10. Towards the Standard Model: Weak Interactions

The best-known weak-interaction process is neutron decay
\[ n \rightarrow p + e^- + \bar{\nu}_e, \]
where \( \bar{\nu}_e \) is the electron antineutrino. Related processes are
\[ n + \nu_e \rightarrow p + e^- \quad \text{and} \quad n + e^+ \rightarrow p + \bar{\nu}_e. \]

These can all be described in an effective theory by Feynman diagrams like
\[ n \rightarrow p \quad \text{Cross sections for such processes involve} \ \left| M \right|^2, \]
where \( M \) is the amplitude, \( \left| M \right|^2 \) describes the probability of the observable process.

\[ G_F = 1.17 \times 10^{-5} (6.6)^{-2} \text{ is the Fermi coupling constant and} \ k_i \text{ and} \ k_f \text{ are scalar products of the 4-momenta of the various fields entering the 4-point vertex.} \]

From this, one obtains a neutron half-life \( T_n = 886 \text{ seconds}, \)
very long by nuclear physics standards. So the relevant process is weak.

In addition to such lepton-hadron interactions, there are also purely leptonic weak-interaction processes such as
\[ \nu_e + e^- \rightarrow \mu^- + \bar{\nu}_\mu, \]
which is related to muon decay, \( \mu^- \rightarrow e^- + \nu_e + \bar{\nu}_e \).

At low energies, all known weak-interaction processes are described by local 4-point interactions of the relevant spinorial fermionic fields. This is the Fermi theory.

Following the discovery by Lee and Yang that weak decays violate parity conservation, Feynman and Bell-Mann proposed a more specific form for the 4-fermion interactions:
\[ \mathcal{L} = \frac{G_F}{\sqrt{2}} J_\ell(x) J_\had(x), \]
where \( J_\ell(x) = \bar{\ell}(x) \gamma \ell(x) \) is a current constructed from fermionic fields, \( J_\h (x) \) is the lepton part, and \( J_\h(x) \) is the hadron part. The key feature of the Feynman & Bell-Mann proposal is the chiral nature of these currents. In particular,

The lepton part of the current is purely left-handed:
\[ J_\ell(x) = \frac{1}{2} \bar{\ell}(x) \gamma ^\mu (1 + \gamma ^5) \ell(x) = \frac{1}{2} \bar{\ell}(x) \gamma ^\mu \ell(x), \]
where all indices on the spinor fields are suppressed. \( \ell(x) \) and \( \bar{\ell}(x) \) are electron and muon spin \( \frac{1}{2} \) fields while \( \ell(x) \) and \( \bar{\ell}(x) \) are their associated neutrino spinor fields. The explicit form of the hadronic current \( J_\h(x) \) is complicated by features of the strong interactions, but one can say much about the theory from the structure of the lepton part \( J_\ell(x) \) alone.
Chiral currents like $\mathbf{J}_5$ are not parity invariant, and would not have been considered prior to the discovery that parity is not conserved by the weak interactions. Under spatial reflections $x^i \rightarrow -x^i$, $i=1, 2, 3$, one finds that $\mathbf{J}_5 = (\mathbf{J}_5 \gamma^5)^3$ is no longer invariant as it is under $\text{Spin}(3,1)$ transformations, but instead $\mathbf{J}_5 \rightarrow -\mathbf{J}_5$. Aside: For theories involving spinors that are invariant under parity, there exists an extension of $\text{Spin}(3,1)$ called $\text{Pin}(3,1)$, whose purely bosonic analogue is $\text{O}(3,1)$. So the precise statement is that weak-interaction theory is not $\text{Pin}(3,1)$ invariant.

The Fermi four-spinor interaction theory, and in particular its refinement by Feynman and Bell-Mann, works well at low energies. Nowadays, it is what we call an effective theory, good for a certain limited energy range. However, the Fermi theory runs into serious problems when considered as a fundamental theory. An important indication of trouble lies in the high-energy behavior of reactions such as $\nu_e + e^{-} \rightarrow \mu^{-} + \nu_{\mu}$. The cross section $\sigma$ for such a process is $\sigma \sim |M|^2 \varepsilon^2$, where $M$ is the 4-particle amplitude and $\varepsilon$ is the energy in the center of mass frame. Consequently, $\sigma \sim G^2 \varepsilon^2$ grows quadratically with energy in the Fermi theory.

The unbounded growth of cross sections is contradicted, however, by an important physical principle: unitarity of the $S$ matrix, $S^\dagger S = 1$. This does not allow for the unbounded growth of probability for any particular physical process; the sum or integral of probabilities for anything at all to happen must equal unity. The coefficient $G^2$ of the quadratic growth in $\varepsilon$ also gives a hint of the energy range where the Fermi theory must start to break down. Above energies of order $(G^2)^{-1/2} \approx 300$ GeV, something else must happen.

A problem closely related to that of unitarity is that of renormalizability, which also concerns high-energy behavior. Relativistic quantum field theories are plagued by the problem of
ultraviolet divergences in Feynman diagrams containing loops of particle lines. Consider a rough estimate of the behavior of a 4-fermion amplitude like \( \frac{1}{(k^2 + m^2)} \) in a momentum-space calculation. Since the Dirac equation \( (\gamma^\mu \partial_\mu + m) \psi = 0 \) is of first order in derivatives, fermionic propagators in momentum space are\( \frac{-i\gamma^\mu \partial_\mu + m}{(k^2 + m^2)} \), i.e. at high momenta \( k^\mu \) they go like \( (\text{momentum})^{-1} \). Consider the behavior of a loop amplitude where the loop-momentum integral \( \int d^4k \) is cut off at some scale \( \Lambda \) (e.g. after Wick rotation and limiting the Euclideanized integral to momenta inside a 4-ball of radius \( \Lambda \)). Consequently, for a loop amplitude with two fermionic propagators, the overall high-momentum behavior will be like \( \Lambda^{-2} \), quadratically divergent. The correction to the "tree-level" amplitude then goes like \( G_F (G_F \Lambda^2) \). This is dimensionally consistent since \( G_F \Lambda^2 \) is dimensionless.

The problems with ultraviolet divergences get worse and worse as one goes up in loop order. For example, at the two-loop order, the correction \( \propto G_F \) behaves at large \( \Lambda \) like \( G_F (G_F \Lambda^2)^2 \), and so on. Detailed study of the quantum corrections also reveals divergences at the one-loop level of structure \( G_F (\ln \Lambda)^2 \pi_i \pi_j \) where the \( \pi_i \) are the momenta on the external lines. At higher loops, one then gets also behaviors like \( G_F (\ln \Lambda)(G_F \Lambda^2)^2 \pi_i \pi_j \) or \( G_F^2 (\ln \Lambda)^2 \pi_i \pi_j \pi_k \pi_l \) with various contractions on the \( i,j \) indices labeling external lines.

In sum, the ultraviolet behavior of the Fermi theory, when considered as a fundamental theory, is a mess. The proliferation of ultraviolet divergences is clearly linked to the \( (\text{mass})^{-2} \) dimensionality of the Fermi constant \( G_F \).

Theories with only a limited number of ultraviolet divergence structures that can be removed by a process of a finite number of subtractions of infinite terms are called renormalizable. Fermi theory is decidedly non-renormalizable.
A suggestion for how to attack the renormalizability problem was to replace four-fermion interactions with a split process involving only three-point vertices without coupling constants carrying inverse-mass dimensions, but joined by an intermediate massive vector field to reproduce the $J^p_k J^p_k$ form of the interaction. Consider neutron decay $n \rightarrow p + e^- + \bar{\nu}_e$ as an example. Instead of the 4-point process

\[ n \rightarrow p + e^- + \bar{\nu}_e \]

(whence arrows indicate natural time flow, do the inflowing arrow on the neutrino line is interpreted as an outgoing antineutrino), one can have a sequence of two three-point processes:

\[ n \rightarrow W^- \rightarrow W^+ \rightarrow p + e^- + \bar{\nu}_e \]

The coupling terms in the Lagrangian describing such 3-point interactions are natural current-vector couplings

\[ \frac{g}{\sqrt{2}} (J^p_k W^0 + J^p_k W^0) \]

where the coupling constant $g$ is dimensionless and $J^p = J^0 + \eta^0$ is the fermionic current introduced by Feynman and Gell-Mann. Now, in our earlier discussion of the massive vector Proca theory, we saw that the field equations are

\[ \partial^\mu J^\mu = F_\nu^\mu \]

(providing now a source $J^0$ on the RHS). In momentum space these become

\[ (k_\mu k^\mu - (k^2 + M^2) J^0) W^\mu(k) = J^0(k). \]

Solve this for $W^\mu(k)$ to obtain the Feynman propagator:

\[ W^\mu = \frac{(J^p + k^p k^0 J^0)}{(k^2 + M^2)} = \frac{(\not{\nu} + \frac{k^p k^0}{M^2}) J^0}{(k^2 + M^2)}. \]

Accordingly, at lowest order the neutron decay amplitude is

\[ M = \frac{1}{4} g^2 \bar{\nu}_e J^p (1 + \frac{P \cdot l}{M}) \nu_e \left\{ -\frac{\not{\nu} + \frac{k^p k^0}{M^2}}{(k^2 + M^2)} \right\} \langle \text{Pion} | \not{\nu} | \text{Neutron} \rangle \]

where evaluation of the hadronic current matrix element $\langle \text{Pion} | \not{\nu} | \text{Neutron} \rangle$ involves strong-interaction physics.
When one considers the neutron decay amplitude at low 4-momenta \( |k| \leq 1 \text{MeV} \), one has \( \frac{1}{k^2 + M^2} \approx \frac{1}{M^2} \) and \( \frac{\text{p} \cdot \text{k}}{M^2} \approx \text{p} \cdot \text{p} \), so the propagator \( \frac{1}{k^2 + M^2} \) becomes just \( -\frac{\text{p} \cdot \text{p}}{M^2} \). Accordingly, if one makes the identification \( G_F = 2 - \frac{\text{p} \cdot \text{p}}{M^2} \), the Fermi theory neutron decay emerges in the low-momentum limit of the \( W^\pm \) exchange amplitude. (The \( 2 - \frac{\text{p} \cdot \text{p}}{M^2} \) factor has a historical origin - \( G_F \) was defined before the recognition of parity violation in the weak interaction.)

Introducing the intermediate vector boson field \( W^\pm \) removes the most obvious source of nonrenormalizability, the dimensional character of \( G_F \). However, UV problems remain. At large momenta, the massive vector propagator \( \frac{1}{(\text{p} \cdot \text{k} + \text{p} \cdot \text{p})/(k^2 + m^2)} \) tends to \( -\frac{1}{k^2 + m^2} \).

This couples to the longitudinal part \( \text{p} \cdot \text{p} \) of the fermionic current and remains of order \( \frac{1}{m^2} \) for large 4-momenta. This asymptotic behavior is worse than that of scalar propagators \( \frac{1}{k^2 + m^2} \) or spinor propagators \( \frac{1}{k^2 + m^2} \).

The worst behavior as the cutoff \( \Lambda \) is taken to infinity is \( \Lambda^4 \Lambda^4 \Lambda^4 \Lambda^4 \Lambda^4 = \Lambda^2 \), i.e. similar to the \( V(16 \times 16 \times 16 \times 16 \times 16) \).

A hint of how the renormalizability problem might be cured comes from the observation that the worst UV behavior above involves \( k \) \( \text{p} \) couplings to the fermionic currents; in position space, this becomes \( 2 \text{(O\text{O})} \). So the suggestion arises that perhaps one might cure the bad UV behavior by ensuring that \( 2 \text{(O\text{O})} = 0 \), i.e. that the fermionic current is conserved. Indeed, Abius Salam showed that this works for neutral intermediate vector boson exchange.

Shaving renormalizability in models with massive vectors carrying non-abelian indices was much harder. The ultimate resolution of the UV problem requires masses obtained via the Higgs effect and spontaneous symmetry breaking. The proof of Yang-Mills renormalizibility was due to 't Hooft and Veltman (1971; Nobel Prize 1999).
Electroweak Theory: $SU(2) \times U(1)$ Symmetry

One part of the solution to the renormalizability problem is to limit the terms in the Lagrangian to terms of mass dimension no more than 4, so that there are no coupling constants with negative mass dimensions. Thus, only $m^2 \phi^2$, $\lambda \phi^4$ and $\phi \phi^3 X$ (where $m$ is of dimension 1 and $\lambda$ & $\phi$ dimensionless), but terms like $\phi^6$ or $\phi \phi^2 X$ are excluded.

Conservation of the fermion currents requires local gauge symmetries that are broken only spontaneously, with vector field masses arising from the Higgs effect. Putting this all together and showing that one can preserve renormalizability in the presence of spontaneous symmetry breaking required the contributions of several people. After 't Hooft and Veltman, the proof was completed in the work of S. Weinberg and B. Lee.

After the discovery by T.D. Lee and C.N. Yang that parity is not conserved in the weak interactions, and in view of the original impression that neutrinos could be massless, they can be separated in the massless limit into chiral spinors. Initially, all neutrinos are left-handed, i.e. they all have $+1$ eigenvalues of $\frac{1}{2}$. Later on, we will discuss neutrino masses, which may be handled by introducing right-handed neutrino states. In any case, the experimentally confirmed theory is a chiral theory involving Weyl spinors.

In weak interaction processes like muon decay, $\mu^- \rightarrow e^- + \nu_e + \nu_\mu$, charged leptons of a given generation (here, the muon generation) can convert into their corresponding electrically neutral neutrinos (here $\nu_e$). Consequently, the intermediate vector boson $W^-$ must carry electric charge, and its antiparticle $W^+$ must carry the opposite electric charge. Thus, at least two gauge bosons are needed. Moreover, since $W^\pm$ are charged, the corresponding group generators cannot commute with the electromagnetic $U(1)$ generator, i.e. the kinetic terms for $W^\pm$ must involve the electromagnetic field $A_\mu$ in the overall Yang-Mills field strength. Thus, the gauge group cannot be Abelian.
The simplest gauge group that can accommodate \( W^+_\mu \) is \( SU(2) \). Indeed, it plays a central role in the weak interactions; it is called weak isospin. Now, \( SU(2) \) has three generators, and the third generator must correspond to an electrically neutral vector field. Could this be the photon? No — all three \( SU(2) \) generators act on neutrinos, so the gauge field corresponding to the third \( SU(2) \) generator must couple to neutrinos, which the photon doesn’t do. But we have just seen that the weak isospin group doesn’t commute with the electromagnetic \( U(1) \), since \( W^+_\mu \) are charged. So the photon’s generator must somehow be incorporated into the overall gauge group. The next simplest group is \( SU(2) \times U(1) \), which turns out to be the correct choice.

One is led to consider the following structure. The left-handed electron \( e^-_L \) can be converted into the left-handed electron neutrino \( \nu_e \). These two spinor fields form an \( SU(2) \) doublet, transforming as \( \nu^e = \begin{pmatrix} \nu_e \\ \not{e} \end{pmatrix} \rightarrow U \nu^e \hspace{1cm} U \in SU(2) \).

As far as the Lorentz structure is concerned, one may represent these left-handed chiral spinor fields either as Weyl 4-component or 2-component spinors, according to taste. In most of the literature, one sees them considered as 4-component Weyl spinors, so we shall do the same in writing Lagrangians here.

Although the neutrino \( \nu_e \) is initially being treated as a massless field, the electron \( e^- \) definitely has a mass. Accordingly, unlike for a massless neutrino, there is always a Lorentz boost that will turn the left-handed electron \( e^-_L \) into a right-handed electron \( e^-_R \). Only massless fields can be purely of just one chirality. However, with the neutrino \( \nu_e \) massless, \( e^-_R \) has no \( SU(2) \) partner. Thus, \( e^-_R \) must be an \( SU(2) \) singlet. It’s not a beautiful picture, but it works.
11. Salam-Weinberg Theory: The Standard Model

The U(1) factor in the Standard Model electroweak symmetry group \( SU(2) \times U(1) \) is known as weak hypercharge. We have seen in the discussion of the Higgs effect that vector fields will remain massless for all generators of the unbroken stability subgroup \( H \). Vector fields corresponding to the generators of the full symmetry \( G = SU(2) \times U(1) \) that lie outside \( H \) will acquire masses. In electroweak theory, we want to retain only one massless vector field, for the photon. So \( SU(2) \times U(1) \) must break down to \( H = U(1)_{\text{em}} \). This unbroken \( U(1)_{\text{em}} \) group is generated by a mixture of the hypercharge generator, denoted \( Y \), and some \( SU(2) \) generator, which one may take to be \( T^3_1 = \frac{1}{2} \delta^3 \) for the left-handed spinors.

Since \( U(1)_{\text{em}} \) commutes with all the \( SU(2) \) generators in the direct product \( SU(2) \times U(1)_{\text{em}} \), the \( U(1)_{\text{em}} \) charges of all members of an \( SU(2) \) multiplet must be the same. Since \( c_i \) and \( c^e \) belong to the same \( SU(2) \) doublet, it is clear that \( U(1)_{\text{em}} \) cannot be \( U(1)_{\text{em}} \). Let the surviving \( U(1)_{\text{em}} \) generator \( Q_{\text{em}} \) be

\[
Q_{\text{em}} = T^3_1 + Y.
\]

(Warning: conventions vary — older books often took \( Q_{\text{em}} = T^3_1 + \frac{1}{2} Y_{\text{el}} \).) Thus, for the \( Y_{\text{el}} = (e^e) \) left-handed doublet, one can use the known electromagnetic charges of \( e^e \) and \( e^- \) (viz. \( 0 \) and \( -1 \) in units of \( e \)) to read off the \( Y \) eigenvalue:

\[
Y(e^-) = \frac{1}{2}(e^e).
\]

Likewise, for the \( e^e \) right-handed singlet electron, for which \( T^3_1 = 0 \), one must have \( Ye^e = (-1)e^- \), since \( y = -y_c e^- \).

Similarly, for the quarks, one notes that the proton (\( uudd \) quarks) has \( q_{\text{em}} = +1 \) in units of \( e \), while the neutron (\( uudd \) quarks) has \( q_{\text{em}} = 0 \), so the \( u \) quark must have \( q_{\text{em}} = +\frac{2}{3} \) and the \( d \) quark must have \( q_{\text{em}} = -\frac{1}{3} \). Accordingly, the left-handed quark doublet \( (u^e_l, d^e_l) \) must have \( y = +\frac{1}{6} \). The right-handed quarks are singlets just like \( e^e \), so using \( T^3_1 = 0 \) one finds that \( u^e \) has \( y = q_{\text{em}} = +\frac{2}{3} \) while \( d^e \) has \( y = q_{\text{em}} = -\frac{1}{3} \).
Although the strong interactions, with gauge group SU(3), are not our main focus at this point, we note the SU(3) representation assignments for the leptons and quarks: the leptons (\( \nu_e, e^+; e^- \)) are SU(3) singlets, while the quarks (\( u_L, d_L; u_R; d_R \)) are SU(3) fundamental 3 reps, i.e., triplets (young tableaux 3).

An incompletely understood feature of elementary particle physics is the fact that all the above structure, called the first generation, gets repeated twice over with leptons and quarks at progressively higher mass levels, making three generations in all (as far as we know). Moreover, there are small amounts of mixing between the three generations, so the "physical" quarks of the various generations are slightly misaligned with respect to the generational assignments of the leptons. Thus, we need another label for the generation, \( m=1,2,3 \), and also need to be alert to the effects of quark mixing between generations. So, in all, we have an (initial, before introducing neutrino masses) Standard Model spectrum consisting of left-handed leptons \( L_m = (\nu^m_L, e^{+m}_L, e^{-m}_L) \), right-handed leptons \( e^{Rm} \), left-handed quarks \( q_{LmA} = (u^{LmA}_L, d^{LmA}_L) \) and right-handed quarks \( u^{RmA} \) and \( d^{RmA} \). The primes on the quarks anticipate eventual small realignments to account for the generational mixing.

Summarize the SU(3) x SU(2) x U(1) \(_Y\) representation assignments:

<table>
<thead>
<tr>
<th>Field</th>
<th>SU(3)</th>
<th>SU(2)</th>
<th>U(1) (_Y)</th>
<th>eigenvalue</th>
</tr>
</thead>
</table>
| \( L_m = (\nu^m_L) \) | 1 | 2 | -\( \frac{1}{2} \) | \{ leptons \}
| \( e^{Rm} \) | 1 | 1 | -1 | \( m=1,2,3 \) generation
| \( q_{LmA} = (u^{LmA}_L, d^{LmA}_L) \) | 3 | 2 | +\( \frac{1}{6} \) | \{ quarks \}
| \( u^{RmA} \) | 3 | 1 | +\( \frac{2}{3} \) | \( A=1,2,3 \) colour
| \( d^{RmA} \) | 3 | 1 | -\( \frac{1}{3} \) | SU(3) index |
Dirac conjugates of all these spinor fields are necessary to write the Standard Model Lagrangian. The Dirac conjugates of these chiral fields have opposite chirality (opposite $\tilde{I}_5$ eigenvalues) and transform in conjugated $SU(3) \times SU(2) \times U(1)_Y$ representations. Thus, for the left-handed lepton doublet, one has $\psi_L^b = (\psi_R^b)_{\tilde{b}}$, $b = 1, 2$ and the $U(1)_Y$ eigenvalue is flipped $y \to -y$, giving $+\frac{1}{2}$ for the conjugated field. The left-handed conjugate is an $SU(2)$ (as well as $SU(3)$) singlet, but its $U(1)_Y$ eigenvalue flips to $+1$. For the quarks, one has to take care also with the strong-interaction $SU(3)$ representations as well. Thus, $Q_{LM}^A = \left( \begin{array}{c} Q \end{array} \right)_{LM}^A$ (i.e. $Q_{LM} = (Q_{LM})_{\tilde{b}}$) transforms in the $(3^*; 2^*; 1)$ representation of $SU(3) \times SU(2) \times U(1)_Y$, while $U_{RM}^A = (U_{RM})_{\tilde{b}}$ and $d_{LM}^A = (d_{LM})_{\tilde{b}}$ transform in the $(3^*; 1)$, and $(3^*; 1)$, representations respectively.

The detailed transformations of these spinor fields follow the general pattern that we have described earlier. Thus, under $SU(3)$, $Q_{\lambda} \to U^{(3)}_{\lambda} B Q_{\lambda}$ with $U^{(3)}_{\lambda} = \exp \left( \frac{i}{2} \Theta_{\lambda} \right)_B$; under $SU(2)$, $L_{\beta} \to U^{(2)}_{\beta} L_{\beta}$ with $U^{(2)}_{\beta} = \exp \left( \frac{i}{2} \Theta_{\beta} \right)_e$; and under $U(1)_Y$, $\Psi \to \exp \left( i y \Psi \right) \Psi$, where $y$ is the $U(1)_Y$ eigenvalue of the field $\Psi$.

The Standard Model gauge group $SU(3) \times SU(2) \times U(1)_Y$ is non-simple, so in constructing covariant derivatives one can have different coupling constants $g_3, g_2, g_1$ for the three simple factor groups. To construct covariant derivatives, one needs gauge fields for $SU(3)$, $SU(2)$ and $U(1)_Y$. Denoting these by the corresponding adjoint representations (but recalling that they really need to be gauge fields, not just adjoint tensors), one has $\left( \begin{array}{c} \psi \end{array} \right)_{\tilde{b}}$ for $SU(3)$ $\sim \left( \begin{array}{c} 3 \end{array} \right)_{\tilde{b}}$, $W^{\mu}_{\nu}$ for $SU(2)$ $\sim \left( \begin{array}{c} 1 \end{array} \right)_{\tilde{b}}$, and $B_{\mu}$ for $U(1)_Y$ $\sim \left( \begin{array}{c} 1 \end{array} \right)_{\tilde{b}}$.

Then one can construct covariant derivatives for all the spinor fields, noting that $(T^{(3)}_{\psi})_{\lambda}^B = \frac{1}{2} (\lambda^i)^{\lambda} b$, $i = 1, 2, 3$ and $(T^{(2)}_{\psi})_{\lambda}^b = \frac{1}{6} (\delta^i)^{\lambda} b$, $i = 1, 2, 3$.
\[ D_{\mu} L_m = \partial_{\mu} L_m + \left( -\frac{i}{2} g_2 W_{\mu}^a \sigma + \frac{i}{2} g_1 B_{\mu} \right) L_m \]
\[ D_{\mu} R_m = \partial_{\mu} R_m + i g_1 B_{\mu} R_m \]
\[ D_{\mu} Q_{Lm} = \partial_{\mu} Q_{Lm} + \left( -\frac{i}{2} g_3 G_{\mu}^a \lambda - \frac{i}{2} g_9 G_{\mu}^a \gamma_5 \right) Q_{Lm} \]
\[ D_{\mu} W_{Rm} = \partial_{\mu} W_{Rm} + \left( -\frac{i}{2} g_3 G_{\mu}^a \lambda - \frac{2i}{3} g_9 B_{\mu} \right) W_{Rm} \]
\[ D_{\mu} d_{Rm} = \partial_{\mu} d_{Rm} + \left( -\frac{i}{2} g_3 G_{\mu}^a \lambda + \frac{1}{3} g_9 B_{\mu} \right) d_{Rm} \]

The kinetic terms for the gauge fields are built using standard Yang-Mills field strengths for each gauge factor:

\[ G_{\mu}^a = \partial_{\mu} A_{\mu}^a - \frac{1}{2} g_3 f_{abc} A_{\mu}^b A_{\mu}^c \]
\[ W_{\mu}^i = \partial_{\mu} W_{\mu}^i - \frac{1}{2} g_9 f_{abc} W_{\mu}^a W_{\mu}^b \]

The gauge field and spinor-field kinetic terms are then standard:

\[ \text{gauge} = \frac{1}{4} G_{\mu}^a G^{a\nu} - \frac{1}{4} W_{\mu}^i W^{i\nu} - \frac{1}{4} B_{\mu}^a B^{a\nu} \]

So far, all we have done is to construct a gauge-coupled system for the non-simple gauge group \( \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \) with a particular set of massless fermions. For the real Standard Model, the \( e_m, \nu_m \) and \( d_{Rm} \) will need to be massive. However, with the choices of group representations that have been made, it is impossible to construct \( \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \) gauge-invariant mass terms for these chiral fields. Every attempt to do so involves something like \( \bar{X}_e \gamma_\mu X_e \) for some spinors \( X_e \) and \( \bar{Y}_e \), and the gauge-group index structures are never such as to permit a legal construction of a gauge-invariant expression. For example, \( \bar{X}_{\nu_m} X_m \) is mismatched in \( \text{SU}(2) \) and \( \text{U}(1) \) transformation types, and similarly for all other candidate mass terms.
The Higgs sector and Yukawa terms

To generate masses for the Standard Model fermions and the intermediate vector bosons, we employ the Higgs effect to break the electroweak sector $SU(2) \times U(1)_Y$ down to $U(1)_{em}$. The simplest system that can do this involves a complex doublet of scalar fields $\phi_a = (\phi^+_a, \phi^-_a)$ transforming as $\left(1, 2 \right)_{\frac{1}{2}}$ under $SU(3) \times SU(2) \times U(1)$. The complex conjugate doublet $\phi^*_a = (\phi^+,*_a, \phi^-,*_a)$ transforms as $\left(1, 2 \right)_{\frac{1}{2}}$ but in $SU(2)$, we can lower the "a" index: $\phi_a = e^{ia\theta} \phi^*_a = (\phi^+,*_a, -\phi^-,*_a)$ which transforms as $(1, 3)_{1/2}$. More complicated systems involving multiple Higgs fields are also possible. So far, only one Higgs state has been seen - presumably a residue of symmetry breaking.

The Lagrangian for the Higgs doublet is standard:

$$
\mathcal{L}_{\text{Higgs}} = -\frac{1}{2} \partial^\mu \phi^\dagger \partial^\mu \phi - V(\phi^1 \phi^2) \quad \phi^+ = (\phi^+,*_a, \phi^-,*_a)
$$

$$
V(\phi^1 \phi^2) = \lambda (\phi^1 \phi^1 - \frac{v^2}{2})^2
$$

where, for the $\phi_a$ gauge transformations

$\phi_a \rightarrow \exp \left( \frac{i}{2} \theta \sigma^a \right) a^b \phi_b \quad SU(2)$

$\phi_a \rightarrow \exp \left( \frac{i}{2} \theta \right) \phi_a \quad U(1)_Y$

The corresponding covariant derivative is

$$(\partial^\mu \phi)_a = \partial^\mu \phi_a - \frac{ie}{2} g_{\mu}^\nu W^\nu \phi_a - \frac{i}{2} g_{\mu} B^\mu \phi_a$$

In order to generate masses for the fermions, we introduce $SU(3) \times SU(2) \times U(1)_Y$ gauge invariant Yukawa interaction terms.

With the Higgs doublet field, we can construct three gauge-invariant types of terms:

- $\bar{L}m e_R \phi = \bar{L}m \phi_a \phi_a$ (hypercharge $\frac{1}{2} - 1 + \frac{1}{2} = 0$),
- $Q_l m d_{\nu} \phi = \bar{Q}_l m A^a \phi_a \phi_a$ (hypercharge $\frac{1}{2} - \frac{2}{3} + \frac{1}{2} = 0$)
- and $Q_l m \nu_R \phi = \bar{Q}_l m B^a \phi_a \phi_a$ (hypercharge $\frac{1}{2} + \frac{2}{3} - \frac{1}{2} = 0$).

Note that we have left the $m$ generation labels open. All one can say a priori is that there should be some matrix of coefficients $m_{mn}$, $h_{mn}$ and $k_{mn}$ for the Yukawa interactions:

$$
Y_{\text{Yukawa}} = -i \left( f_{mn} \bar{L}m e_R \phi + h_{mn} \bar{Q}_l m d_{\nu} \phi + k_{mn} \bar{Q}_l m \nu_R \phi \right) + \text{h.c.}
$$
The full Standard Model Lagrangian is then
\[ \mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{Gauge/sfermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Lepton}}. \]

**Symmetry breaking**

For the chosen potential \( V = \lambda (\phi^* \phi - \frac{\mu^2}{2})^2 \), the vacuum expectation value \( \phi_0 \) of the scalar doublet field may, without loss of generality, be taken to be
\[ \phi_0 = \left( \begin{array}{c} 0 \\ \mu \end{array} \right) \quad \text{with} \quad v^2 = \frac{\mu^2}{\lambda}, \quad \phi^* \phi = v^2. \]

Now find the stability subgroup \( H \) of unbroken gauge symmetry by making infinitesimal \( SU(2) \times U(1) \) transformations of the vacuum \( \phi_0 \):
\[ \delta \phi = \frac{i}{2} (\xi \sigma + ip) \phi = \frac{i}{2} \left( \begin{array}{cc} \xi_1 + i \xi_2 & -i \xi_2 \\ i \xi_2 & \xi_1 - i \xi_2 \end{array} \right) \left( \begin{array}{c} 0 \\ v \end{array} \right). \]

Requiring \( S_3 \phi = 0 \), one finds \( \xi_1 = 5_2 = 0 \) and \( \xi_3 = 0 \).

Accordingly, the stability subgroup \( H = U(1) \) is generated by \( T^3 + Y \), as anticipated.

Now expand \( \phi \) about its \( \phi_0 \) vacuum value using the exponential parametrization used earlier:
\[ \phi = \exp \left( \frac{i}{2v} \xi(x) k^i \right) \left( \begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array} \right), \]

where the \( k^i \) are the broken generator combinations \( \frac{1}{2} T^1 \), \( \frac{1}{2} T^2 \) and \( \frac{1}{2} T^3 - Y \) and the \( \xi(x) \) are the would-be Goldstone bosons. The \( \xi(x) \) are eliminated by going into unitary gauge where \( \xi(x) = 0 \) and
\[ \phi = \text{unitary} \left( \begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array} \right). \]

To work out the boson field masses, we need the covariant derivative of \( \phi \) in detail in the unitary gauge:

\[ D_\mu \phi = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array} \right) - \frac{i}{2 \sqrt{2}} \left( \begin{array}{c} g_1 W^3_0 + g_1 B^3_\mu - g_2 W^2_0 - g_2 B^2_\mu \\ g_1 W^3_0 + g_2 W^2_0 \end{array} \right). \]
Expanding the kinetic term for $\phi$, one has

$$-D_\mu \phi^+ D^\mu \phi =
- \frac{1}{2} \partial_\mu H^+ H - \frac{1}{8} (v + H)^2 g_2^2 (W^1_\mu + i W^2_\mu) (W^1_\mu + i W^2_\mu)
- \frac{1}{8} (v + H)^2 (g_2 W^3_\mu + g_1 B_\mu) (g_2 W^3_\mu + g_1 B_\mu)

while the potential term becomes

$$-V = -\frac{\lambda}{4} \left[ (v + H)^2 - \frac{\mu^2}{\lambda} \right]^2
= -\frac{\lambda}{4} \left( 2vH + H^2 \right)^2
= -\left( \lambda v^2 H^2 + 2\lambda v H^3 + \frac{\lambda}{4} H^4 \right)$$

From these results, we read off the masses of the bosonic fields. For the Higgs field $H$, one has the expansion

$$-V \text{ out to second order with a standard mass term } -\frac{1}{2} m_H^2 H^2$$
giving $m_H^2 = 2\lambda v^2 = 2\mu^2$. 

For the spin-one particles, the relevant terms in $S_{\text{Higgs}}$ are

$$-\frac{1}{8} g_2^2 v^2 (W^1_\mu - i W^2_\mu) (W^1_\mu + i W^2_\mu) - \frac{1}{8} v^2 (g_2 W^3_\mu + g_1 B_\mu) (g_2 W^3_\mu + g_1 B_\mu).$$

The fields $W^1_\mu$ and $W^2_\mu$ appear nicely here in the diagonal combination $-\frac{1}{8} g_2^2 v^2 (W^1_\mu + W^2_\mu)$. Comparing to the standard form

$$-\frac{1}{2} M^2 W^1_\mu W^1_\mu - \frac{1}{2} M^2 W^2_\mu W^2_\mu,$$

we identify $M^2 = g_2^2 v^2 / 4$. 

It is no accident that $W^1_\mu$ and $W^2_\mu$ have the same mass; they transform into each other under the unbroken stability subgroup $U(1)_\text{em}$ generated by $T^3 + Y$; set $S_3 = 5_2 = 0$ and $S_3 = 0$ to find $\delta (W^1_\mu) = \delta (0 \ 1) (W^2_\mu)$. 

Accordingly, $W^{1,2}_\mu$ are charged under $U(1)_\text{em}$. Combine them into a conjugated pair of complex vector fields $W_\mu^+ = \frac{1}{\sqrt{2}} (W^1_\mu + i W^2_\mu)$. These transform as $W^{+} \rightarrow C^{+} \cdot W^{+}$ for finite $U(1)_\text{em}$ transformation. Compare the mass term to the standard $-M^2 W^+ W^- W^\mu$ to obtain again $M^2 = M^2_1 = M^2_2 = g_2 v^2 / 2$. 

We now know $m_H = 125 \text{ GeV}$.
The remaining vector fields, $W^3_\mu$ and $B_\mu$, appear in the combination $-g_1 B_\mu + g_2 W^3_\mu$ in the mass term. So define a normalized combination

$$Z_\mu = \frac{-g_1 B_\mu + g_2 W^3_\mu}{\sqrt{g_1^2 + g_2^2}}.$$ 

One can also write $Z_\mu = W^3_\mu \cos \theta_W - B_\mu \sin \theta_W$

where $\cos \theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}$ and $\sin \theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}$.

$\theta_W$ is the weak mixing angle, or Weinberg angle. In terms of $Z_\mu$, the kinetic term has the standard form

$$-\frac{1}{4} (D_\mu Z^\nu - D^\nu Z_\mu)(D^\nu Z_\mu - D_\mu Z^\nu)$$

while the mass term $-\frac{1}{2} M_Z^2 Z_\mu Z^\mu$ is $-\frac{1}{8} v^2 (g_1^2 + g_2^2) B_\mu Z^\mu$.

So we identify $M_Z = \frac{1}{2} v \sqrt{g_1^2 + g_2^2}$.

The remaining vector field combination, orthogonal to $Z_\mu$, is

$$A_\mu = W^\mu_\mu \sin \theta_W + B_\mu \cos \theta_W = g_1 W^3_\mu + g_2 B_\mu.$$ 

$A_\mu (x)$ has no mass term. This vector field combination is the gauge field corresponding to the unbroken $U(1)_L$ gauge symmetry. So we identify $A_\mu$ as the photon field.

**Fermion masses and vector-field couplings**

In addition to the masses of $W^\pm_\mu$ and $Z_\mu$, spontaneous symmetry breaking also generates masses for the fermions, via the Yukawa interactions. Inserting the unitary-gauge values $\Phi = (\frac{1}{\sqrt{2}} v + \Pi)$ and $\Phi' = (\frac{1}{\sqrt{2}} v + \Pi')$ into the Yukawa terms in the SM Lagrangian, one obtains

$$\text{Yukawa} = -\frac{i}{2} (v + \Pi)(\bar{e} L e e_n + \mu \nu L L \nu L e_n + K_m n \nu n + \bar{W}_m W_n W_n) + \text{h.c.}.$$ 

The fermion masses arise from the $\frac{1}{2} \Pi$ terms, while the other terms are trilinear Higgs-lepton and Higgs-quark couplings.
The mass terms \( \frac{-i}{\sqrt{2}} (\overline{e_m} e_m + \overline{\nu_m} \nu_m + \overline{d_m} d_m) \) involve general mixtures of the three \( m = 1, 2, 3 \) generations for general \( f_{mm}, k_{mn} \) and \( h_{mm} \). However, we can simplify the structure of the mass terms by making non-symmetry \( U(3) \) unitary transformations independently for all the lepton and quark species: \( e_m \rightarrow U_{\ell m} e_m, \)
\( e_m \rightarrow U_{\nu m} \nu_m, \)
\( e_m \rightarrow U_{d m} d_m \), \( U_{\ell m} = U_{\nu m} = U_{d m} = I \)
\( d_m \rightarrow d_m = U_{\nu m} d_m \), \( d_m \rightarrow d_m = U_{d m} d_m \). Note that we are now defining the unprimed quarks. The Dirac conjugates of each of these transform contragrediently: \( e_m \rightarrow \overline{e}_m U^{\dagger}_{\ell m} \), etc. The effects of these field-redefinition transformations on the Yukawa coupling matrices are unitary transformations \( f_{mm} \rightarrow f_{mm} U_{h m} U_{h m} \), and similarly for \( k_{mn} \) and \( h_{mn} \). Since the unitary matrices acting on the first and second indices are independent, one can use these transformations to make \( f_{mm}, k_{mn} \) and \( h_{mn} \) diagonal and with positive real values: \( f_{mm} = \text{diag} (f_1, f_2, f_3), k_{mn} = \text{diag} (k_1, k_2, k_3), h_{mn} = \text{diag} (h_1, h_2, h_3) \). Then the mass terms become simply
\[
\frac{-i}{\sqrt{2}} (\overline{e}_m e_m + \overline{\nu}_m \nu_m + \overline{d}_m d_m)
\] 
So the lepton and quark masses are \( m_{\ell m}^{(e)} = \frac{1}{2} f_{m m} \overline{e}_m e_m, m_{\nu m}^{(\nu)} = \frac{1}{2} k_{m m} \overline{\nu}_m \nu_m, m_{d m}^{(d)} = \frac{1}{2} h_{m m} \overline{d}_m d_m \). Note that the neutrinos have not yet developed masses because we have not yet introduced right-handed species for them.

The six non-symmetry \( U(3) \) transformations used to diagonalize the mass terms cannot be made scot-free in the rest of the Lagrangian; however — we need to take account of their effects on the kinetic and gauge-coupling terms. The pure derivative kinetic terms like \( \frac{1}{2} \overline{\ell}_m \ell_m \overline{\ell}_m \ell_m \) or \( \frac{1}{2} \overline{\nu}_m \nu_m \overline{\nu}_m \nu_m \) are actually invariant under such \( U(3) \) transformations. Moreover, although no neutrino masses have yet been generated, there is no reason not to transform \( \nu_{\ell m} \) in exactly the same way as \( e_{\ell m} \). Then, for the left-handed leptons one has a uniform transformation \( \nu_{\ell m} \rightarrow U_{\nu m} \nu_{\ell m} \), and for such a rigid \( U(3) \) transformation, \( D_{\nu m} \rightarrow U_{\nu m} D_{\ell m} \).
Consequently the full left-handed lepton kinetic term $-\frac{1}{2} \overline{E_L} \gamma^\mu \gamma^5 E_L$, including the gauge-field interactions, is invariant under $U_{\nu \nu}$. This is the reason why we did not bother putting primes on $E_L$. Similarly, for the right-handed leptons one finds that $-\frac{i}{2} \overline{E_R} \gamma^\mu \gamma^5 E_R$ is invariant under $U_{\nu \nu}$.

For the quarks, one needs to be more careful. To diagonalize the mass matrix, one needs to make independent $U(3)$ transformations $U_{\nu \nu}$ and $U_{d \nu}$, so for the left-handed quarks, there is no uniform transformation of $Q_{dn}$. The pure derivative kinetic terms $-\frac{1}{2} \overline{U_{\nu \nu}} D_{\mu} \gamma^\mu U_{\nu \nu}$ and $-\frac{1}{2} \overline{d_{\nu \nu}} D_{\mu} \gamma^\mu d_{\nu \nu}$ are the naturally diagonal in the $U_{\nu \nu}$ and $d_{\nu \nu}$ quark states themselves invariant. So are the other naturally $U_{\nu \nu}$ and $d_{\nu \nu}$ diagonal terms coupling the quarks to $W_3^\pm$ and $B_\nu$ (i.e. to $Z$ and $A_\nu$). But the gauge-coupling terms involving $W_3^\pm$ are not all invariant. These charged-current interactions are

$$L_{CC} = \frac{i g_2}{12} \left[ W_3^+ (\overline{U_{\nu \nu}} \gamma^\mu \gamma^5 U_{\nu \nu} + \overline{d_{\nu \nu}} \gamma^\mu d_{\nu \nu}) + W_3^- (\overline{E_L} \gamma^\mu \gamma^5 E_L + \overline{d_{\nu \nu}} \gamma^\mu d_{\nu \nu}) \right].$$

As we have seen, transforming $V_{\nu \nu}$ and $E_L$ by the same $U_{\nu \nu}$ leaves the lepton terms invariant. For the quark terms, define $V_{\nu \nu} = (U_{\nu \nu} + U_{d \nu})_{\nu \nu}$. $V^\dagger V = 1$, so $V_{\nu \nu}$ is also unitary. Then, after diagonalizing the fermion mass matrices, the charged-current interactions become

$$L_{CC} = \frac{i g_2}{12} \left[ W_3^+ (\overline{V_{\nu \nu}} \gamma^\mu \gamma^5 V_{\nu \nu} + \overline{V_{\nu \nu}} \gamma^\mu V_{\nu \nu}) + W_3^- (\overline{E_L} \gamma^\mu \gamma^5 E_L + \overline{V_{\nu \nu}} \gamma^\mu V_{\nu \nu}) \right].$$

The matrix $V_{\nu \nu}$ is the Cabibbo–Kobayashi–Maskawa (CKM) matrix. A priori, such a 3×3 unitary matrix would depend on 9 real parameters. However, there is still some simplification one can make with non-symmetry field redefinitions. One can make pure phase $e^{i \alpha}$ transformations of the 3×2 = 6 quark species, leaving the pure derivative $\gamma$ kinetic terms and the mass terms invariant (since for these, e.g., $E_L$ and $E_R$ transform the same way, so $\overline{E_L} E_R$ is invariant). These can be used to remove some of
the parameters determining $V_{\text{mn}}$. But only the non-symmetry phase transformations can be used this way, and there exists one phase symmetry of the whole Lagrangian: an overall transformation of all six quarks by the same phase. So only $6 - 1 = 5$ parameters determining $V_{\text{mn}}$ can be removed. Left over, we have 4 physically important parameters. For unknown reasons, they are small: $V_{\text{mn}}$ is real.

Had $V_{\text{mn}}$ turned out to be a real, orthogonal $3 \times 3$ matrix, it would have depended on $3 \times 2/2 = 3$ real parameters. Since $V_{\text{mn}}$ actually depends on 4 physically significant parameters, there is just one that reveals the complex nature of $V_{\text{mn}}$: a phase. The non-zero value of this phase is related to the breakdown of CP symmetry, and was a key element in the 2008 Nobel Prize for Nambu, Kobayashi, and Maskawa.

Count the physically important parameters in the original (i.e., prior to neutrino masses) Standard Model. One has 10 masses (3 leptons, 6 quarks and 1 Higgs); 4 CKM matrix parameters; 3 Higgs self-couplings $\lambda$ and 3 coupling constants $g_3, g_2$ & $g_1$. That makes 18 visible at the classical level. There is one more that arises at the quantum level (the "vacuum angle" $\theta_3$ in the term $-\frac{g_3^2}{2m_W^2} G^\mu \phi^\dagger \phi^\mu$). So there are 19 parameters in total. That may seem a lot, but consider the precision of the resulting theory. Taking electromagnetic SM effects into account, the anomalous gyromagnetic ratio $\frac{9-2}{2}$ for the electron is calculated to be $0.0011596521594 (230)$. This should be compared with the experimental value $0.0011596521884 (43)$, a fantastic agreement between theory and experiment.

Neutrino masses

In 1967, when the Standard Model was formulated, there was no evidence for neutrino masses, so they were treated as massless. Subsequent experiments by Ray Davis (Homestake Gold Mine, Lead SD, 10.5 gallons of dry cleaning fluid) and Masatoshi Koshiba (Super Kamiokande) led to the discovery of oscillations between...
The three generations of neutrinos. This explained why the Davis experiment saw only about 1/3 of the expected flux of solar neutrinos, because it was sensitive only to νe. In 1968, Bruno Pontecorvo and V.N. Gribov had proposed that neutrino masses would lead to such oscillations. Since 2001, results from the Sudbury Neutrino Observatory show that only about 35% of solar neutrinos are νe when they reach the earth. These observations give evidence for neutrino masses, in particular, for differences in the νe, νμ and ντ masses, but don’t give any single value. Nonetheless, we have from them evidence that at least one mass is at least 0.04 eV; miniscule with respect to the other masses in electroweak physics.

One popular way to incorporate neutrino masses into the Standard Model is to add right-handed spinor fields νRN that are singlets with respect to all S.M. Lepton groups: (1, 1, 0). This allows for two new types of term in the Lagrangian. The first is a new type of Yukawa coupling: \(-i\overline{\Psi}_{\mu\nu} L_{\mu\nu} \bar{\nu}_{\nu} \Phi + \text{h.c.} \) (hypercharge: \(\pm 1 \otimes -1 = 0\)). Taken alone, this is not very good physically: if \(\nu_{\mu\nu}\) are of the same order as the other Yukawa coefficients \(\nu_{\mu\nu}\), then neutrino masses come out too large.

The second new type of term in the Lagrangian has a paradoxical effect. Since the \(\nu_{\mu\nu}\) are SU(3) x SU(2) x U(1) singlets, we can, for them and only for them, construct a normal bilinear mass term:
\[
\langle \nu_{\mu\nu} \rangle = \langle \nu_{\mu\nu} \rangle^c \Sigma \nu_{\mu\nu},
\]
Another way to write this mass term is to repackage the \(\nu_{\mu\nu}\) as Majorana spinors \(\nu^{\text{Maj}}_{\mu\nu} = (\nu_{\mu\nu})^T \) (a convenient step for \(\nu_e\) since it is a gauge group singlet) and write \(-iM \overline{\nu}_{\mu\nu} \nu^{\text{Maj}}_{\mu\nu}\). This “Majorana mass” has no reason to be small — it could, for example, be at the TeV scale.

Now consider what happens after symmetry breaking and choice of the unitary gauge. The mass term arising from \(-i\nu_{\mu\nu} L_{\mu\nu} \bar{\nu}_{\nu} \Phi + \text{h.c.}\) is \(-i\nu_{\mu\nu}(\overline{\nu}_{\nu} \nu_{\nu} \nu_{\nu} + \nu_{\nu} \nu_{\nu} \nu_{\nu})\). Adding to this its conjugate, one has \(-\frac{1}{2} \nu_{\mu\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu} \nu_{\nu}\).
Thus, the Yukawa interaction produces off-diagonal mass terms between left and right neutrino species after electroweak symmetry breaking.

To understand better what one will see physically with such a system, consider just a single generation, e.g. \( n = 1 \). Phase rotation of \( \nu_L \) can be used to make \( p_{\nu_{\ell}^c} \) real and positive. For simplicity of presentation, repackage both \( \nu_L \) and \( \nu_R \) spinors into Majorana spinors:

\[
\nu_1 = \begin{pmatrix} \nu_{e1}^L \\ \nu_{e1}^R \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} \nu_{e2}^L \\ \nu_{e2}^R \end{pmatrix}.
\]

Then combine the two types of mass term:

\[
- \frac{i}{2} \left( \begin{array}{cc} M & \frac{\sqrt{2} v}{\sqrt{2} M} \\ \frac{\sqrt{2} v}{\sqrt{2} M} & \frac{1}{2} \end{array} \right)
\]

The vanishing element in the \( 11 \) position reflects the impossibility of having a mass for \( \nu_L \) spinors in the original Standard Model. The true eigenvalues \( \lambda \) of such a mass matrix are given by solving

\[
- \frac{\lambda}{2} (M - \lambda) - \frac{\sqrt{2} v^2}{2} = 0 \quad \text{giving} \quad \lambda = \frac{M}{2} \left( 1 \pm \sqrt{1 - \frac{\sqrt{2} v^2}{2M^2}} \right).
\]

Thus, if \( M \) is very large compared to \( \frac{\sqrt{2} v^2}{2} \), i.e. for \( \frac{\sqrt{2} v^2}{2M^2} \ll 1 \), we have one large eigenvalue \( \lambda_+ \approx M \), corresponding to a very heavy neutrino species (so far undiscovered) and also a small eigenvalue

\[
\lambda_- \approx \frac{M}{2} \left( 1 - \left( 1 - \frac{\sqrt{2} v^2}{4M^2} \right) \right) = \frac{\sqrt{2} \nu^2}{8M}.
\]

So if \( \frac{\sqrt{2} v}{\sqrt{2} M} \ll M \), one has \( \lambda_- \ll \frac{\sqrt{2} v}{\sqrt{2} M} \ll \frac{1}{2} \), i.e. this mechanism can give a mass that is nonzero but much smaller than the other lepton masses. This mechanism is called the Seesaw mechanism. It remains speculative – much remains to be understood about neutrino masses and the mixing of neutrino generations.

Some resources on neutrinos:

- John N. Bahcall, "Solving the Mystery of the Missing Neutrinos"
- www.nobelprize.org/nobel-prizes/physics/articles/bahcall
- Hitoshi Murayama, "The origin of neutrino mass"
- hitoshi.berkeley.edu/neutrinos/PhysicsWorld.pdf
12. Grand Unification

The Standard Model successfully accounts for known electroweak phenomena, but together with the strong interactions the $\text{SU}(3) \times \text{SU}(2)_L \times U(1)_Y$ structure is only marginally a "unified" theory. A much more appealing unification would be obtained if the gauge group were a simple group. Then there would be only a single gauge coupling and one could hope to obtain the Standard Model's $g_3, g_2$, and $g_1$ couplings from some dynamical mechanism after a high energy symmetry breaking down to $\text{SU}(3) \times \text{SU}(2)_L \times U(1)_Y$.

The simplest group into which $\text{SU}(3) \times \text{SU}(2)_L \times U(1)_Y$ can be embedded is $\text{SU}(5)$. This is the basis for the Georgi - Glashow model. Consider the $\text{SU}(5)$ fundamental (defining) representation; schematically the hermitean generators are:

$$\left(\begin{array}{c|c}
A & B \\
\hline
C & D \\
\end{array}\right)_{AB} = \left(\begin{array}{cccc}
-\frac{1}{2} & 1 & 1 & 0 \\
1 & -\frac{1}{2} & 0 & 1 \\
0 & 0 & -\frac{1}{2} & 1 \\
0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 1 & -\frac{1}{2} \\
\end{array}\right)$$

Identify the upper left "a" block as the $\text{SU}(3)$ subgroup generators; the lower right "c" block as the $\text{SU}(2)_L$ generators. The $\text{SU}(5)$ generators must be traceless; there is then a diagonal traceless generator of $\text{SU}(5)$ that commutes with both the $\text{SU}(3)$ and $\text{SU}(2)_L$ subgroups:

$$Y = \text{diag} \left( -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2} \right).$$

(The normalization of $Y$ is chosen to give the correct hypercharge assignments for quarks and leptons.)

It is striking that the known leptons and quarks can be assembled into representations of $\text{SU}(5)$. With the above identification of the $\text{SU}(3) \times \text{SU}(2)_L \times U(1)_Y$ subgroup, the fundamental $\text{SU}(5)$ 5 representation decomposes as

$$5 \to (3, 1) \oplus (1, 2) \oplus \text{triv}$$

under $\text{SU}(3) \times \text{SU}(2)_L$. Label the 5 rep. components as $\left( f_1, f_2, f_3, h_1, h_2 \right)$. Write the tracelessness of the $\text{SU}(5)$ generators requiring that
the diagonal $U(1)_Y$ generator in this representation satisfy

$$3y_f + 2y_R = 0.$$  

Thus, choosing $y_f = -\frac{1}{3}$ so as to agree with the hypercharge assignment for known SM fermions transforming under $SU(3)_c \times SU(2)_L$ as $(3, 1)$, i.e. for the $d_R^m$ quarks, one finds $y_R = +\frac{1}{2}$ as given above. Which particles can these be? The whole internal symmetry structure needs to be in a direct product with the $SU(2)_c$ Lorentz symmetry, so the $(h_1, h_2)$ doublet must also correspond to right-handed spinors. Looking at the available SM species, we recall that the $L^c_R$ leptons transform as $(1, 2)^{-\frac{1}{2}}$, so their charge conjugates $(L^c_R)^c$ are right-handed $(1, 2)^{+\frac{1}{2}}$. However, in $SU(2)_c$ one can lower an index with $\epsilon_{ab}$: $(L^m_{1c})^c = \epsilon_{ab} (L^R_m)^b$. So the fundamental $SU(5)$ representation is $(d_R^m, L^m_c)$.

What about the right-handed electron singlet and the $u_r^m$ and $d_r^m$ quarks? Consider $Q^{mR}$ transforming in the SM representation $(3, 2)^{-\frac{1}{2}}$. Since this has both $SU(3)_c$ and $SU(2)_L$ indices, it cannot come from a single-index $SU(5)$ fundamental rep. Try instead a $SU(5)$ tensor-product representation $\chi_{[48]}$. This is a $5 \times 5 = 10$ dimensional complex representation of $SU(5)$, and it does the trick:

$$|A> = (\hat{a}, a); \hat{a} = 1, 2, 3 \quad a = 1, 2$$

$$\chi_{ABm} = \left( \begin{array}{cc} \epsilon_{abc} (\lambda_{Rm})^b - \sigma_{abc} \lambda_{Rm} & \psi_{Rm} \\ \sigma_{abc} \lambda_{Rm} & \epsilon_{abc} (\lambda_{Rm})^b \end{array} \right)$$

where every component is $SU(2)_c$ left-handed.

Let's see how the hypercharge assignments work out for this representation. For $M^A_B \in SU(5)$, $\chi_{AB}$ transforms as

$$\chi_{AB} \rightarrow M^A_B \chi_{EF},$$

so for a transformation $M^A_B = (e^{i\theta})^A_B$ with $\chi$ obeying $\left( -\frac{1}{2} - \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$, the hypercharges corresponding to the $A$ and $B$ indices on $\chi_{AB}$ simply add, so for the components of $\chi_{AB}$ we get hypercharge assignments

$$\left( \begin{array}{c} -\frac{1}{2} - \frac{1}{2} = -\frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2} = 0 \\ +\frac{1}{2} + \frac{1}{2} = 1 \end{array} \right)$$

which are correct for the above SM fermions remembering that $(\lambda_{Rm})^c$ has $y = -\frac{1}{3}$ and $(\lambda_{Rm})^c$ has $y = +\frac{1}{3}$.
Thus, taken together with the fundamental rep \( Y_{\text{fund}} = (d', \bar{r}_m a, \varepsilon a)^T \), which when charge conjugated gives a 5 representation \( (Y_{\text{mc}})^T = (d', \bar{r}_m c, l_m)^T \), which is composed of left-handed SU(2), the ordinary Standard-Model fermions fit precisely into SU(5) representations \( \overline{5} \oplus 10 \) for each \( m = 1, 2, 3 \) generation.

If one wants to generate neutrino masses by the see-saw mechanism, one then needs to include an additional SU(5) singlet for the right-handed neutrinos \( \nu_{\text{R}} \),

Then the SU(5) fermion representations are \( \overline{5} \oplus 10 \oplus 1 \).

Can one make things even more unified by picking a larger grand unification gauge group? Well, yes. Before the Georgi-Glashow SU(5) model, Howard Georgi realized that the group Spin(10) (double covering of SO(10)) has a spinor representation 16, which decomposes under \( \text{SU}(5) \times \text{U}(1) \) as \( 16 \rightarrow 10 \oplus \overline{5} \oplus 15 \), giving the right SU(5) representations, including all the left-handed SM fermions of a given generation \( (m = 1, 2, 3) \) into a single irreducible representation of Spin(10). The Spin(10) adjoint \( \rightarrow \) adjoint of SO(10) decomposes as \( 46 \rightarrow 24 \oplus 10 \oplus 10^* \).

Before one gets too excited about all of this, one must recognize the potential downside to all this unification dreaming. Unification is not cost-free: in positing the existence of more and more gauge fields for these larger groups, one must take care that they don't produce unwanted physical effects.

Consider the gauge fields of the SU(5) theory. The SU(5) adjoint decomposes under \( \text{SU}(3) \times \text{SU}(2)_L \times \text{U}(1)_Y \) as \( 46 \rightarrow (3, 1, 1) \oplus (1, 3, 1) \oplus (1, 1, 3) \oplus (3, 2)_{-5/3} \oplus (3, 2, 1)_{5/3} \).

The art of making unified models is to arrange for further symmetry breaking to make the extra non-SM fields massive.
In order to break the SU(5) gauge symmetry down to SU(3) x SU(2) x U(1) in the first instance (with SU(2) x U(1) subsequently breaking to U_{em}(1)), one needs to incorporate further Higgs–type fields. As in the Standard Model itself, the resulting massive X vector fields will mediate dimension 6 processes, with an effective coupling $g_6^2 / M^2$ where $M$ is the mass of the X vector field.

Now comes the dangerous part of this game: the X̅μνμμ̅abb vector bosons carry both SU(3) colours and SU(2) indices. So, there can be interactions like

$$X_{\mu a} X_{\nu a} X_{\rho \mu b} X_{\sigma \nu b} e^{\mu \nu} \epsilon_{a b} \left( y : \frac{1}{3} - \frac{2}{3} + \frac{1}{2} = 0 \right)$$

and

$$X_{\mu a} d_{\mu a} X_{\nu \mu b} \bar{L}_{\nu b} \left( y : \frac{1}{6} - \frac{1}{3} - \frac{1}{3} = 0 \right)$$

Consequently, one can have diagrams like

Which give rise to processes like $W^+ \rightarrow d_c e^+$ ($\gamma : \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$), which can give rise to proton decay: $p^{0} \rightarrow u d e$ ($\pi^{0}$ decays electromagnetically to gamma rays, $\pi^{0} \rightarrow 2 \gamma$).

The problem with the Georgi–Glashow model was thus proton decay. Detailed analysis shows that the proton lifetime in the SU(5) model cannot be more than $10^{31}$ years. However, proton decay observation experiments at the Super Kamiokande Lab in Japan (1728 kilotons of pure water) shows the proton lifetime to have a lower limit of $10^{34}$ years. So the SU(5) Georgi–Glashow model disappointingly fails because it predicts too short a proton lifetime.

Another way to view the chances for SU(5) grand unification is from the bottom up. In order to try to push the proton lifetime as high as possible,
Keeping the $g_s$ coupling constant in a reasonable range for perturbation theory, $g_s \approx 0.1$, the mass of the $X$ vector boson must be very large, $M_X \gg 10^{15}$ GeV.

Instead of assuming a unified model based on some gauge group corresponding to the $10^{15}$ GeV scale unification, consider just the Standard Model but evolved in energy scale from the $100$ GeV electroweak scale up to $10^{15}$ GeV. As a consequence of renormalization, gauge coupling constants effectively change with energy scale $\mu$:

$$
\frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu_0)} - b_1 \ln \left( \frac{\mu^2}{\mu_0^2} \right)
$$

This is in fact the main reason why one can even attempt to make a grand unified model. At energy scales larger than the grand unification scale $\approx 10^{15}$ GeV, the grand unification gauge group would be effectively unbroken and there would be a single overall coupling constant, e.g. $g_5$. At energy scales below the grand unification scale, this gauge group would be broken to the Standard Model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. With the broken symmetry, the three gauge coupling constants evolve differently:

$$
\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(\mu_0)} - b_1 \ln \left( \frac{\mu^2}{\mu_0^2} \right)
$$

$$
\frac{1}{\alpha_2(\mu)} = \frac{1}{\alpha_2(\mu_0)} - b_2 \ln \left( \frac{\mu^2}{\mu_0^2} \right)
$$

$$
\frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(\mu_0)} - b_3 \ln \left( \frac{\mu^2}{\mu_0^2} \right)
$$

Because $M_X$ is so large, $g_1$, $g_2$ and $g_3$ can evolve to very different values at the electroweak scale. Conversely, in a "bottom-up" approach, one can take the known experimental values for $\alpha_1$, $\alpha_2$ and $\alpha_3$ at the electroweak scale (say $\mu_0 = M_Z = 91.2$ GeV) and see what values they run to at $10^{15}$ GeV. If there is a unified theory based upon a simple gauge group, there should be some scale $\mu$ where $\alpha_1$, $\alpha_2$ and $\alpha_3$ cross at a point.
In making this exercise, it is important to get the normalization right for the $\alpha (\mu)$ coupling as compared to $\alpha_2$ and $\alpha_3$. The upshot is that one should compare $5x^{1/3}$ to $\alpha_2$ and $\alpha_3$. Plotting the inverses of these versus $\log(\mu^2)$ one gets this picture:

![Graph showing the inverses of the couplings versus \log(\mu^2)]

Is this the end for grand unified models? No: 

Supersymmetric gauge theories introduce a "superpartner" for every species present in the Standard Model (electrons, sneutrinos, gauginos, higgsinos, ...). These change the renormalization coefficients $b_1, b_2, b_3$ in a useful way providing the scale of supersymmetry breaking, is not too much larger than the electroweak scale (current hope: $\mathcal{M}_{\text{susy}} \approx 1\,\text{TeV}$).

Here is the picture in the Minimal Supersymmetric Extension of the Standard Model (MSSM) with $\mathcal{M}_{\text{susy}} = 1\,\text{TeV}$:

![Graph showing MSSM couplings versus \log(\mu^2)]

Thus, the best hope for a truly unified theory of particle physics currently lie in the Supersymmetric models. Supersymmetric SU(5) and SO(10) models manage to have sufficiently long proton lifetimes to be consistent with current data.

Of course, that scenario would require eventually discovering superpartners in the TeV range: exactly the energies now accessible in the LHC.
The Running of Coupling Constants

At tree level in quantum field theory, one has vertices like those from matter with coupling constant $g$. At one loop, there are matter diagrams contributing to the same process (i.e., with the same external lines) like

These generally are divergent and require renormalization of $g$.

In the procedure of renormalization, one introduces a regulator, e.g., a maximum scale $\Lambda$ for the momentum integrals, and expresses the renormalized coupling in terms of an unrenormalized coupling $g_0$ and $\Lambda$, referred to a subtraction scale $\mu$. For example, in electrodynamics one has

$$ e_R^2(\mu) = e_0^2 - \frac{e_0^4}{12\pi^2} \ln \left( \frac{\Lambda^2}{\mu^2} \right) . $$

When amplitudes are rewritten in terms of $e_R$ (and other renormalized quantities like the electron mass), the $\Lambda$ dependence of the divergent diagrams cancels against the $\Lambda$ dependence introduced by $e_R$. Consequently, one may take the limit $\Lambda \to \infty$ and obtain non-divergent results. However, this procedure introduces dependence on the subtraction scale $\mu$.

Once again considering the example of electrodynamics, if one compares the values of $e^2_0(\mu)$ and $e^2(\infty)$, one finds (to one-loop order, further corrected at higher orders)

$$ \frac{1}{e^2_0(\mu)} = \frac{1}{e^2(\infty)} + \frac{1}{12\pi^2} \ln \left( \frac{\Lambda^2}{\mu^2} \right) , $$

or, defining $\alpha = e^2/4\pi$, $\frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\infty)} + 5 \ln \left( \frac{\Lambda^2}{\mu^2} \right) $,

where $\beta$ is the renormalization coefficient. Its value depends on the field species participating in the dynamics.
Returning to the running of the Standard Model
couplings, one finds
\[ b = \frac{1}{12\pi} \left[ T(R_0) + 2T(R_2) - 11T(A) \right] \]
where \( T(R) \) is the Dynkin index for a representation \( R \)
and \( R_0, R_2 \) and \( A \) refer to the representations carried by
spin-zero fields, spin-half fields, and the adjoint (for the
contributions to \( b \) from the gauge fields themselves). The
Dynkin index is defined in terms of the trace of group \( R \)
representation generators \( T^i \) by \( tr(T^iT^i) = T(R) \delta^{ij} \). In
these conventions, \( T(R) = \frac{1}{2} \) for the fundamental representation
of \( SU(N) \) and \( T(A) = N \) for the \( SU(N) \) adjoint.

When it comes to Weyl hypercharge, care needs to
be taken to normalize the generator in calculating \( b \),
consistently with the conventions for the nonabelian factor
groups \( SU(2)_L \) and \( SU(3)_c \). For them, in the fundamental
representation one has \( tr(T^iT^i) = \frac{1}{2} \delta^{ij} \). So to make the \( Y \)
normalization consistent with this, one should define \( \hat{Y} = \frac{1}{\sqrt{6}} Y \)
(for \( Y = \text{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}) \), for which \( tr(\hat{Y}^2) = \frac{3}{5}\cdot \frac{5}{6} = \frac{1}{6} \).

Consequently, the three couplings that should evolve
to come together at some scale \( \mu \) are \( g_3, g_2 \) and \( \tilde{g}_1 = \frac{1}{\sqrt{3}} g_1 \)
(so that \( g_1 Y = \tilde{g}_1 \hat{Y} \)). Or, in terms of \( \alpha \)'s, one looks
for a convergence at the same value for \( \alpha_3, \alpha_2 \) and \( \frac{5\alpha_1}{3} \).