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NONLINEAR REALIZATIONS OF INTERNAL AND SPACE-TIME SYMMETRIES[1]

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[1] Original page numbering (117-141) has been modified.

Introduction

In field theory the symmetries are realized as field transformations under which the theory has to be invariant. The good old fashioned symmetries are represented by linear and homogeneous field transformations

$$\psi'(x) = S^{-1}(\Lambda)\psi(\Lambda x + a) \qquad \text{(The Poincaré group)}$$

$$\psi'(x) = e^{ie\alpha}\psi(x); \quad A'_\mu = A_\mu \qquad \text{(Gauge invarince of the first kind)}$$

$$\psi'(x) = e^{i\vec{\theta}\,\vec{t}}\psi(x) \qquad \text{(Charge independence of strong interactions)}$$

etc. $\cdots$

These symmetries lead to conservation laws of energy-momentum, of total angular momentum, of charge, of isotopic spin, etc. ..., and all fields and state vectors have to be classified according to representations of the corresponding groups: particles of spin $0,\ 1/2,\ 1\ \cdots$ for the Poincaré group, of isotopic spin $0,\ 1/2,\ 1$ for the isotopic symmetry, etc. They give restrictions on the S-matrix elements between states with fixed number of particles. We shall call these symmetries the algebraic ones.

But in field theory symmetries of a different nature exist also, which are realized nonhomogeneously and even nonlinearly in fields. For example, gauge invariance of the second kind

$$\psi'(x) = e^{ie\alpha}\psi(x); \qquad A'_\mu = A_\mu + \partial_\mu\alpha$$

The chiral symmetry transformations, e.g., the chiral transformations of π-mesons in $SU(2)\times SU(2)$

$$\delta\vec{\pi} = \frac{F_\pi}{2}\vec{\epsilon}\,(1-\vec{\pi}^2) + F_\pi^{-1}(\vec{\pi}\vec{\epsilon}\,)\vec{\pi}$$

the invariance under the group of general transformations of coordinates in general relativity and so on.

These groups of transformations do not give any new conservation laws beside these implied by their algebraic subgroups. They do not lend to classification of the particle states. Instead these symmetry groups have a strong influence on the possible form of dynamics; the low energy theorems for photons and pions in the cases of gauge invariance and chiral $SU(2)\times SU(2)$ correspondingly, the Adler-Weisberger relation, the equivalence principle in general relativity and so on. Because of this such symmetries are called the dynamical. The concepts of algebraic and dynamical symmetries are brilliantly discussed in Weinberg's Brandeis lectures [1] to which I refer.

I shall discuss in these lectures the method of constructing the nonlinear realizations for Internal and space-time symmetry groups with given, algebraic subgroups e.g. the isotopic group, the Lorentz group, etc. This method uses extensively the Cartan differential forms and its development in field theory is due to Coleman, Wess, Zumino [2] and especially to D.V. Volkov [3, 4].

To illustrate the nonlinear realizations of space-time symmetries I shall discuss the Klein affine group, with the Lorentz group as its algebraic subgroup and I will show the importance of this group. In combination with the conformal group, this group leads to Einstein's theory of general relativity.

Nonlinear realizations. Internal symmetries

It would be instructive to begin with the simplest case, the case of internal symmetry. Let G be the dynamic symmetry group, U being its algebraic subgroup. We denote the generators of U by $V_i (i = 1, \cdots, k)$ and the remaining generators of G by $Z_a (a = k+1, \cdots, n)$. The commutation relations are

$$[V_i, V_j] = c_{ijk} V_k \tag{1.a}$$
$$[V_i, Z_a] = c_{iab} Z_b \tag{1.b}$$
$$[Z_a, Z_b] = c_{abc} Z_c + c_{abi} V_i \tag{1.c}$$

where c are structure constants of the group G.

If second term in (1.c) is absent the algebra of G has an automorphism $V \to V, Z \to -Z$. Note that this case corresponds to the symmetric group space. A general group element of G can be represented in some vicinity of unity in the form

$$g = e^{i\alpha_a Z_a} e^{iu_i V_i} \in G \tag{2}$$

In order to obtain the nonlinear realization of G/U we have to introduce the fields $\xi_a(x)$ corresponding to generators Z_a and having the same transformation properties under U as Z_a's. These fields $\xi_a(x)$ can be called preferred or Goldstone fields.

To define the action of the group G on $\xi_a(x)$ we form $e^{i\xi_a Z_a}$. Note that this expression is an element of G. Then

$$g e^{i\xi_a Z_a} = e^{i\xi'_a(\xi,g) Z_a} e^{iu_i(\xi,g) V_i} \tag{3}$$

In fact we are considering the coset space G/U. Now we define

$$g: \quad \xi_a \to \xi'_a = \xi'_a(\xi, g) \tag{4}$$

This is a group transformation law:

$$\begin{aligned} g_1 g_2 e^{i\xi Z} = g_1 e^{i\xi'(\xi,g_2) Z} e^{iu'(\xi,g_2)V} =& e^{i\xi''(\xi'(\xi,g_2),g_1)Z} e^{iu''(\xi'(\xi,g_2),g_1)V} e^{iu'(\xi,g_2)V} = \\ =& e^{i\xi''(\xi'(\xi,g_2),g_1)Z} e^{iu(\xi,g_1 g_2)V} \end{aligned} \tag{5}$$

because the generators V belong to subgroup U. So we have

$$\xi'(\xi, g_1 g_2) = \xi''(\xi'(\xi, g_2), g_1) \tag{6}$$

and this proves the group property of the transformation law (4). This transformation law is nonlinear.

For all other fields one can define the transformation law as follows. Let ψ be the arbitrary field belonging to some linear representation D of the algebraic subgroup U. Then for any element $g \in G$ we define

$$g: \quad \psi \to \psi' = D\left(e^{iu(\xi,g)V}\right)\psi \tag{7}$$

However D is a representation of U and we have eq. (6). This proves that eq. (7) defines the group transformation law, nonlinear in Goldstone fields $\xi(x)$. If an element G belongs to the subgroup U, the (4) and (7) become linear representations of U. In fact, if $g = e^{iuV}$ (u are constant parameters), then

$$e^{iuV}e^{i\xi Z} = e^{iuV}e^{i\xi Z}e^{-iuV}e^{iuV} = \exp\left[i\xi_a\left(e^{iuV}Z_a e^{-iuV}\right)\right]e^{iuV} = e^{i\xi_a\varphi_{ab}Z_b}e^{iuV}$$

because due to eq. (1.b) the generators Z form some representation of the group $U : e^{iuV}Z_a e^{-iuV} = \varphi_{ab}Z_b$. So for $g = e^{iuV}$ the transformation laws (4) and (7) are reduced to linear ones independent of Goldstone fields:

$$\xi'_a = \varphi_{ab}\xi_b, \qquad \psi' = D\left(e^{iuV}\right)\psi. \tag{8}$$

Note that in the case when there exists the automorphism R: $V \to V, Z \to -Z$ one can easily find the linearly transformed quantity. Taking the inverse of (3), applying the automorphism and making product with (3) we obtain the linear transformation law for $e^{2i\xi Z}$:

$$ge^{2i\xi Z}R\left(g^{-1}\right) = e^{2i\xi' Z} \tag{9}$$

Duo to the dependence of transformation laws on the Goldstone fields one has to have some special prescription for obtaining the covariant derivatives of fields, i.e. such quantities, which transform under G as representations of the algebraic subgroup U with ξ-depending parameters. The Cartan differential forms provide us a very powerful tool. Let us consider $e^{-i\xi Z}de^{i\xi Z}$, where the differential d acts on $\xi(x)$. Decomposing this expression into the complete system of generators V and Z we have ($\partial\omega_a^Z$ and $\partial\omega_i^V$ are the Cartan differential forms.)

$$e^{-i\xi(x)Z}de^{i\xi(x)Z} = iZ_a\partial\omega_a^Z + iV_i\partial\omega_i^V \tag{10}$$

For the transformed fields we obtain using (3)

$$\begin{aligned} e^{-i\xi' Z}de^{i\xi' Z} =& e^{iu(\xi,g)V}e^{-i\xi Z}g^{-1}\ d\ ge^{i\xi Z}e^{-iu(\xi,g)V} = \\ =& e^{iu(\xi,g)V}\left(e^{-i\xi Z}de^{i\xi Z}\right)e^{-iu(\xi,g)V} + e^{iu(\xi,g)V}de^{-iu(\xi,g)V} \end{aligned}$$

because the element g is independent of ξ. For the Cartan differential forms it follows that

$$Z\partial\omega^{Z\prime} = e^{iu(\xi,g)V}Z\partial\omega^Z e^{-iu(\xi,g)V}$$

$$\partial\omega_a^{Z\prime} = \varphi_{ab}(\xi,g)\partial\omega_b^Z = \left(D^Z\left(e^{iu(\xi,g)V}\right)\partial\omega^Z\right)_a \tag{11}$$

i.e. the form $\partial\omega_a^Z$ is transformed according to a linear representation of the algebraic subgroup U with ξ-depending parameters, and

$$V\partial\omega^{V\prime} = e^{iu(\xi,g)V}V\partial\omega^V e^{-iu(\xi,g)V} - ie^{iu(\xi,g)V}de^{-iu(\xi,g)V} \tag{12}$$

Now we can define the covariant derivative of the Goldstone field $\xi_a(x)$ in the form

$$\nabla_\mu\xi = \frac{\partial\omega_a^Z(\xi)}{\partial x_\mu}; \qquad \nabla_\mu\xi' = D^Z\left(e^{iu(\xi,g)V}\right)\nabla_\mu\xi \tag{13}$$

For an arbitrary field ψ transforming under the algebraic subgroup U according to the representation $D(U)$ we form the covariant differential

$$\Delta\psi = \partial\psi + iV_j \partial\omega_j^V \psi \tag{14}$$

where V are the generators of U in the representation $D(U)$ appropriate to the field ψ. Using eq. (12) one obtains

$$\Delta\psi' = D\left(e^{iu(\xi,g)V}\right)\Delta\psi \tag{15}$$

and one can define the covariant derivative of ψ in the form

$$\nabla_\mu\psi = \frac{\partial\psi}{\partial x_\mu} + iV_j \frac{\partial\omega_j^V}{\partial x_\mu}\psi \tag{16}$$

The covariant derivatives $\nabla_\mu\psi$ transform under G in the same way as ψ itself does eq. (8);

$$\nabla_\mu\psi' = D\left(e^{iu(\xi,g)V}\right)\nabla_\mu\psi \tag{17}$$

Since $\psi, \nabla_\mu\psi$ and $\nabla_\mu\xi$ all have the transformation rules (8), (17) and (13) any Lagrangian constructed from just these quantities will be invariant under the full group G if it is invariant under the algebraic subgroup U.

The Goldstone fields $\xi(x)$ have to enter a Lagrangian only through their covariant derivatives $\nabla_\mu\xi$. At the same time the vacuum state and particle states are invariant only under the algebraic subgroup U, so the full symmetry G is spontaneously broken.

We note however, that with additional requirements on the behaviour of the tree diagrams at high energies, consistent with the Regge behaviour, some algebraic properties of the full group G reappear [5–7].

Namely the one-particle states must form irreducible or reducible linear representations of the full group G. The mass-matrix m^2 must belong to definite representations of G for all cases investigated; $SU(2)\times SU(2)/SU(2)$ [5], $SU(3)/SU(2)\times Y$ [6], $SU(3)\times SU(3)/SU(2)\times Y$. The results are very good, they give the Gell-Mann-Okubo mass-formula, Gell-Mann-Oakes-Renner formula and so on.

Nonlinear realizations. Combined space-time and internal symmetries

Now we turn to the combined space-time and internal symmetries. We shall use the coordinates $x_i(i = 1, \cdots, 4)$ where $x_4 = ict$. In this case the group G under consideration must contain the translations represented by the generators P_μ. Let the full set of generators of the group G consist of P_μ, the Lorentz group generators $L_{\mu\nu}$ the generators of some good internal symmetry group V_i (say, isotopic one) and let us denote the remaining generators of G by Z_a. The subscript a may be some combination of the Lorentz indices μ, ν and the internal symmetry indices. The Lorentz group is a good one so it is necessarily contained in the algebraic subgroup U, which can contain also come good internal symmetry, represented by the generators V.

We have to find the nonlinear representation of some group G which become linear on its algebraic subgroup U. To this end one can represent the action of G in a form analogous to eq. (3)

$$g e^{ixP} e^{i\xi(x)Z} = e^{ix'P} e^{i\xi'(x')Z} u(\xi(x), g) \tag{18}$$

$\xi(x)$ are Goldstone fields whose the transformation properties under the Lorentz and a good Internal symmetry group are determined by the corresponding transformation properties of the generators Z.

$$u(\xi(x), g) = e^{iu_i(\xi(x),g)V_i} e^{iu_{\mu\nu}(\xi(x),g)L_{\mu\nu}}$$

is an algebraic subgroup element with ξ-depending parameters. Let us emphasize that in the right hand side of eq. (18) in all places stands just the transformed coordinate x' and not x. It is easy to see quite analogously to the case of internal symmetry considered above that if G belongs to the algebraic subgroup U, then the translation of ξ become linear, e.g. the Lorentz transformation, under which $x' = \Lambda x$, $\xi'(x') = S^{-1}(\Lambda)\xi(x)$. Also for a translation, due to the special place occupied by e^{ixP} in eq. (18) we have for e^{iPa}; $x' = x + a$; $\xi'(x') = \xi(x)$. For the general group element g eq. (18) determines the nonlinear transformation law for $\xi_\alpha(x)$:

$$\xi'_a(x') = \xi'_a(\xi(x), g) \tag{19}$$

(Let us forget that the argument of $\xi'(x')$ is x' not x!) Let ψ be an arbitrary field, which is transformed under the algebraic subgroup according to representation $D(u)$ with the generators $L_{\mu\nu}$ and V_i appropriate to ψ. The transformation law for $\psi(x)$ under the group G is given by

$$\psi'(x') = D\left(u\left(\xi(x), g\right)\right)\psi(x) \tag{20}$$

So $\psi(x)$ transforms according to its representation D but with parameters depending on the Goldstone fields $\xi(x)$. This dependence disappears if g belongs to the algebraic subgroup U. The proof of the group property of the transformation laws (19) and (20) is quite analogous to that for the internal symmetry case (see above) and we omit it.

In order to obtain Cartan differential forms we consider now

$$e^{-i\xi(x)Z} e^{-ixP}\, d\, e^{ixP} e^{i\xi(x)Z} = iP_\mu\, \partial\omega^P_\mu(x) + iZ_a\, \partial\omega^Z_a(x) + iV_i\, \partial\omega^V_i(x) + iL_{\mu\nu}\, \partial\omega^L_{\mu\nu}(x) \tag{21}$$

Now just as above (see eq. (11)) we obtain the transformation laws for these forms

$$P_\mu\, \partial\omega^{P\prime}_\mu(x') = u\left(\xi, g\right) P_\mu\, \partial\omega^P_\mu(x) u^{-1}\left(\xi, g\right) \tag{22.a}$$

$$Z_a\, \partial\omega^{Z\prime}_a(x') = u\left(\xi, g\right) Z_a\, \partial\omega^Z_a(x) u^{-1}\left(\xi, g\right) \tag{22.b}$$

$$\begin{aligned} V_i\, \partial\omega^{V\prime}_i(x') + L_{\mu\nu}\, \partial\omega^{L\,\prime}_{\mu\nu}(x') =& u\left(\xi, g\right)\Big(V_i\, \partial\omega^V_i(x) + L_{\mu\nu}\, \partial\omega^L_{\mu\nu}(x)\Big) u^{-1}\left(\xi, g\right) - \\ &- iu\left(\xi, g\right) du^{-1}\left(\xi, g\right). \end{aligned} \tag{22.c}$$

We see that the forms $\partial\omega_\mu^P(x)$ and $\partial\omega_a^Z(x)$ are transformed under G according to the representations D^P and D^Z of the algebraic subgroup U with parameters depending on the Goldstone fields $\xi_\alpha(x)$. E.g. $\partial\omega_\lambda^P(x)$ transforms as a Lorentz vector

$$\partial\omega_\lambda^{P\prime}(x') = \left(e^{iu_{\mu\lambda}(\xi,g)L_{\mu\nu}}\right)_{\lambda\sigma}\partial\omega_\sigma^P(x) \tag{23}$$

where $L_{\mu\nu}$ -are matrices of the vector representation of the Lorentz group (because P_μ is a Lorentz vector and a scalar under any internal symmetry). Now we can construct out of these forms the covariant derivatives. The appropriate term is $\frac{\partial}{\partial\omega_\mu^P(x)}$. (recall that $\frac{\partial}{\partial x_\mu}$ has bad transformation properties).

The covariant derivatives of the Goldstone fields are defined as

$$\nabla_\mu\xi_a(x) = \frac{\partial\omega_a^Z(x)}{\partial\omega_\mu^P(x)} \tag{24}$$

The covariant differential $\Delta\psi$ of any other field ψ transforming according to a representation D of the algebraic subgroup U is defined as

$$\Delta\psi(x) = \partial\psi(x) + iV\partial\omega^V\psi(x) + iL\partial\omega^L\psi(x) \tag{25}$$

where V_i and $L_{\mu\nu}$ are the internal symmetry and Lorentz matrices appropriate to representation D. Then

$$\Delta\psi'(x') = D(u)\Delta\psi(x) \tag{26}$$

Therefore the covariant derivative of ψ is

$$\nabla_\mu\psi(x) = \frac{\partial\psi(x)}{\partial\omega_\mu^P(x)} + iV\frac{\partial\omega^V(x)}{\partial\omega_\mu^P(x)}\psi(x) + iL\frac{\partial\omega^L(x)}{\partial\omega_\mu^P(x)}\psi(x) \tag{27}$$

and it transforms under G according to some reducible representation of the algebraic subgroup U with ξ -depending parameters. Now we can repeat the statement made above (under eq. (17)):

Since $\psi, \nabla_\mu\psi$ and $\nabla_\mu\xi$ transform according to representations of the algebraic subgroup U, any Lagrangian L will be invariant with regard to the full group G, if it is invariant under the algebraic subgroup U.

However, the volume element $-idx_1dx_2dx_3dx_4$ is non adequate. To obtain the correct volume element we have to construct it out of the differential forms $\partial\omega_\mu^P(x)$ in an invariant manner. Note that we can write

$$\partial\omega_\mu^P(x) = \omega_{\mu\nu}dx_\nu \tag{28}$$

Now the invariant volume ia given by the external product

$$dV = -i\partial\omega_1^P \wedge \partial\omega_2^P \wedge \partial\omega_3^P \wedge \partial\omega_4^P = -i\ \mathrm{Det}(\omega)dx_1dx_2dx_3dx_4 \tag{29}$$

The action principle is $\delta\int \mathcal{L}dV = 0$. As in the case of internal symmetries below, all these nonlinear realizations G/U correspond to spontaneously broken symmetries:

the Lagrangian and the equations of motion are invariant with respect to these nonlinear transformations but the vacuum and particle states are invariant only with regard to the algebraic subgroup.

Note that nonlinear realizations of conformal symmetries were considered in several papers, e.g. by Isham, Salam, Strathdee [8, 9] and Volkov [4]. Volkov [4] described also a general theory, of nonlinear realizations with the use of Cartan differential forms. In a remarkable paper [10] he constructed a nonlinear realization of space-time algebra which includes the spinor anticommuting generators.

Let us point that all formulas above determine the transformation laws and the Lagrangian up to a change of field variables.

Examples. Affine group

Let us consider the very interesting case of the affine group of Klein. The affine group is the group of transformations under which all straight lines transform into straight lines. These transformations are general linear transformations of space-time:

$$x'_\mu = a_{\mu\nu}x_\nu + e_\mu. \tag{30}$$

The corresponding group G is the semi-direct product

$$G \sim P_4 \otimes GL(4,R)$$

Now we shall consider nonlinear realizations of this group which become linear on an algebraic subgroup of it - the Lorentz group, so we deal with the symmetry G/L. Here I follow the paper of A. Borisov and myself. The full set of generators of G is: four translation generators P_μ, six Lorentz generators $L_{\mu\nu}$ and ten symmetric generators $R_{\mu\nu}$ (the trace of $R_{\mu\nu}$ gives dilatation). The commutation relations are:

$$\begin{aligned}
&\frac{1}{i}\left[L_{\mu\nu}, L_{\rho\sigma}\right] = \delta_{\mu\rho}L_{\nu\sigma} + \delta_{\nu\sigma}L_{\mu\rho} - (\mu \leftrightarrow \nu)\\
&\frac{1}{i}\left[L_{\mu\nu}, R_{\rho\sigma}\right] = \delta_{\mu\rho}R_{\nu\sigma} + \delta_{\nu\sigma}R_{\mu\rho} - (\mu \leftrightarrow \nu)\\
&\frac{1}{i}\left[R_{\mu\nu}, R_{\rho\sigma}\right] = \delta_{\mu\rho}L_{\nu\sigma} + \delta_{\nu\sigma}L_{\mu\rho} + (\mu \leftrightarrow \nu)\\
&\frac{1}{i}\left[L_{\mu\nu}, P_{\sigma}\right] = \delta_{\mu\sigma}P_{\nu} - (\mu \leftrightarrow \nu)\\
&\frac{1}{i}\left[R_{\mu\nu}, P_{\sigma}\right] = \delta_{\mu\sigma}P_{\nu} + (\mu \leftrightarrow \nu)
\end{aligned} \tag{31}$$

In this case the Z, generators are $R_{\mu\nu}$ and therefore the Goldstone field is a symmetric tensor field $h_{\mu\nu}$. The action of the group G is defined according to the general prescription (18) which reads in one case as

$$ge^{ixP}e^{ih(x)R} = e^{ix'P}e^{ih'(x')R}e^{iu_{\mu\nu}(h,g)L_{\mu\nu}} \tag{32}$$

Then one gets for infinitesimal special-linear transformations with parameters $\alpha_{\mu\nu}$ (i.e. $g = \exp\left(i\alpha_{\mu\nu}R_{\mu\nu}\right)$)

$$\delta h_{\mu\nu} = h'_{\mu\nu}(x') - h_{\mu\nu}(x) = \sum_{m,n} b_{mn}\Big(h^m \alpha h^n\Big)_{\mu\nu} \tag{33}$$

$$u_{\mu\nu} = \sum_{m,n} c_{mn}\Big(h^m \alpha h^n\Big)_{\mu\nu} \tag{34}$$

where $\Big(h^m \alpha h^n\Big)_{\mu\nu}$ is:

$$h_{\mu\sigma_1}h_{\sigma_1\sigma_2}\cdots h_{\sigma_{m-1}\sigma_m}\alpha_{\sigma_m\rho_1}h_{\rho_1\rho_2}\cdots h_{\rho_n\nu}$$

and coefficients b_{mn} and c_{mn} are given by generating functions:

$$\sigma_1(x,y) = \sum_{m,n} b_{mn}x^m y^n = 2(x-y)\ \mathrm{cth}\ 2(x-y) \tag{33'}$$

$$\sigma_2(x,y) = \sum_{m,n} c_{mn}x^m y^n = -\mathrm{th}\ (x-y) \tag{34'}$$

For calculations we can use the vector representation in which

$$\Big(R_{\mu\nu}\Big)_{\lambda\sigma} = i\Big(\delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\lambda}\Big)$$

Then

$$\Big(e^{ih_{\alpha\beta}R_{\alpha\beta}}\Big)_{\mu\nu} = \Big(e^{-2h}\Big)_{\mu\nu} = \delta_{\mu\nu} - 2h_{\mu\nu} + \frac{4h_{\mu\sigma}h_{\sigma\nu}}{2!} - \cdots \tag{35}$$

The automorphism $L \to L,\ R \to -R$ gives (in matrix representation) the relation for special linear transformations

$$e^{-2\alpha}e^{-4h(x)}e^{-2\alpha} = e^{-4h'(x')} \tag{36}$$

which means that $\Big(e^{-4h(x)}\Big)_{\mu\nu}$ is transformed linearly. Analogously the second linearly transformed quantity is $\Big(e^{4h(x)}\Big)_{\mu\nu}$:

$$e^{2\alpha}e^{4h(x)}e^{2\alpha} = e^{4h'(x')} \tag{36'}$$

Note also the useful matrix relations, for infinitesimal transformations which follow from (32)

$$\left\{u, e^{\pm 2h}\right\} = \pm\left[\alpha, e^{\pm 2h}\right]$$
$$\delta e^{\pm 2h} = \pm\left\{\alpha, e^{\pm 2h}\right\} + \left[u, e^{\pm 2h}\right] \tag{37}$$

Now we can write the general infinitesimal transformation law for any field $\psi(x)$ under $P_4 \otimes GL(4,R)$

$$\delta\psi(x) = \psi'(x') - \psi(x) = iu_{\mu\nu}(h)L_{\mu\nu}\psi(x) \tag{38}$$

where $u_{\mu\nu}$ is given by eq. (34) and $L_{\mu\nu}$ are matrices representing the Lorentz group generators on the field ψ. So for spinors $L_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$ and we have

$$\delta\psi(x) = iu_{\mu\nu}(h)\frac{1}{2}\sigma_{\mu\nu}\psi(x) \tag{39}$$

Analogously for the four-vectors a_μ:

$$\Big(L_{\alpha\beta}\Big)_{\mu\nu} = -i\Big(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}\Big)$$

and we obtain

$$\delta a_\mu(x) = a'_\mu(x') - a_\mu(x) = 2u_{\mu\nu}(h)a_\nu. \tag{39'}$$

This quantity transforms nonlinearly in the Goldstone field $h_{\mu\nu}$ under a special affine transformation being a four-vector with regard to the Lorentz transformation. However, in the case of integer spins one can define some linearly transformed quantities by means of a change of field variables. For any given a_μ one can introduce a covariant vector as

$$A_\mu = \Big(e^{2h}\Big)_{\mu\nu}a_\nu \tag{40}$$

and a contravariant vector as

$$A^\mu = \Big(e^{-2h}\Big)_{\mu\nu}a_\nu = \Big(e^{-4h}\Big)_{\mu\nu}A_\nu \tag{41}$$

These quantities transform linearly. Using equations (37) one can easily see that under special affine transformation these quantities transform according to

$$\begin{aligned} \delta A_\mu &= 2\alpha_{\mu\nu}A_\nu \\ \delta A^\mu &= -2\alpha_{\mu\nu}A^\nu \end{aligned} \tag{42}$$

Analogously one can introduce covariant and contravariant linearly transforming tensors for any representations (p,q) $(p+q = 0,1,2,\cdots)$ of the Lorentz group. We wish to stress that this cannot be done if $p+q = \frac{1}{2},\frac{3}{2},\cdots$

According to the general prescriptions given above the Cartan differential form (22.a)

$$P_\mu\ \partial\omega^P_\mu(x) = e^{-ih_{\mu\nu}R_{\mu\nu}}P_\lambda dx_\lambda e^{ih_{\mu\nu}R_{\mu\nu}} = P_\mu\Big(e^{2h}\Big)_{\mu\nu}dx_\nu$$

Therefore

$$\partial\omega^P_\mu(x) = \Big(e^{2h}\Big)_{\mu\nu}dx_\nu \tag{43}$$

and

$$\frac{\partial}{\partial\omega^P_\mu(x)} = \Big(e^{-2h}\Big)_{\mu\nu}\frac{\partial}{\partial x_\nu} \tag{44}$$

For invariant volume we obtain (29)

$$dv = \mathrm{Det}\Big(e^{2h}\Big)\ dx^4 = \sqrt{\mathrm{Det}\Big(e^{4h}\Big)}\ dx^4 \tag{45}$$

The covariant derivatives according to eqs. (24) and (27) can be written down as:

for the Goldstone fields

$$\nabla_\lambda h_{\mu\nu} = -\frac{1}{4}\Big(e^{-2h}\Big)_{\lambda\sigma}\Big\{e^{2h}, \frac{\partial}{\partial x_\sigma}e^{-2h}\Big\}_{\mu\nu}, \tag{46}$$

for an arbitrary field ψ with Lorentz generators $L_{\mu\nu}$:

$$\nabla_\lambda \psi(x) = \Big(e^{-2h}\Big)_{\lambda\sigma} \cdot \partial_\sigma \psi(x) + iL_{\mu\nu}V_{\mu\nu,\lambda}(h)\psi(x) \tag{47}$$

with

$$V_{\mu\nu,\lambda}(h) = \frac{1}{4}\Big(e^{-2h}\Big)_{\lambda\sigma}\Big\{e^{2h}, \frac{\partial}{\partial x_\sigma}e^{-2h}\Big\}_{\mu\nu} - c\Big(\nabla_\mu h_{\lambda\nu} - \nabla_\nu h_{\lambda\mu}\Big) \tag{48}$$

where c is an arbitrary constant.

The first term here is given by the general procedure above and the second is a non-minimal addition. Just as in the case of electrodynamics one can introduce $F_{\mu\nu}$ into the definition of covariant derivative $\partial_\mu - ieA_\mu - c\gamma_\mu F_{\mu\nu}$ and the last term gives a nonminimal gauge invariant electromagnetic coupling.

Now the invariant action principle can be obtained by constructing out of all fields $\psi, \nabla_\mu\psi$ and $\nabla_\lambda h_{\mu\nu}$ a Lorentz scalar and by integrating it over the invariant volume dv.

The strong resemblance of this theory to the Einstein general relativity theory is observed on each step. The Goldstone field is the symmetric tensor field. The transformation laws for covariant and contravariant vectors (42) are just the same as in general relativity (and also in all other tensor fields). Moreover the nonlinear spinor transformation law (39) coincides with that derived in gravitation theory by Polubarinov and by myself [11, 12] (up to the field variable change), if one identifies the metric tensor $g_{\mu\nu}$ with $\Big(e^{4h}\Big)_{\mu\nu}$. This identification follows also from the form of the invariant volume (43) and from the relation between covariant and contravariant vectors (41). The covariant derivative (47) would coincide with that in general relativity if the constant c in eq. (48) was equal to 1. It can be verified that with $c = 1$ the covariant derivative (47) gives rise to affine connections for linearly transformed tensors and it coincides with the covariant derivative for spinors derived in [11, 12]. The form (47) has some advantages since it is written down in a unified manner for all the representations of the Lorentz group.

The whole correspondance with the Einstein theory is achieved if one considers simultaneously the $P_4 \otimes GL(4, R)$ group and the conformal group, It is so because the following statement is correct and can be proved: <u>The closure of conformal and affine algebra is just the algebra of general coordinate transformations.</u>

<u>The Einstein gravitation theory is deduced if one considers the nonlinear realization of the affine group and the conformal group simultaneously. It is remarkable that gravitons play a twofold role - they are Goldstone particles for the Klein group and they are gauge particles according to the group of general coordinate transformations.</u>

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