nu^\prime$, the matrix $M_{\nu}$ transforms into $U^T M_{\nu} U$.

By the process of Takagi Factorization, any complex symmetric matrix $M$ can be decomposed as $M = UD U^T$, where $D$ is a real nonnegative diagonal matrix and $U$ is unitary. Hence one can find $U$ such that $U^T M_{\nu} U = D$ is real, diagonal and nonnegative. Accordingly, the neutrino mass matrix $M_{\nu}$ may be diagonalized by $\nu^\prime_L = \nu^\prime L e^* \nu^\prime$ for some $U(3)$ matrix $U$.

With the diagonalization of the neutrino mass matrix comes a change in the charge-current couplings if $U = U^T$. As in the case of the quark gauge couplings, couplings to $A_\mu$ and $Z_\mu$ do not mix top and bottom components of $SU(2)_L$ doublets, so those remain undisturbed.
The situation is now different for lepton couplings to the charged $\nu^\pm$. Recall that in the minimal Standard Model without neutrino masses, the couplings of $W^\pm$ to the leptons

$$ \mathcal{L}_{\text{int}} = \frac{g_2}{\sqrt{2}} \left[ W^+_\mu \bar{\nu}_\mu \gamma^\mu \nu_e + W^-_\tau \bar{\nu}_\tau \gamma^\mu \nu_e \right] $$

remained unchanged under the $\mathbf{U}^e(3)$ transformations needed to diagonalize the electron-muon-tau mass matrix, because one could without other consequence transform the $\nu_3$ neutrinos with the same matrix $\mathbf{U}^e(3)$, causing the lepton charged current transformations to cancel. After including neutrino masses and the need to diagonalize them, however, the charged current couplings transform to

$$ \mathcal{L}_{\text{int}} = \frac{g_2}{\sqrt{2}} \left[ W^+_\mu \bar{\nu}_\mu \gamma^\mu \nu_e + W^-_\tau \bar{\nu}_\tau \gamma^\mu \nu_e + V^e_{\mu \tau} \bar{\nu}_\mu \gamma^\mu \nu_e + V^e_{\mu \tau} \bar{\nu}_\tau \gamma^\mu \nu_e \right] $$

where $V^e_{\mu \tau} = U^e_{\mu \tau} U^e_{\tau \nu}$. The PMNS mixing matrix $V^e_{\mu \tau}$ is the PMNS mixing matrix. As for the CKM mixing matrix for the quarks, $V^q_{\mu \tau}$ is unitary.

Now count the number of physically important parameters needed to determine $V_{\mu \tau}$ for a system with $N$ generations. The $N \times N$ unitary matrix $V^e_{\mu \tau}$ depends a priori on $N^2$ real parameters. However, as in the CKM case, one may use non-symmetry phase transformations to put $V^e_{\mu \tau}$ into some standard form. Note, however, that the $V^e_{\mu \tau}$ structure of the neutrino mass terms do not allow for phase transformations of the $V_{\mu \tau}$ leaving the neutrino mass terms invariant. There are, however, phase changes of the $V_{\mu \tau}$ together with identical phase changes of the $V^e_{\mu \tau}$ that leave the $V^e_{\mu \tau}$ mass terms invariant. So there are just $N$ phases that can be used to put the PMNS matrix into a standard form. Thus, for $N = 3$ generations, there are $N^2 - N = 9 - 3 = 6$ physically important parameters. Of these, 3 correspond to orthogonal $O(3)$ transformation parameters, similarly to the CKM case for the quarks. In addition, there are 3 phases determining the complex structure of $V^e_{\mu \tau}$. 
Note that in the count of physically relevant parameters for the PMNS matrix, an overall phase transformation of all fermion has not been excluded from the number of adjustable phases. This is because the $M_{
u}V_{
u\ell}$ neutrino mass term structure does not admit such an overall phase symmetry. The count of $N$ adjustable phases from the $\ellLM, \ellLM$ fields remains maintained, unlike the situation for the CKM matrix.

A standard parametrization of the lepton charged current mixing matrix \( V_{\nu\ell}^{lep} \) is

\[
V_{\nu\ell}^{lep} = U_{PMNS} \times \frac{\text{diag}(\cos \theta_1, \cos \theta_2, \cos \theta_3)}{1}
\]

where the PMNS matrix $U_{PMNS}$ has the same structure as the CKM matrix for the quark couplings:

\[
U_{PMNS} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} c_{13} \\
0 & -s_{23} & c_{23} c_{13}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
-c_{12} & s_{12} & 0 \\
s_{12} & -c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\end{pmatrix}
\]

where $c_{mn} = \cos \theta_{mn}$ and $s_{mn} = \sin \theta_{mn}$.

The additional two phases in $V_{\nu\ell}^{lep}$ occurring in the diag \((\cos \theta_1, \cos \theta_2, \cos \theta_3)\) factor arise because of the Majorana mass structure of $M_{\nu}V_{\nu\ell}$, which prevents these from being adjusted to zero. [Note, however, that one phase, $\theta_{23}$, has conventionally been rotated away by making a common phase rotation of the charged leptons $\ellLM, \ellLM$.] The two phases $\theta_1$ and $\theta_2$ can only be observed by considering processes that are sensitive to the difference between Majorana and Dirac mass terms. In particular, neutrino oscillations are not sensitive to this difference.

Another aspect of the extra $\theta_1, \theta_2$ phases is the violation of lepton number by the Majorana mass term structure.
Lepton number \( L \) is a Noether charge corresponding to overall \( U(1) \) phase rotations of all leptons: \( L(e^{\mu}_L) = L(e^{\tau}_L) = L(\nu^e_L) = 1 \) and \( L(e^{\mu}_R = e^{\tau}_R) = L(e^{\mu}_R = \bar{e}^{\mu}_R) = 0 \). Non-leptonic particles are assigned lepton number zero. Lepton number conservation is an approximate rigid symmetry consequence. It is violated by anomalies at the quantum level, but so far there is no experimental evidence for lepton number non-conservation. The \( e^{\mu}_L \) and \( e^{\tau}_L \) phases are not observable in processes that conserve total lepton number \( L \), such as neutrino oscillations.

**Neutrino Oscillations**

Most of the information known about neutrino masses and mixings comes from the phenomenon of neutrino oscillations. Neutrinos are usually created by charged-current interactions, produced in association with a charged lepton. Accordingly, they start off in a flavour eigenstate \( \nu^{\mu}_L \). For example, muon decay \( \mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \) (Note: lepton number conservation) produces muon neutrinos and electron antineutrinos. Neutrinos are typically detected by observing the charged lepton in another charged-current reaction. In this way one may observe the final flavour state and find whether it differs from the initial state. The production and capture processes involve interactions of flavour eigenstates. For a generic neutrino mass matrix, however, the interaction eigenstates don't coincide with mass eigenstates.

Consider for simplicity just a two generation system \( m=1,2 \) with the mixing matrix:

\[
\begin{pmatrix}
C & S \\
-S & C
\end{pmatrix}
\begin{pmatrix}
e^{i\Delta_1} & 0 \\
0 & e^{i\Delta_2}
\end{pmatrix}
\]

The flavour basis is related to the mass basis for neutrinos by:

\[
|\nu_e> = \begin{pmatrix} e^{i\Delta_1} C\cos \theta \sin \phi \\ -e^{i\Delta_1} S\cos \theta \sin \phi \end{pmatrix} |\nu_1> + e^{i\Delta_2} \sin \theta \sin \phi |\nu_2>
\]

\[
|\nu_\mu> = \begin{pmatrix} -e^{i\Delta_1} C\sin \phi \\ e^{i\Delta_1} C\cos \theta \end{pmatrix} |\nu_1> + e^{i\Delta_2} \cos \theta \sin \phi |\nu_2>
\]

|\nu_e>, |\nu_\mu> : Flavour eigenstates

|\nu_1>, |\nu_2> : Mass eigenstates
The mass eigenstates have the standard quantum mechanical evolution in time:

\[ |\nu_n, t\rangle = e^{iHt} |\nu_n, 0\rangle = e^{i\frac{p^2}{2m^2}t} |\nu_n, 0\rangle \]

Consequently, an initially electron-generation neutrino state \( |\nu_e, 0\rangle \) with spatial momentum \( p \) evolves after time \( T \) to the state

\[ |\nu_e, T\rangle = e^{i\frac{p^2}{2m_e^2}T} |\nu_e, 0\rangle + e^{i\frac{p^2}{2m_\mu^2}T} |\nu_\mu, 0\rangle \]

The overlap amplitude of this state with the initial interaction
flavour eigenstate \( |\nu_e, 0\rangle \) is

\[ \langle \nu_e, 0 | \nu_e, T \rangle = \left( e^{i\frac{p^2}{2m_e^2}T} + e^{i\frac{p^2}{2m_\mu^2}T} \right) \left( e^{i\frac{p^2}{2m_e^2}T} - e^{i\frac{p^2}{2m_\mu^2}T} \right) \]

\[ = \cos^2\theta e^{i\frac{p^2}{2m_e^2}T} + \sin^2\theta e^{i\frac{p^2}{2m_\mu^2}T} \]

Hence the probability of finding an electron neutrino in the final state is

\[ P(\nu_e \rightarrow \nu_e) = \left| \langle \nu_e, 0 | \nu_e, T \rangle \right|^2 = \left| \cos^2\theta e^{i\frac{p^2}{2m_e^2}T} + \sin^2\theta e^{i\frac{p^2}{2m_\mu^2}T} \right|^2 \]

\[ = \cos^2\theta + \sin^2\theta + 2\cos^2\theta \sin^2\theta \cos \left( \frac{p^2}{2m_e^2}T - \frac{p^2}{2m_\mu^2}T \right) \]

\[ = 1 - \sin^2\theta \sin^2 \frac{T}{2} \left( \frac{p^2}{2m_e^2} - \frac{p^2}{2m_\mu^2} \right) \]

Now make the approximation that the neutrino mass values \( m, m_\mu \) are much smaller than \( p \), and that \( p = E \), up to corrections of order \( m^2/p^2 \). Then \( \frac{p^2}{2m_e^2} - \frac{p^2}{2m_\mu^2} \approx (m_\mu^2 - m_e^2)/2E \).

Express the resulting probability in terms of the distance travelled \( L \) instead of the time \( T \), noting \( L = T \) for highly relativistic particles (with \( c = 1 \)). Thus

\[ P(\nu_e \rightarrow \nu_e) = 1 - \sin^2\theta \sin^2 (\Delta m^2 L/4E) \]

and since probabilities must add up to 1, one also has

\[ P(\nu_e \rightarrow \nu_\mu) = \sin^2\theta \sin^2 (\Delta m^2 L/4E) \]

where \( \Delta m^2 = (m_\mu^2 - m_e^2) \).

Some points to note about this neutrino oscillation result:

- The Majorana phases \( e^{i\theta_e}, e^{i\theta_\mu} \) have cancelled out. One cannot distinguish between Dirac and Majorana mass term structures on the basis of neutrino oscillation experiments.
- The oscillation probability involves only \( \Delta m^2 \) differences between squares of masses, not the masses directly.
A more general treatment gives the probability of a neutrino being produced as a flavour eigenstate $\nu_i$ with energy $E$ and then detected in flavour eigenstate $\nu_j$, a distance $L$ away:

$$P_{\nu_i \rightarrow \nu_j}(E, L) = |U_{ij}|^2 |\Psi_i(E, 0)|^2$$

$$= \frac{E}{\pi} \sum_{k} e^{-(m_k - m_j)^2 L^2/2E} U_{ik} \overline{U}_{kj}$$

in which the Majorana phases $\varphi_i$ have cancelled out, so only the PMNS mixing matrix $U = U_{PMNS}$ appears in the oscillation probability. The sums over $i$ and $j$ arise from inserting a complete basis of mass eigenstates in the probability amplitude.

Defining the "oscillation length" $\lambda(E) = \frac{2E}{\Delta m^2}$, or $\lambda(E) = \frac{500m(E/1GeV)(E/\text{MeV})}{\Delta m^2}$, and noting that the observed neutrino mass splittings are bounded by $\Delta m^2 \leq 5 \times 10^{-3} \text{eV}^2$ with typical energies $E \approx 10^{6} \text{eV}$, oscillation lengths typically need to be in ranges of 10^5 cm to get useful information about $\Delta m^2$ values. Note that for lengths $L \ll \lambda(E)$, the probability $P_{\nu_i \rightarrow \nu_j}$ tends to $P_{\nu_i \rightarrow \nu_j}(E, 0) = \delta_{ij}$. Accordingly, neutrino oscillation experiments involve very long-baseline muon setups, e.g. 245 km for the T2K experiment with a neutrino beam from J-PARC in Tokai to detector in Kamioka. The T2K experiment has a range of muon neutrino energies around 600 MeV, chosen to get the highest $P_{\nu_\mu \rightarrow \nu_e}$ over a distance of 295 km.

Two distinct types of neutrino oscillations have been clearly observed. The first type concerns the disappearance of $\nu_\mu$ neutrinos coming from the Sun as well as the disappearance of $\nu_\tau$ neutrinos coming from reactors. Considered in the two-flavour case (a good approximation even with $N = 3$ flavours), such experiments give $30^\circ < \theta_2 < 238^\circ$, $\Delta m^2 = m_2^2 - m_1^2 = 7.53 \times 10^{-5} \text{eV}^2$, (99% confidence level) $\sin^2 \theta_2 = 0.846 \pm 0.021$, $\theta_3 = \theta_\tau$. 
The second type of observed neutrino oscillation involves the disappearance of $\nu_e$ and $\bar{\nu}_e$ neutrinos from cosmic-ray showers in the upper atmosphere or of $\nu_x$ neutrinos from long-baseline beam experiments. Again, interpretation in terms of a two-flavour system proves to be a good approximation.

Results give $38^\circ < \delta_{\text{atm}} < 52^\circ$, $\Delta m^2_{\text{atm}} = |m^2_3 - m^2_2| = 2.44 \times 10^{-3} \text{eV}^2$ (90% C.L.) $\sin^2 2\theta_{\text{atm}} = 0.92$ $\delta_{\text{atm}} = \phi_{31}$. Since $\sin^2 2\theta_{\text{atm}}$ is consistent with unity, such oscillations are close to maximal.

From these experimental results, one sees that the splitting between the two neutrino species involved in the solar-neutrino oscillations is much smaller than the splitting between either of them from the third neutrino species. Common practice is to label the neutrino-mass eigenstates so the small splitting is between $\nu_1$ and $\nu_2$. Since $\Delta m^2_{12} = \Delta m^2_{23}$ determines the solar-neutrino oscillations, which involve the disappearance of $\nu_e$ neutrinos, either $\nu_1$ or $\nu_2$ must have a significant overlap with the $\nu_e$ flavour basis neutrino. Conventionally, one chooses $\nu_1$ to be the mass eigenstate with this significant overlap.

Note that since the neutrino oscillations involve $|\Delta m^2_{ij}|$ differences, they do not determine the values of the individual masses. This leaves open two possibilities for the neutrino mass hierarchy. If the nearly degenerate ($\nu_1, \nu_2$) pair is less massive than $\nu_3$, the mass pattern is called the "normal" hierarchy. Alternatively, if the ($\nu_1, \nu_2$) pair is more massive than $\nu_3$, the hierarchy is called the "inverted" hierarchy.

The results of experiment may be summarized in an approximate PMNS matrix obtained by setting $\theta_{13} = 0$, maximal $\theta_{23} = \pi/4$ (so $\delta_{13} = 0$), $c_{13} = 1$, $s_{23} = c_{23} = \sqrt{2}/2$:

$$
W^{\text{PMNS}} \approx \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & 1/\sqrt{2} \\ s_{12}/\sqrt{2} & -c_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.
$$
CP invariance and breakdown

Owing to the chiral nature of the Standard Model, i.e. the key role its plays in the definition of the fermion field representations, parity is clearly not a symmetry. No is charge conjugation. However, the combined CP discrete transformation is a symmetry of the gauge and leptonic sector provided neutrino masses are not turned on. We will come to what happens with quarks and neutrino masses in due course (the stories are similar, and involve CKM and PMNS matrices). First, we need to establish how parity and charge conjugation act on spinor fields.

We have seen that the proper Lorentz group $SO(3,1)$ is related to the Spin $(3,1)$ group by $Spin(3,1) \cong SU(2,1)$ and $SO(3,1) \cong SU(2,1)/\mathbb{Z}_2$, i.e. $Spin(3,1)$ is the “double cover” of $SO(3,1)$. When one wants to extend $SO(3,1)$ to $O(3,1)$ by the inclusion of transformations $L^M$ with $det (L) = -1$, defining the corresponding extension of $Spin (3,1)$ requires some care. The corresponding group is called $Spin (3,1)$ — a mathematician’s joke, attributed to J.P. Serre (according to Atiyah, Bott & Shapiro), although when asked about the name, would not confirm his role in it.

Although the “bosonic” Lorentz groups $O(1,3)$ and $O(3,1)$ are isomorphic — making the choice of “mostly plus” or “mostly minus” formalisms simply one of preference — this is not the case for $Pin (1,3)$ and $Pin (3,1)$, which are not isomorphic.

The key difference between Pin(3,1) and Spin(3,1) is the existence of group elements built from odd powers of gamma matrices. The generators of Spin(3,1) are represented by gamma-matrix commutators: $[\sigma_{\mu}, \sigma_{\nu}] = \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \sigma^{\rho\sigma}$, which satisfy the algebra $[\sigma_{\mu}, \sigma_{\nu}] = i (\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu}) = -i (\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu})$. [Check infinitesimal transformation $\delta V^\rho = \frac{i}{2} (\omega^{\mu\nu} \sigma^\mu) V^\nu = \omega^\rho_{\mu} V^\mu$] Group element representations exp(i $\omega^\mu_{\nu} \sigma^\mu$) clearly involve only even powers of the $\gamma$ matrices.

Now consider how parity inversions may be realized when acting on spinor fields. When acting on a scalar field $\phi(x)$, one has $\phi(x) \rightarrow \eta_0 \phi(x)$, where for space inversion $x^\mu = (x^0, x^i)$, $i=1,2,3$ and $\eta_0$ is the intrinsic parity of the field $\phi$; $\eta_0 = 1$ for scalars, $\eta_0 = -1$ for pseudoscalars. When it comes to spinor fields, however, it is not just a matter of changing the field's argument $x^\mu \rightarrow x^\mu$ and supplying a phase. One needs to ensure that the Dirac equation remains valid after a parity transformation. For a massive spinor field $\Psi(x)$, one has $\eta_0 (x + m) \Psi(x) = 0$. Let the parity operation be $\Psi(x) \rightarrow \Psi(x) = \Lambda_p \Psi(x)$, where $x^\mu = (x^0, x^i)$ and $\Lambda_p$ is some unitary matrix. Since $2i: \Psi(x) = -\frac{1}{2} i: \Psi(x)$, one has $(\gamma^0 \gamma_{\alpha} - \gamma^i \gamma_{3x: + m}) \Psi(x) = 0$. So the flip of sign in the $\gamma^0 \gamma_{\alpha}$ term must be reversed by the $\Lambda_p$ operator. Requiring $[\Lambda_p, \gamma_{\alpha}] = 0$ and choosing $\Lambda_p$ hermitian leads to $\Lambda_p = \pm i: \gamma^0$. Pick $\Lambda_p = i: \gamma^0$. Then $\Psi(x) = \eta_0 i: \gamma^0 \Psi(x)$, and the Dirac equation is preserved: $(\gamma^0 \gamma_{\alpha} + m) \Psi(x) = 0$. The phase $\eta_0$ is called the intrinsic parity, and needs to be chosen carefully in the context of a given theory to achieve overall invariance, including interactions if possible.
The Dirac conjugate spinor \( \bar{\psi} = \psi^+ \gamma^0 \) consequently transforms under parity as \( \bar{\psi}(x) \rightarrow \bar{\psi}^*(x) = \gamma^0 \bar{\psi}(x) \gamma^0 \), and since \( \psi \) is a phase, \( |\psi|^2 = 1 \), one obtains the following transformations of spinor bilinears under parity:

- \( \bar{\psi}(x) \gamma^2 \psi(x) \rightarrow \bar{\psi}(x) \gamma^2 \psi(x) \) scalars
- \( \bar{\psi}(x) \gamma^5 \psi(x) \rightarrow -\bar{\psi}(x) \gamma^5 \psi(x) \) pseudoscalars
- \( \bar{\psi}(x) \gamma^0 \psi(x) \rightarrow \bar{\psi}(x) \gamma^0 \psi(x) \) change density
- \( \bar{\psi}(x) \gamma^1 \psi(x) \rightarrow -\bar{\psi}(x) \gamma^1 \psi(x) \) current density.

Thus, one finds that a standard kinetic term for a non-chiral Dirac spinor \( \frac{1}{2} \bar{\psi} (\gamma^0 \partial_0 - \gamma^a \partial_a) \psi \) is parity invariant, as is a mass term \( \frac{1}{2} m \bar{\psi} \psi \).

For chiral spinor fields, however, parity is broken:

- \( \bar{\psi} \gamma^0 \psi = \frac{1}{2} (1 + \gamma^5) \bar{\psi} \psi \rightarrow \frac{1}{2} (1 - \gamma^5) \bar{\psi} \psi \), i.e., parity switches left and right projections \( \bar{\psi}_L \rightarrow \bar{\psi}_R \), and
- \( \bar{\psi} \gamma^i \psi = \frac{i}{2} (\gamma^i \gamma^5 + \gamma^5 \gamma^i) \bar{\psi} \psi \rightarrow \frac{i}{2} (\gamma^i \gamma^5 - \gamma^5 \gamma^i) \bar{\psi} \psi \).

In the Standard Model, with its chiral representations and couplings, parity is accordingly not a good symmetry.

Next consider change conjugation. We have already introduced the notion of the change conjugate \( \bar{\psi}(x)C = (\gamma^i \psi(x))^T \), where in the Weyl representation of the gamma matrices, numerically one has \( C = -\gamma^0 \gamma^5 \), becoming in 2-component notation \( C = \begin{pmatrix} 0 & i e^3 \beta \\ e^{-i \alpha} & 0 \end{pmatrix} \). [Note: relative to Weinberg, \( C = e^{-i \beta} \bar{C}_{\text{Weinberg}} \). This is because for Weinberg \( \bar{C}_{\text{Weinberg}} = 4^+ \beta \) where \( \beta = i \gamma^0 \).] Consider now the Dirac equation for a charged spinor field:

\( (\gamma^a \partial_a + i e A^a \gamma^a + m) \psi = 0 \).

Take the adjoint:

\[ \bar{\psi} \gamma^a \partial_a \psi + i e A^a (m \bar{\psi} \psi + \bar{\psi} \psi) = 0 \]

Then use \( \psi^+ \gamma^5 \bar{\psi} \psi = -\bar{\psi} \gamma^5 \psi \) to get:

\[ -\bar{\psi} \gamma^a \partial_a \psi + i e A^a (m \bar{\psi} \psi + \bar{\psi} \psi) = 0 \]

Then transpose:

\[ -\psi^+ (\partial_a \bar{\psi})(\bar{\psi})^T + i e A^a (m \bar{\psi} \psi + \bar{\psi} \psi) = 0 \]

and note \( C \psi^+ = -\psi^+ C \) for \( C = -\gamma^5 \gamma^0 \). So for \( \bar{\psi} \psi = (4\psi)^T \), one finds

\[ (\gamma^a \partial_a - i e A^a \gamma^a + m) \psi = 0 \], i.e., the Dirac equation goes into itself but with \( e \rightarrow -e \).
In a system with a variety of spinor fields, it may be necessary to include also change conjugation phases, so the change conjugation transformation can more generally be written \( \psi \rightarrow \gamma \psi C(\Phi)^T \). Now consider the effect of a change-conjugation transformation on the Lagrangian for a Dirac spinor field \(-i \overline{\psi} \gamma^\mu (\partial_\mu + i e A_\mu) \psi /4\). As we have seen, \( \overline{\psi} = \overline{\psi}_c \), so the transform \(-i \overline{\psi}_c \gamma^\mu (\partial_\mu + i e A_\mu) \psi_c /4\), noting that the phases \( \gamma^\mu \psi_c \) cancel out, may be rewritten \(-i \overline{\psi}_c \gamma^\mu (\partial_\mu + i e A_\mu) + m \gamma^4 \psi_c /4\), noting that \( \gamma^4 \psi = \psi \). Then, using \( \gamma^\mu \gamma^0 = - \gamma^0 \gamma^\mu \) and \( (\gamma^\mu)^T = - \gamma^\mu \), one finds \( \overline{\psi} \gamma^\mu \psi_c = \overline{\psi} \gamma^\mu \psi_c = \overline{\psi} \gamma^\mu \psi_c \), so also \( \overline{\psi}_c \gamma^\mu \psi_c = \overline{\psi} \gamma^\mu \psi_c \). Then \( \overline{\psi} \gamma^\mu \psi = - \overline{\psi} \gamma^\mu \psi_c \) so \(-i \overline{\psi}_c \gamma^\mu (\partial_\mu + i e A_\mu) + m \psi_c /4\) and integrating \( A_\mu \) by parts in the action makes this into \(-i \partial_\mu (\overline{\psi} \gamma^\mu \psi_c + m \psi_c /4\), corresponding exactly to the effect of the change-conjugation transformation on the field equation. Invariance under such a transformation consequently would require transforming \( A_\mu \rightarrow -A_\mu \), treating \( e \) as a fixed coupling constant. This works in QED, which consequently has a change-conjugation symmetry (as well as parity). In more complicated models, one just has to search for spinor field \( \gamma^\mu \) phases and vector-sign flips to see if invariance of the action can be achieved.

In order to investigate the degree to which change-conjugation symmetry works in the Standard Model, we need to know its effect on the various types of terms in the Lagrangian.

\( C = - \sqrt{\gamma} \). It may be useful to make a summary of relations:

- \( \overline{\psi} = \overline{\psi}_c \gamma^0 = \overline{\psi}_c \gamma^0 \) (or \( \gamma^0 \overline{\psi} = - \gamma^0 \overline{\psi}_c \))
- \( \overline{\psi} \gamma^\mu = \overline{\psi}_c \gamma^\mu \gamma^0 = - \overline{\psi}_c \gamma^\mu \)
- \( \gamma^\mu \psi_c = \gamma^\mu \psi_c \gamma^0 = \gamma^\mu \psi_c \)
- \( \overline{\psi}_c = - \overline{\psi} \gamma^0 \), \( (\overline{\psi} \gamma^0) = - \overline{\psi}_c \), \( (\overline{\psi}_c \gamma^0) = - \overline{\psi} \gamma^0 \)

From these follow a number of transform relations for fermion bilinears which are needed to find the effect of change conjugation on the various types of terms in the SM action.
\[ \overline{\psi} \psi \rightarrow \overline{\gamma} \psi = \gamma X \psi \]
\[ \overline{\psi} \gamma^a \psi \rightarrow \overline{\gamma^a} \psi = - \overline{\gamma^a} X \psi \]
\[ \overline{\psi} \gamma^5 \psi \rightarrow \overline{\gamma^5} \psi = \overline{\gamma^5} X \psi \]
\[ \overline{\psi} \gamma^a_5 \psi \rightarrow \overline{\gamma^a_5} \psi = \overline{\gamma^a_5} X \psi \]
\[ \overline{\psi} \gamma^{a_5} \psi \rightarrow \overline{\gamma^{a_5}} \psi = - \overline{\gamma^{a_5}} X \psi = \overline{\gamma^{a_5}} X \psi \]

The above relations are valid for arbitrary Dirac (general) spinor fields, which anticommute. In performing charge conjugation transformations, only fields transform - quantities like \( \gamma^a \) in bilinears are not directly transformed, although it may be useful to consider charge conjugates of products of \( \gamma^a \) acting on spinor fields in rearranging the result of a transformation. We have already done this above in discussing the transformation of the gauge coupled fermion kinetic term. Another example could be \( \overline{\psi} \gamma^5 \psi \rightarrow \overline{\gamma^5} \psi \) (transformation just of the fields) followed by rearrangement: \( \overline{\psi} \gamma^5 \psi = \overline{\psi} \gamma^5 \psi \) (using \( \overline{\psi} = \overline{\psi} \)), then \( \overline{\psi} \gamma^5 \psi = - \overline{\gamma^5 \psi} = - \overline{\gamma^5 \psi} \), so one has \( - \overline{\gamma^5 \psi} = - X^t \gamma^5 \gamma^5 \psi \), as above.

Now, consider the effect of a charge conjugation transformation on the pure derivative part of a kinetic term for spinor fields carrying a representation of some group \( G \), e.g. \( SU(3) \). For a field \( \psi_A \), where \( A \) is an index corresponding to a complex representation of the group \( G \), the charge conjugation transformation sends \( \psi_A \) to \( \overline{\psi} \), with a raised index for the conjugate \( \overline{\psi} \) representation. For now, let \( \psi_A \) be a general Dirac spinor field. The charge conjugation transformation then transforms the kinetic term for \( \psi_A \) as \( - \frac{i}{2} \psi_A \gamma^a \psi_A \overline{\psi} = \frac{i}{2} \psi_A \gamma^a \overline{\psi_A} \), noting that the \( \gamma^5 \) phases cancel between the transformations of \( \psi_A \) and \( \overline{\psi} \).

Then, using the above rules, the kinetic term can be reorganised to \( + \frac{i}{2} \psi_A \gamma^a \overline{\psi} \psi_A = \frac{i}{2} \psi_A \gamma^a \overline{\psi} \psi_A \). When such a Lagrangian is integrated over in the action, one may integrate the \( \psi \) derivative term by parts giving back \( - \frac{i}{2} \psi \gamma^a \psi_A \), so such pure-derivative kinetic terms are \( C \)-invariant.

Next, consider what happens when there is nonabelian gauge coupling to the spinor fields. We have seen in the QED...
The case that the gauge field may itself need to transform under charge conjugation in order to achieve invariance. The key requirement is to preserve covariance. Consider what happens for some general putatively covariant derivative $D_{\mu}$ acting on a spinor field $\psi^A$ carrying a representation of the gauge group $G$ denoted by the index $A$. [For example, for the group SU(3), still consider $\psi^A$ to be a general Dirac spinor field.] The charge conjugate transformation sends $\psi^A \rightarrow \psi^{A\dagger}$ (again ignoring phases $\eta_c$). The covariant derivative of $\psi^A$ is $D_{\mu}\psi^A = \partial_{\mu}\psi^A + ig A_{\mu}^a \Gamma^{a\mu} \psi^A$ (assuming a simple group $G$ for which one has a single coupling constant $g$; for non-simple groups, one can have separate $g_a$ for each simple factor).

Applying a charge conjugation transformation now to $\psi^A$ (making only the group $G$ index manifest; $\psi^A$ has also spinor indices), one obtains $\Gamma^{a\mu} \psi^A = \Gamma^{a\mu} \psi^{A\dagger}$, which seems very noncovariant. However, we need to see what can be achieved by careful choice of $A_{\mu}^a$. For a unitary representation $U_B = \exp(i \phi B)$ of $G$, the generators $T^B$ are hermitian, so $(T^B)^A_B = T^A_A$. The desired, covariant, form of the C-transformed $(\psi^{A\dagger})^A_A$ is

$A_{\mu}^a T^a_{\mu} = -A_{\mu}^c T^c_{\mu}$.

So we see that what is required of the gauge field is

$A_{\mu}^a T^a_{\mu} = -A_{\mu}^c T^c_{\mu}$, i.e.,

for $T^a_{\mu} = T^a_{\mu}$ symmetric, one must have $A_{\mu}^a = -A_{\mu}^c$;

for $T^a_{\mu} = -T^a_{\mu}$ antisymmetric, one has $A_{\mu}^a = A_{\mu}^c$.

These transformations of the $A_{\mu}$ are just what is needed to make the nonabelian field strength $F_{\mu
u}$ transform coherently under charge conjugation with $A_{\mu}^c$.

$F_{\mu
u} = \frac{i}{g} [D_{\mu}, D_{\nu}]$, so $F_{\mu\nu}$ flips sign if $A_{\mu}$ does, and

stays invariant if $A_{\mu}$ does.
Check whether this transformation of the gauge fields $A_\mu$ under $C$ agrees with what one finds in QED, where $C = U(2)$. The $U(2)$ unitary representation is just $U = e^{i\varphi}$, so one may take $T = 1$. This is a rather trivial symmetric 1x1 matrix. Accordingly, the gauge field $A_\mu$ in $D_A = (\partial_\mu + i e A_\mu) \gamma^\mu$ must transform $A_\mu \rightarrow -A_\mu$, as we found above. For SU(2), with generators $\frac{1}{2} \sigma^i$ with $\sigma^1 = (0i), \sigma^2 = (0-1), \sigma^3 = (10)$, the $A_\mu$ and $A_\mu^a$ gauge fields need to flip sign, while the $A_\mu^a$ gauge field remains invariant under $C$.

Now consider what charge conjugation does to chiral spinor fields when coupled to gauge fields. For simplicity, consider the chiral coupling of a spinor field $\chi$ to a $U(1)$ gauge field like $B_\mu : \chi \partial_\mu \chi - \frac{1}{2} B_\mu \chi \gamma^\mu (1 - \gamma^5) \chi$. In this example, $\chi$ is the charge of the left-handed part of $\chi$, while the right-handed part of $\chi$ remains uncharged, i.e. $D_\mu \chi^L = \partial_\mu \chi^L - \frac{1}{2} B_\mu \chi^L, D_\mu \chi^R = \partial_\mu \chi^R$. Under charge conjugation, the pure derivative kinetic term $\partial^\mu \chi \chi = -2(\bar{\chi} \gamma^\mu \chi + \chi \gamma^\mu \bar{\chi})$ is $C$-invariant, but for the chiral gauge coupling $-\frac{1}{2} B_\mu \bar{\chi} \gamma^\mu (1 + \gamma^5) \chi - \frac{1}{2} B_\mu \bar{\chi} \gamma^\mu (1 - \gamma^5) \chi$. Considering just the spinor field $\chi$ transformation $\chi \rightarrow \chi^* \gamma^\mu$. Rearrange using the earlier rules: $\bar{\chi} \gamma^\mu \chi = -\gamma^\mu \chi$, $\bar{\chi} \gamma^\mu \bar{\chi} = +\gamma^\mu \bar{\chi}$, so the gauge coupling transforms to $-\frac{1}{2} B_\mu \bar{\chi} \gamma^\mu (1 - \gamma^5) \chi$, i.e. charge conjugation has transformed a left-handed chiral coupling into a right-handed chiral coupling. Note that, unlike the nonchiral QED coupling case, there is no transformation $B_\mu \rightarrow -B_\mu$ that can achieve invariance. The expected $B_\mu \rightarrow -B_\mu$ works for the $\chi^\mu \chi$ term, but not for the $\chi^\mu \bar{\chi}$ term in the coupling.

It makes things clearer to consider the effect of charge conjugation in $2$-component notation.
For a Dirac field, $\psi = (\lambda \chi^\alpha, \chi^\alpha)$, one has $\psi_e = (\chi^\alpha, \chi_e^\alpha)$, so the charge-conjugation transformation is $(\lambda \chi^\alpha) \to \psi_e (X_e^\alpha)$, i.e. $\lambda \chi^\alpha \to \psi_e \chi_e^\alpha$, $\chi^\alpha \to \psi_e^* \lambda \chi^\alpha$.

In a chiral gauge coupling, C-invariance can not be achieved because the couplings of $\lambda \chi^\alpha$ and $\chi \chi_e^\alpha$ are not symmetrical. [In the above example, $\chi \chi_e^\alpha$ does not couple at all.]

An even more incompatible situation for C-invariance occurs if one chirality is simply absent for some spinor field type. This is the case for the original Standard model without right-handed neutrinos.

Consider the effect now of a C-transformation on the charged-current couplings in the Standard Model:

$$-\frac{g_2 e}{2 \Delta} \left[ V_{mn} W_{mn}^{\ast} \bar{u}_m \gamma^\mu (1 + \gamma_5) d_n + (\gamma^\mu)_{mn} W_{mn} d_m \gamma^\mu (1 + \gamma_5) u_n \right]$$

Under charge conjugation, this becomes

$$+\frac{g_2 e}{2 \Delta} \left[ \bar{u}_m \gamma^\mu (1 + \gamma_5) d_n \right. \left. + (\gamma^\mu)_{mn} W_{mn} d_m \gamma^\mu (1 + \gamma_5) u_n \right]$$

where we have used the anticipated $W_e \leftrightarrow -W_e^\ast$ based on the symmetric/antisymmetric nature of the SU(2) generators. Then $\bar{u}_m \gamma^\mu (1 + \gamma_5) d_n = - \bar{d}_n \gamma^\mu (1 - \gamma_5) u_m$ as we have seen, so the coupling cannot be charge-conjugation invariant: there has been a $P \to P^\ast$ switch of projectors.

Now consider the effect of a parity transformation on this coupling. Under parity, the spatial components of gauge fields will have to switch sign in order to make couplings like $A_i \chi^\alpha \gamma^\mu \gamma^\nu$ invariant. The time components don't flip sign, however, just as the charge density $\bar{d}_m \gamma^\mu \gamma^\nu$. So $A_i \chi^\alpha \gamma^\mu \gamma^\nu$ terms remain parity invariant. For $\bar{d}_m \gamma^\mu$ coupling terms, 
However, there is a sign flip: $A^+ \rightarrow A^-$. Consequently, $A^+ (1 + i \xi) \rightarrow A^- (1 - i \xi)$, also producing a $P \rightarrow P$ switch of projectors.

Consequently, combining $C$ and $t$ transformations, it is possible to arrange for invariance of kinetic and gauge coupling terms in the standard model as long as the mixing between generations doesn't cause a problem, e.g., if $V_{mn}$ were just $\delta_{mn}$.

Under the combined $C \& P$ transformation for the charge current coupling, one obtains

$$\frac{-g^2}{2 \sqrt{2}} \left\{ (\eta_{\mu n}) (\eta_{\mu n})^* V_{mn} W^+ \gamma_5 (1 + i \xi) \bar{d}_m \right\}$$

If the phases can be chosen to make $(\eta_{\mu n}) (\eta_{\mu n})^* = 1$ and if the CKM matrix $V_{mn}$ can be chosen to be real, then the above term becomes the complex conjugate of the original term before the $CP$ transformation. Note the inversion of the $(m,n)$ indices in the fermion bilinears, which can be corrected by rewriting the $CP$-transformed charged current coupling

$$\frac{-g^2}{2 \sqrt{2}} \left\{ (\eta_{\mu n})^* (\eta_{\mu n}) (\eta_{\mu n})^* V_{mn} W^+ \gamma_5 (1 + i \xi) \bar{d}_m \right\}$$

which agrees with the starting coupling terms provided $(\eta_{\mu n}) (\eta_{\mu n})^* (\eta_{\mu n})^* V_{mn} = V_{mn}$. Clearly, this can be achieved if $(\eta_{\mu n}) (\eta_{\mu n}) = 1 = (\eta_{\mu n})^* (\eta_{\mu n})^*$ for all $m,n$ and if $V_{mn} = V_{nm}$.

An easier way to see that complex coefficients in the SM Lagrangian imply $CP$ violation is to...
consider the implications of CPT invariance. A basic theorem on the relation between spin and statistics (appeared first implicitly in the work of Schrödinger, then more explicitly in work by Lüders & Pauli; see Strebecker & Wightman, “PCT, Spin and Statistics, and all that”) shows that CPT violation implies breaking of Lorentz symmetry. On piners, a CPT transformation corresponds to an orientation preserving Lorentz transformation. Consequently, in a Lorentz invariant field theory any term in the Lagrangian must be CPT invariant, whether it has real or complex coefficients, and T invariance implies CP invariance. The T transformation, however, is antunitary, so it sends i → -i in addition to whatever it does to fields. Thus, if a given Lagrangian term is T invariant for real coefficients, it must be T violating for imaginary coefficients. Then by CPT invariance, one concludes that imaginary coefficients imply CP violation.

In the Standard Particle Data Group presentation of the CKM matrix V_{mn}, this means that a non-zero value of the phase \delta implies CP violation.

Discussing CP violation in terms of a particular phase in one version of the CKM matrix is not optimal, because different choices of basis or phases of V_{mn} can alter the discussion without altering the physics. It is better to have a basis-independent measure of CP violation. For example, in the flavor basis where the W^+ interactions are flavor diagonal, V_{mn} = \delta_{mn}, but the quark mass matrix is then nondiagonal and complex, and CP is still violated. Conversely, if the quark mass matrix were diagonal and V_{mn} were complex, there could still be an absence of CP violation if there were some remaining phase change that could eliminate the complex character of V_{mn}. If one of the quarks is massless, this is the case.
Recall that the original Yukawa coupling matrices are related to the diagonal quark mass matrices:

\[ y_\text{(u)}_{mn} = k_m U_{us}^* U_{ut}^* U_{en} \text{ for upquarks, } y_\text{(d)}_{mn} = k_m U_{ds}^* U_{dt}^* U_{en} \text{ for downquarks} \]

where \( y_\text{(u)} \equiv \frac{\nu_1}{\nu_2} \text{ diag } (k_1,k_2,k_3) \) and \( y_\text{(d)} \equiv \frac{\nu_1}{\nu_2} \text{ diag } (h_1,h_2,h_3) \)
are the diagonal mass matrices in mass basis. The CKM matrix is \( V_{mn} = (U_{le} (U_{\text{le}}^*)^T)_{mn} \). Thus, if \( U_{\text{le}} = U_{\text{ed}} \), then \( V = I \) with no flavour mixing or CP violation.

Instead of rotating \( U_\text{le} = U_{\text{ut}} U_{\text{le}}, \quad U_\text{ed} = U_{\text{ut}} U_{\text{le}}, \quad U_\text{et} = U_{\text{de}} U_{\text{et}} \) to make the mass matrix diagonal, one could first let \( U_{\text{le}}^\dagger \text{le} = U_{\text{le}}^\dagger \text{le} \) unchanged, but take \( U_{\text{ed}}^\dagger \text{le} = (U_{\text{ut}}^\dagger + U_{\text{ue}}^\dagger) U_{\text{en}} \) to make \( y_\text{(u)}^{\dagger} = (U_{\text{ut}}^\dagger + U_{\text{ue}}^\dagger) U_{\text{en}} \)
which is hermitian, and similarly take \( U_{\text{et}}^\dagger \text{et} = U_{\text{et}}^\dagger \text{et} \)
\( U_{\text{et}}^\dagger \text{et} = (U_{\text{de}}^\dagger + U_{\text{de}}^\dagger) U_{\text{en}} \) so \( y_\text{(d)}^{\dagger} = (U_{\text{de}}^\dagger + U_{\text{de}}^\dagger) U_{\text{en}} \) also hermitian.

So consider \( y_\text{(u)} \) and \( y_\text{(d)} \) to be hermitian without loss of generality.

If \( y_\text{(u)} \) and \( y_\text{(d)} \) could be simultaneously diagonalized, then one would have \( V = I \) and there would be no CP violation. A hermitian matrix is diagonalized by a unitary transformation \( Y = U Y U^\dagger \). In order for the hermitian matrices to be simultaneously diagonalizable (with the same \( U \)), they must commute. So the amount of CP violation is encoded in the commutator

\[ -i K = [y_\text{(u)}, y_\text{(d)}] = [U_{\text{ut}}^\dagger U_{\text{ue}}^\dagger U_{\text{en}}^\dagger, U_{\text{de}}^\dagger U_{\text{de}}^\dagger U_{\text{en}}^\dagger] = U_{\text{ut}}^\dagger [D_{\text{(u)}}, V_{\text{(d)}}^\dagger] U_{\text{ed}} \]

The matrix \( K \) is traceless and hermitian, because \( y_\text{(u)}^{\dagger} \) and \( y_\text{(d)}^{\dagger} \) are hermitian.

The natural basis-invariant quantity is its determinant:

\[ \det K = -i \det [D_{\text{(u)}}, V_{\text{(d)}}^\dagger V_{\text{(d)}}] \]

It turns out that

\[ \det K = -2^N J (m_e - m_c) (m_t - m_u) (m_c - m_u) (m_b - m_s) (m_s - m_d) \]

where \( J = \text{Im} (V_{ij} V_{jk}^\star V_{il} V_{lk}^\star V_{jl}^\star V_{lk}^\star) = -\text{Im} (V_{ij} V_{jk} V_{il} V_{lk}^\star) = \text{Im} (V_{ij} V_{jk} V_{il} V_{lk}) = \text{etc.} \)

or, in general
\[ \text{Im} (V_{ij} V_{jk} V_{il} V_{lk}^\star V_{jl} V_{lk}^\star) = J \sum_{\text{perm}} \text{Im} (V_{ij} V_{jk} V_{il} V_{lk}) \]

where \( J \) is the Jarlskog invariant.
The different products giving expressions for \( J \) are all
equal owing to the unitarity of the CKM matrix \( V \). In the
standard Particle Data Group parametrization,
\[
y = s_{12}s_{23}s_{31} c_{12}c_{23}c_{31} s_{13}
\]
As we saw earlier, \( J \) is twice the area of the unitarity
triangle. All the weak CP violation in the Standard Model
is proportional to \( K \), i.e. proportional to \( \text{Im} \det [y^{\dagger}, y^{\dagger}] \).
If \( V \) is real, we have seen that there is no CP
violation, \( J = 0 \) and so \( \det K = 0 \). Moreover, if any
two up-type or two down-type quarks are degenerate in
mass, then \( \det K = 0 \). If there are such degenerate
masses, an extra phase rotation becomes possible to
remove the CP phase. And we have already seen that
if there were only two generations, there would be no CP
violation because one could then remove all phases from \( V \).
Accordingly, CP-violating effects must involve all three
generations. The CKM elements coupling either of the first
two generations to the third are small, so the amount of
CP violation is suppressed by products of such small coefficients.
The first measurement of CP violation was through
decays of neutral kaons. Their quark content is
\[
K^0 = s_d \quad \text{and} \quad \bar{K}^0 = 
\]
which are CP conjugates of each other, mixing generations 1 and 2. The CP eigenstates are
\[
K_1 = \frac{K^0 + \bar{K}^0}{\sqrt{2}}, \quad K_2 = \frac{K^0 - \bar{K}^0}{\sqrt{2}}.
\]
\( K_1 \) is CP even, while \( K_2 \) is CP odd. Were CP a perfect
symmetry, only \( K_1 \) could decay to \( \pi^+ \pi^- \pi^0 \), which are CP even.
\( K_2 \) could in that case only decay to \( \pi^+ \pi^- \pi^0 \) or \( \pi^+ \pi^- \pi^0 \).
This makes \( K_2 \) much longer lived than \( K_1 \) (0.089 ns), however.
Christenson, Cronin, Fitch and Turlay found in 1964 that \( K_2 \) can
decay to \( \pi^+ \pi^- \pi^0 \), about 0.2% of the time, indicating CP violation.
Continuous and approximate symmetries

In addition to the $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y$ local gauge symmetry of the Standard Model, there can be additional rigid (aka "global") symmetries and also approximate symmetries that appear when certain SM parameters are considered to be small. In the bosonic sector, we have already seen an example of such an approximate symmetry in the Higgs sector, which develops an $\text{SO}(4) \cong \text{SU}(2) \times \text{SU}(2)$ (up to $\mathbb{Z}_2$ double covering) when the $\text{U}(1)_Y$ gauge coupling is turned off. This is the origin of the $\text{SU}(2)$ custodial symmetry.

In the fermionic sector, the $\text{SU}(3)_c$ gauge group representations for $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)_Y$ are $(3,2)_0 \oplus (3,1)_{-2/3} \oplus (1,2)_{1/2} \oplus (1,1)_1$, when all spinor fields are presented in their left-handed chiral versions. Thus, instead of listing right-handed e.g. $\text{C}_{\text{LM}}$ as $(1,1)_{-1}$, the left-handed charge conjugate field $\bar{\text{C}}_{\text{LM}}$ is listed as $(1,1)_{1}$. Similarly for $\text{L}_{\text{LM}}$ and $\bar{\text{L}}_{\text{LM}}$ $(3,1)_{-1/3}$. The point of presenting the spinor fields this way is similar to that of presenting the Higgs $\phi_a$ as 4 real fields: to see if there are any degeneracies of representations that could give rise to additional, non-gauged, symmetries. However, all 5 of these irreducible representations for left-handed chiral fields are inequivalent. So there are no accidental symmetries relating fermions within a single generation. However, if the various $\text{f}_{\text{LMM}}, \text{h}_{\text{LMM}}, \text{k}_{\text{LMM}}$ Yukawa couplings are turned off, each of these 5 sets of spinor fields develops an independent $\text{U}(3)$ rigid symmetry in generation space. So in that limit, there is an accidental ($\text{U}(3)$) rigid symmetry.

When the Yukawa interactions are turned on, and incorporating experimental information that none of the eigenvalues of $\text{f}_{\text{LMM}}, \text{h}_{\text{LMM}}$, or $\text{k}_{\text{LMM}}$ vanish or are degenerate, the candidate rigid accidental symmetry group is reduced to a product of independent $\text{U}(1)$ phases for the various fermi species, subject to the requirement that they leave
fmn, lmn and kmn invariant. This requirement imposes the relations $U_e = U_{e^*} = U_{eL}$ and $U_L = U_{eR}$. These constraints reduce the accidental symmetry of the lepton sector to

$$U_L = U_{eR} = \begin{pmatrix}
e^{i\theta_1} \\
e^{i\theta_2} \\
e^{i\theta_3}
\end{pmatrix}$$

For the quarks, a phase transformation symmetry must preserve the form of the charged-current interactions when written in the mass basis (i.e. with diagonal mass matrices). A candidate rigid accidental symmetry must therefore commute with the CKM matrix. For a general unitary CKM matrix, the only such candidates must be proportional to the unit matrix in generation space. This leaves only a single $U(1)$:

$$U_3 = U_{uL} = U_{uR} = \begin{pmatrix}
e^{i\theta_1/3} \\
e^{i\theta_2/3} \\
e^{i\theta_3/3}
\end{pmatrix}$$

The factors of $1/3$ are chosen so that the charges of all quarks are $1/3$. Bound states of quarks need to be $SU(3)$ singlets, and thus contain a multiple of 3 quarks. Consequently, they have integer charges (0 or ±1) under this symmetry.

Note that with the above constraints, the left and right handed components of each flavour of leptons (6 flavours) or quarks (6 flavours) transform the same way, e.g. $\psi_L \rightarrow e^{i\theta_L} \psi_L$, $\bar{\psi}_R \rightarrow e^{i\theta_R} \bar{\psi}_R$. Antiparticles have opposite transformations, hence opposite charges, e.g. $\psi^+_L \rightarrow e^{-i\theta_L} \psi^+_L$, $\bar{\psi}^-_R \rightarrow e^{i\theta_R} \bar{\psi}^-_R$. Note that antiparticles are CPT transforms of particles. There are four such accidental symmetries, which to date appear to be experimentally conserved to a high degree.

- Electron number: $L_e(e^-) = L_e(\nu_e) = +1$, $L_e(e^+) = L_e(\bar{\nu}_e) = -1$ otherwise.
- Muon number: $L_\mu(\mu^-) = L_\mu(\nu_\mu) = +1$, $L_\mu(\mu^+) = L_\mu(\bar{\nu}_\mu) = -1$ otherwise.
- Tau number: $L_\tau(\tau^-) = L_\tau(\nu_\tau) = +1$, $L_\tau(\tau^+) = L_\tau(\bar{\nu}_\tau) = -1$ otherwise.
- Baryon number: $B(q) = \frac{1}{3}$ for all quarks, $B(q) = -\frac{1}{3}$ for antiquarks.

The sum $L = L_e + L_\mu + L_\tau$ is known as lepton number.
It is a striking feature of the Standard Model that both its $SU(3) \times SU(2) \times U(1)$ gauge symmetry plus the four $U(1)$ accidental symmetries correspond to the conservation of the corresponding quantum numbers as observed experimentally. At least, that was the situation until neutrino masses and neutrino oscillations were discovered, which show that the individual lepton family numbers have very small violations. It is now known that rare muon decays like $\mu \rightarrow e^+\nu$ are possible.

The total lepton number $L = L_e + L_\mu + L_\tau$ is conserved to a very high degree, but it too is subject to small amounts of violation. One type of $L$ violation arises from Majorana neutrino mass terms: mass terms with structure $\nu_i \nu_j \nu_i$ have net lepton number $-2$, and the conjugates $\bar{\nu}_i \nu_j \bar{\nu}_i$ have net lepton number $+2$. We shall also see that lepton number is subject to anomalies.

Baryon number $B$ is similarly conserved to a very high degree, but, like lepton number, is subject to anomalies. No experimental evidence for baryon number violation has been observed. A classic example of baryon number violation would be proton decay.

The difference $(B-L)$ is even more robust. Rigid $(B-L)$ symmetry, unlike baryon number alone or lepton number alone, is not broken by chiral anomalies. If neutrinos are given mass via the Seesaw mechanism, $|\Delta(B-L)| = 2$ is possible, but the experimental limits are very stringent ($\gamma < 10^{28}$ years) ($\text{Proton} (B=1, L=0) \rightarrow \ell^+ (8=0, L=0) + \ell^- (8=0, L=0)$ conserves $B-L$).

Searches for violations of any of these rigid accidental symmetries of the Standard Model are important ways to probe for hints of physics beyond the Standard Model. Such symmetry violations could point toward Grand Unification, or some non-minimal versions of supersymmetry.
Low energy effective theories and approximate symmetries

We have already encountered the notion of an effective field theory at lower energies obtained by integrating out higher energy processes in the discussion of neutrino masses. This is a general procedure, pioneered by Ken Wilson. The content and interaction strengths of an effective field theory vary according to energy scale. For a given effective low-energy field content, there will be an effective cutoff above which new higher-energy processes begin to take place (e.g. $W^\pm$ and $Z^0$ creation above their mass scales $\sim 100 \text{ GeV}/c^2$). Below a given cutoff, interaction strengths vary according to renormalization group flow, which we will come back to later on.

For the strong interactions, there is a natural scale below which an effective field theory description becomes relevant. Cuing to asymptotic freedom, the strong interaction coupling decreases as energy scale increases. At energies of the order of the $Z^0$ boson mass ($\sim 91 \text{ GeV}/c^2$), one finds

$$\alpha_s(M_Z^2) = \frac{\alpha_s^2}{4\pi} = 0.12.$$ At lower energies, $\alpha_s$ increases and actually $\frac{\alpha_s}{4\pi}$ tends to infinity at energies $\sim \Lambda_{\text{QCD}} = 250 \text{ MeV}/c^2$. This doesn't actually mean that couplings diverge, what it does mean is that perturbation theory becomes unreliable for the strong interactions below $\Lambda_{\text{QCD}}$.

What can one do to describe QCD physics below the $\Lambda_{\text{QCD}}$ scale? Since $\Lambda_{\text{QCD}}$ is about 270 of the electroweak scale $\sqrt{M_W M_Z}$ (or of the Higgs mass, which we now know to be 125 GeV/c$^2$), it is appropriate to integrate out all the electroweak $\mathcal{O}(100 \text{ GeV}/c^2)$ structure. Moreover, quark masses vary significantly according to flavour:

$\mu \sim 2.3 \text{ MeV}/c^2$, $m_d \sim 4.8 \text{ MeV}/c^2$, $m_s \sim 95 \text{ MeV}/c^2$ "light quarks"

$M_c \sim 1.3 \text{ GeV}/c^2$, $M_b \sim 4.2 \text{ GeV}/c^2$, $m_t \sim 173 \text{ GeV}/c^2$ "heavy quarks."

For energies between $m_u$, $m_d$, $m_s$, and $\Lambda_{\text{QCD}}$ an approximation
becomes relevant: freeze out all $M_w, M_t, M_H$ physics and the $C, \beta$ and $T$ quarks, and treat the "light" $u, d$ and $s$ quarks as massless.

The resulting low energy theory has a leading-order effective Lagrangian consisting just of the strong interaction gluons and the massless $u,d,s$ quarks:

$$ \mathcal{L}_{\text{eff}} = -\frac{1}{4} G_{\mu
u} G^{\mu\nu} = \left( \bar{u} \gamma_\mu (\not\partial + \not g) u \right) \left( \bar{d} \gamma_\mu (\not\partial + \not g) d \right) \left( \bar{s} \gamma_\mu (\not\partial + \not g) s \right) $$

where now the covariant $\not D = \partial^\mu D_\mu = \partial^\mu D_\mu - \frac{1}{2} g_3 (\not q \otimes \not A) \partial^\mu$ involves only the SU(3) gauge fields $G^\mu_{\alpha} \otimes \otimes A^\mu$, $\alpha = 1, \ldots, 8$.

In this massless-quark limit, an accidental $U(3)$ rigid symmetry has clearly appeared, corresponding to flavoured rotations of the $u,d,s$ quarks. The SU(3) subgroup of this $U(3)$ is what Murray Gell-Mann called the eightfold way (alluding to the Noble Eightfold Path of Buddhism).

This SU(3) symmetry was independently proposed by Yuval Ne'eman while he was a student of Abdus Salam at Imperial [M. Gell-Mann & Y. Ne'eman, "The Eightfold Way" (Benjamin, 1964)].

Of course, this $U(3)$ is broken by the non-zero quark masses, but it does give a reasonably accurate description of QCD physics. Since $U(3) = SU(3) \times U(1)$, the SU(3) structure and the $U(1)$ symmetry can be discussed separately. The $U(1)$ subgroup is none other than baryon number symmetry $U(1)$. It is the only part of $U(3)$ which survives when one includes the rest of the Standard Model.

The reason for the $U$ subscripts is that there is actually more accidental rigid symmetry in $\mathcal{L}_{\text{eff}}$ than just the $U(3)$. The Dirac spinor fields $u, d$ and $s$ are reducible under the $\text{Spin}(3,1)$ group; they may be separated into left- and right-chiral projections, so the spinor terms in $\mathcal{L}_{\text{eff}}$ may be separated into $\left( \bar{u} \gamma_\mu (\not\partial + \not g) u \right) \left( \bar{d} \gamma_\mu (\not\partial + \not g) d \right) \left( \bar{s} \gamma_\mu (\not\partial + \not g) s \right)$.
Hence, the full accidental symmetry of \( S_{\text{LQCD}} \) is actually \( U_{c}(3) \times U_{v}(3) \). \( U_{v}(3) \) is the diagonal subgroup of this, under which the \( k \) and \( \bar{p} \) chiral projections of each light quark are rotated by the same \( U_{v}(3) \) matrix.

How much of this accidental symmetry is actually seen in particle experiments? \( U_{v}(3) \) is baryon symmetry \( U_{v}(3) \), and is good to a high degree. Ignoring the breakings caused by quark mass musses, the \( SU_{v}(3) \) symmetry organises baryonic bound states of the quarks into \( SU_{v}(3) \) representations. The lightest set of such baryons is the octet, built from three quarks in the \( 8\times8 \) representation of \( SU_{v}(3) \).

The mixed permutation symmetry \( \tilde{V} \) implies a contraction between the spinor indices of the maximal helicity state: \( \tilde{e}^{ABC} q_{1} q_{2} q_{3} \) must be antisymmetric in \( \{ a \} \) indices, where \( \tilde{e}^{ABC} \) contracted with the \( SU_{v}(3) \) strong interaction indices \( A, B, C \) produce a colour singlet. So this \( SU_{v}(3) \) approximate multiplet has spin \( \frac{3}{2} \). A heavier baryon multiplet is the decuplet, in the \( 10 \times 10 \) representation of \( SU_{v}(3) \). The maximal helicity state of this multiplet, formed from \( \tilde{e}^{ABC} q_{1} q_{2} q_{3} q_{4} \) must have totally symmetrized \( \{ a \} \) spinor indices, so this multiplet has spin \( \frac{5}{2} \). \( SU_{v}(3) \times U_{v}(3) \) agree well with particle data, e.g. the multiplets:

- Octet: \( p^{+}(uud), n^{0}(udd), \bar{p}(uds), Z^{+}(uus), Z^{0}(uus), Z^{-}(dds), \Xi^{0}(uus), \Xi^{-}(dds) \)
- Decuplet: \( \Delta^{++}(uuu), \Delta^{+}(uud), \Delta^{0}(udd), \Delta^{-}(uuu), \Delta^{++}(uus), \Delta^{+}(uus), \Delta^{0}(uus), \Delta^{-}(udu), \Omega^{0}(uus), \Omega^{-}(uus) \)

What about the remainder of the \( U_{c}(3) \times U_{v}(3) \) symmetry of \( S_{\text{LQCD}} \)? None of it fits with experimentally observed particle multiplet structure. We shall see that one of the \( U_{c}(3) \times U_{v}(3) \) generators has an anomaly, and so is removed as a candidate approximate symmetry. Write the action of \( U_{c}(3) \times U_{v}(3) \) transformations on the \( u, d, s \) quarks as

\[
S \left( \frac{U}{d} \right) = \left[ \frac{1}{3} \lambda_{3} \lambda_{9} \tilde{p}_{2} + \frac{2}{3} \lambda_{2} \lambda_{9} \tilde{p}_{2} \right] \left( \frac{d}{s} \right)
\]

where \( \lambda_{3} \), \( \theta = 0, 1, \ldots, 8 \) are the Gell-Mann matrices (so \( SU_{v}(3) \) generators are \( \frac{1}{2} \theta \)) extended by \( \lambda_{0} = \frac{1}{\sqrt{3}} \mathbb{1}_{3} \). Then \( \lambda_{0} \lambda_{9} \tilde{p}_{2} = 2 \tilde{p}_{2} \).
Then define $\omega^0 = \omega^0 + \omega^0$, $\omega^0 = \omega^0 - \omega^0$, so the infinitesimal $U(3) \times U(3)$ transformation becomes

$$S(Y, \psi) = \left[ \begin{array}{c c} \psi^0 & \psi^0 \\ \psi^0 & \psi^0 \end{array} \right] (Y, \psi).$$

Only the axial $\omega^0$ transformations are potentially anomalous. Of them, only one generator is actually anomalous. The anomaly will turn out to involve the trace of the axial generators $\lambda^0 15$, and the only such generator with a non-vanishing trace is $\lambda^0 = 1/3 \tau_3$. Note that while the $\lambda^0$ generators generate a closed $U(3)$ subalgebra, the $\lambda^0 15$ generators have commutators that go into the $U(3)$ generators, see $\gamma^2 = I$. So the $\lambda^0 15$ generators do not in general form a subalgebra.

The exception is $10 15$ and since $\lambda^0$ is proportional to the $3 \times 3$ unit matrix, $\lambda^0 15$ commutes with all other generators; it generates an axial $U_A(1)$ subgroup. This is a normal subgroup, since $\lambda^0 15$ is an ideal, so after it is excluded by the occurrence of anomalies, the remaining generators form a group, which is $SU(3) \times SU(3) \times U_A(1)$.

How much of this chiral accidental symmetry is approximately linearly realized in the experimental particle spectrum? Only the $SU(3) \times U_A(1)$ subgroup. The reason is that the $SU(3)$ strong interactions give rise to condensation of quark bilinears: $\langle \tilde{u}u \rangle = \langle \tilde{d}d \rangle = \langle \tilde{c}c \rangle = v^2$ (recall that the dimension of fermi fields is $3/2$, so these condensates are of dimension 3). The strong interactions moreover give rise to confinement: the fermionic quark states are not directly seen in the spectrum—only $SU(3)$ singlet bound states are observed. This is a classic situation of spontaneous symmetry breaking, where the quark bilinear condensates break the chiral $SU(3) \times SU(3) \times U(3)$ symmetry spontaneously down to $SU(3) \times U_A(1)$. The occurrence of such condensates is an inherently nonperturbative phenomenon in QCD.
A complete derivation of the properties of the QCD vacuum and of colour SU(3) confinement is still lacking. (This problem is an extension of the Clay Mathematics Institute Millennium prize offered for showing confinement in pure Yang-Mills theory without quarks. Success at that would be worth one million dollars.) There are a variety of approaches to the problem, including lattice field theory. In the absence of a full ab initio derivation from microscopic QCD theory, however, one can still approach the analysis of the vacuum and spectrum from general mathematical properties of spontaneously broken symmetries.

Consistency with the observed particle spectrum shows that the axial symmetry transformations generated by $\lambda \Delta i \lambda$, $\lambda = 1, 2, \ldots$ must be spontaneously broken. From Goldstone’s Theorem, we know that there must exist corresponding Goldstone modes. Were the $SU(3) \times SU(3)$ chiral symmetry an exact symmetry, the Goldstone modes arising from its spontaneous breakdown to $SU(3)$ would be exactly massless. However, we know that the chiral symmetry is actually broken by a number of effects: quark masses, couplings to $W$, and $Z$. So the on-shell Goldstone modes are actually only approximately massless. One refers to them as pseudo-Goldstone bosons.

Interpretation of the particle spectrum in terms of spontaneously broken $SU(3) \times SU(3)$ works reasonably well. Experimentally, it is seen that the octet of pseudoscalars is significantly lighter than the next lightest states, which form the octet of vector mesons (including the rho mesons). Consistency of this picture of the pseudoscalar mesons as pseudo Goldstone bosons is given support by lattice QCD calculations which show that the dependence of pseudoscalar mean masses on quark masses is in agreement with the predictions of chiral perturbation theory. Moreover, there are two mean masses, the $\eta$ and the $\eta'$, of which the $\eta$ is a member of the pseudo Goldstone boson octet, and light, while the $\eta'$ is an SU(3) singlet and is significantly heavier. This fits naturally with an associated $\eta'$ with the anomalous $U_A(1)$ axial generator, which is explicitly broken so not protected by Goldstone’s theorem. This is hard to explain at the quark level.