EXAMINATION

MSc IN QUANTUM FIELDS AND FUNDAMENTAL FORCES

TP.7 — Supersymmetry

Wednesday, May 30$^{\text{th}}$ 2001

Answer **Three** of the following **Five** Questions

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title and the number of the question attempted.

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1. Derive the irreducible supermultiplet of states for particle of mass m in their Lorentz rest frame. Do this as follows

   a) Start from the $d = 4$, N-extended supersymmetry algebra in 4-component notation,

   $$\{Q^i, \bar{Q}^j\} = 2\delta^{ij}\gamma^m P_m \qquad i = 1 \ldots, N$$

   where $\bar{Q}^j = (Q^j)^\dagger\gamma^0$. Specialise this algebra to the subspace of mass m states in the rest frame and show that, within this subspace, the supersymmetry algebra in 2-component notation takes the form

   $$\{Q^i_\alpha, \bar{Q}_{\beta j}\} = 2m\delta_{\alpha\beta}\delta^i_j$$
   $$\{Q^i_\alpha, Q^j_\beta\} = 0 \qquad \{\bar{Q}_{\alpha i}, \bar{Q}_{\beta j}\} = 0 \qquad \alpha, \beta = 1, 2 .$$

   To do this, recall that in the Weyl representation, the γ-matrices take the form

   $$\gamma_m = \begin{pmatrix} 0 & \mathrm{i}\sigma_{m\,\alpha\dot{\beta}} \\ \mathrm{i}\bar{\sigma}_m^{\dot{\alpha}\beta} & 0 \end{pmatrix} \qquad m = 0, \ldots, 3$$

   where $\sigma_{m\,\alpha\dot{\beta}} = (\mathbf{1}, \vec{\sigma})$, $\bar{\sigma}_m^{\dot{\alpha}\beta} = (\mathbf{1}, -\vec{\sigma})$ (where $\vec{\sigma}$ are the Pauli matrices), and the translation rule between 4-component and two-component notation for a general Majorana spinor λ^i is given by

   $$\lambda^i = \begin{pmatrix} \lambda^i_\alpha \\ \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}i} \end{pmatrix} \qquad \text{where } \bar{\lambda}_{\dot{\beta}i} = (\lambda^i_\beta)^*.$$

   Include in your discussion an explanation of why one does not need to distinguish between dotted and undotted indices in considering the *rest-frame* supersymmetry algebra.

   b) Give the explicit form of the representation matrices for the Lorentz little group in the chosen Lorentz frame when acting on the Q^i_α and give their commutators with the Q^i_α.

   c) Construct normalised generators such that their algebra becomes a Clifford algebra, and then use this to derive the irreducible supermultiplet of mass m states at rest.

   d) Carry out the construction derived above explicitly for the case of the $d = 4$, $N = 2$ supersymmetry algebra (without central charges). Show in particular how the combinatorics of the construction yields a finite multiplet of states, with the index structure of the next to last level given by

   $$\tfrac{1}{8}\left[\epsilon_{\beta\gamma}\epsilon_{ij}|\lambda_{\alpha k}\rangle + \epsilon_{\gamma\alpha}\epsilon_{jk}|\lambda_{\beta i}\rangle + \epsilon_{\alpha\beta}\epsilon_{ki}|\lambda_{\gamma j}\rangle - \epsilon_{\gamma\beta}\epsilon_{ik}|\lambda_{\alpha j}\rangle - \epsilon_{\alpha\gamma}\epsilon_{ji}|\lambda_{\beta k}\rangle - \epsilon_{\beta\alpha}\epsilon_{kj}|\lambda_{\gamma i}\rangle\right]$$

   for some spinor states $|\lambda_{\alpha i}\rangle$, and the index structure of the last level given by

   $$\tfrac{1}{36}\left[\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}(\epsilon_{ik}\epsilon_{jl} + \epsilon_{jk}\epsilon_{il}) + \epsilon_{\alpha\delta}\epsilon_{\gamma\beta}(\epsilon_{ik}\epsilon_{jl} + \epsilon_{ij}\epsilon_{kl}) + \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}(\epsilon_{il}\epsilon_{jk} + \epsilon_{ij}\epsilon_{kl})\right]|\phi\rangle$$

   for some scalar state $|\phi\rangle$. Show that these combinations have the necessary symmetries and find expressions for the independent states $|\lambda_{\alpha i}\rangle$ and $|\phi\rangle$ using combinations of the Clifford creation operators with indices αi, βj, γk, $\delta \ell$, *etc.* acting on the Clifford vacuum.

   Summarise your results by giving the multiplicities for states of each spin found in the supermultiplet.

2. Consider a supersymmetry algebra in two-component index notation that is modified to make the $\{Q_\alpha, Q_\beta\}$, $\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$ anticommutators nonvanishing:

$$\{Q_\alpha, Q_\beta\} = \frac{1}{R} M_{\alpha\beta} \qquad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = \frac{1}{R} M_{\dot\alpha\dot\beta} \qquad \alpha, \beta = 1, 2,$$

where $M_{\alpha\beta}$ $(= M_{\beta\alpha})$, $M_{\dot\alpha\dot\beta}$ is the complex 2-component form of the angular momentum:

$$M_{\alpha\dot\beta\gamma\dot\delta} = \sigma^m_{\alpha\dot\beta}\sigma^n_{\gamma\dot\delta} M_{mn} = \epsilon_{\alpha\gamma} M_{\dot\beta\dot\delta} + \epsilon_{\dot\beta\dot\delta} M_{\alpha\gamma} \qquad m, n = 0, \ldots, 3.$$

The ϵ tensor satisfies $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha{}_\beta$. The other lowest-dimension commutation relation is the standard one,

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2P_{\alpha\dot\beta}.$$

For the $M_{\alpha\beta}$, one has the standard commutation relations with the Q_α, $Q_{\dot\beta}$ and the $P_{\alpha\dot\beta}$, expressing the Lorentz transformation properties of these generators:

$$\begin{aligned} [M_{\gamma\delta}, P_{\alpha\dot\beta}] &= \tfrac{1}{2}(\epsilon_{\alpha\gamma} P_{\delta\dot\beta} + \epsilon_{\alpha\delta} P_{\gamma\dot\beta}) \\ [M_{\gamma\delta}, Q_\alpha] &= \tfrac{1}{2}(\epsilon_{\alpha\gamma} Q_\delta + \epsilon_{\alpha\delta} Q_\gamma) \\ [M_{\gamma\delta}, \bar{Q}_{\dot\beta}] &= 0, \end{aligned}$$

while the commutation relations with $M_{\dot\gamma\dot\delta}$ follow from $M_{\dot\gamma\dot\delta} = (M_{\gamma\delta})^*$. Thus, a chiral Lorentz transformation of Q_α with complex parameter $\rho^{\gamma\delta} = \rho^{\delta\gamma}$ is given by

$$\delta Q_\alpha = [\rho^{\gamma\delta} M_{\gamma\delta}, Q_\alpha] = \rho_\alpha{}^\delta Q_\delta.$$

a) Use the above and the QQQ Jacobi identity to show that

$$[P_{\alpha\dot\beta}, Q_\beta] = \tfrac{1}{2}\epsilon_{\alpha\beta}[P^\gamma{}_{\dot\beta}, Q_\gamma].$$

b) Use the PQQ Jacobi identity to show that

$$\{Q_\delta, [P^\sigma{}_{\dot\beta}, Q_\sigma]\} = -\frac{1}{R} P_{\delta\dot\beta}.$$

c) Hence, consider the general covariant possibilities and show that

$$[P_{\alpha\dot\beta}, Q_\sigma] = -\frac{1}{4R}\epsilon_{\alpha\sigma}\bar{Q}_{\dot\beta}\,, \qquad [P_{\alpha\dot\beta}, \bar{Q}_{\dot\sigma}] = -\frac{1}{4R}\epsilon_{\dot\beta\dot\sigma} Q_\alpha$$

d) Finally, show that

$$[P_{\alpha\dot\beta}, P_{\gamma\dot\delta}] = -\frac{1}{8R^2}\epsilon_{\alpha\gamma} M_{\dot\delta\dot\beta} - \frac{1}{8R^2}\epsilon_{\dot\delta\dot\beta} M_{\gamma\alpha}\,.$$

(These results, taken together with the $[M, M]$ commutation relations, form the Lie algebra for the anti-De Sitter group $SO(3,2)$. This is the isometry group of anti-De Sitter space, in which the the cosmological constant Λ is given by $\Lambda = R^{-4}$.

e) Show how the above construction limits to the usual super Poincaré algebra when spacetime is required to be flat, and why this then requires the momentum and supersymmetry generators to commute in this limit.)

3. Consider the free field action in $D = 4$ for a real gauge vector field A_m and a massless Majorana spinor field ψ:

$$I = \int d^4x(-\tfrac{1}{4}F_{mn}F^{mn} - \tfrac{\mathrm{i}}{2}\bar{\psi}\gamma^m\partial_m\psi)$$
$$F_{mn} = \partial_m A_n - \partial_n A_m$$

a) Assuming that the supersymmetry transformations for this action are linear, write the most general form of the transformations for A_m and ψ that is consistent with dimensional analysis, Lorentz covariance, reality and gauge invariance, leaving a relative numerical coefficient to be determined.

b) Determine the relative coefficient left unfixed in a) by requiring invariance of the action. Find the conserved supersymmetry current.

c) Verify that, up to terms proportional to the free equation of motion for ψ and a gauge transformation for A_m, your transformations satisfy the supersymmetry algebra:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = -2\mathrm{i}\bar{\epsilon}_1\gamma^m\epsilon_2\partial_m.$$

where ϵ_1 and ϵ_2 are the anticommuting supersymmetry parameters.

Basic relations: $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$; $\gamma_{mn} = \frac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m)$; $\gamma_5^2 = -1$; $\epsilon^{mnpq}\gamma_{npq} = 6\gamma_5\gamma^m$.

Fierz identity for anticommuting α, β, γ: $\beta\bar{\alpha}\psi = -\frac{1}{4}(\bar{\alpha}\gamma_A\beta)\gamma^A\psi$, where

$$\gamma_A = \mathbf{1},\quad \gamma_5, \gamma_m, \gamma_5\gamma_m, \frac{1}{\sqrt{2}}\gamma_{mn}$$
$$\gamma^A = \mathbf{1}, -\gamma_5, \gamma^m, \gamma_5\gamma^m, -\frac{1}{\sqrt{2}}\gamma^{mn}.$$

For anticommuting Majorana α & β, the following hold:

$\bar{\alpha}\beta = \bar{\beta}\alpha \qquad \bar{\alpha}\gamma^m\beta = -\bar{\beta}\gamma^m\alpha \qquad \bar{\alpha}\gamma_5\gamma^m\beta = \bar{\beta}\gamma_5\gamma^m\alpha \qquad \bar{\alpha}\gamma_{mn}\beta = -\bar{\beta}\gamma_{mn}\alpha.$

4. The R-weight w of a chiral superfield $\phi(x, \theta_3, \bar{\theta}_3)$ expresses the phase that the whole superfield acquires under an R-transformation with parameter α:

$$R\phi(x, \theta, \bar{\theta}) = e^{iw\alpha}\phi(x, e^{i\alpha}\theta, e^{-i\bar{\theta}}) .$$

Thus, for example, a standard mass term in the superspace action for the Wess-Zumino model is R-invariant if the superfield ϕ has R-weight $+1$.

a) Consider a model with three chiral superfields ϕ_1, ϕ_2 and ϕ_3 and find the Lagrangian for the O'Raifertaigh model as the most general parity-invariant model containing linear. bilinear and trilinear terms in these superfields that is also R-invariant with weights $w_1 = 2$, $w_2 = 2$, $w_3 = 0$ and that is also invariant under the discrete transformation $\phi_1 \to \phi_1$, $\phi_2 \to -\phi_2$, $\phi_3 \to -\phi_3$. Let the coefficient of the linear term have parameter λ, let the coefficient of the bilinear term have coefficient m and let the coefficient of the trilinear term have coefficient g.

b) Write down the auxiliary field equations for the three complex auxiliary fields of the model. You may assume a normalisation of the kinetic terms such that the auxiliary fields appear in the form $\int d^4x hh^*$. Express the potential for this model in terms of the component physical fields, after elimination of the auxiliary fields. You do not have to be concerned with the overall normalisation of the potential.

c) Show that when $m^2 > 2\lambda g$, the minimum of the potential is at $\mathcal{A}_2 = \mathcal{A}_3 = 0$ and is independent of $\mathcal{A}_1$, where the $\mathcal{A}_i$ are the complex scalar fields, *i.e.* the lowest θ components of the ϕ_i. Letting $\mathcal{A}_i = A_i + iB_i$, find the masses of the A_i and B_i in this case. Explain what happens and find the values of the scalars that minimise the potential when $m^2 < 2\lambda g$.

d) Determine whether supersymmetry is spontaneously broken, independently of the relation between m^2 and $2\lambda g$.

e) Find the fermion mass matrix for the model. Again, you do not have to be concerned with the overall normalisation. Explain whether your result is consistent with what one expects for masses in the ϕ_i supermultiplets.

5. a) Describe the general superspace form of the quantum corrections $\Delta\Gamma$ to the effective action for a Wess-Zumino model with classical action $I_{\rm WZ} = \int d^4x d^4\theta \bar{\phi}\phi + \mathrm{Re}\int d^4x d^2\theta(m\phi^2 + g\phi^3)$, as derived from consideration of the superspace Feynman rules. You should make clear which aspects of $\Delta\Gamma$ are "local."

   b) Show how translation invariance in the bosonic variables x_i^μ, $i = 1 \dots N$ for an N-point Green's function occurring in $\Delta\Gamma$ leads to supersymmetry invariance of $\Delta\Gamma$.

   c) Weinberg's theorem states that the counterterms required to cancel the ultraviolet divergences in $\Delta\Gamma$ have a local structure in the bosonic coordinates. In this context, local means that the counterterms are expressed by a single $\int d^4x$ integral, with an integrand involving the fields corresponding to the external lines of a diagram, together with their derivatives out to a finite order. In momentum space, this translates to having polynomial behaviour in a single momentum.

   Show how this locality constraint, together with the structure of $\Delta\Gamma$ described in part a), gives rise to the non-renormalisation theorem for the Wess-Zumino model.

   d) Show how the WZ non-renormalisation theorem may also be derived using the background field method. Split the total chiral superfield ϕ into a background part φ and a quantum part Q: $\phi = \varphi + Q$. In the background field method, φ appears only on the external lines of the one-particle-irreducible (1PI) diagrams contributing to $\Delta\Gamma$, while the internal lines involve Q. All three superfields, ϕ, φ and Q, are subject to the chirality constraint $\bar{D}_{\dot\alpha}\phi = \bar{D}_{\dot\alpha}\varphi = \bar{D}_{\dot\alpha}Q = 0$.

      i) Solve the constraint on Q by setting $Q = \bar{D}^2X$ and show how the terms actually used in deriving the $\Delta\Gamma$ quantum corrections have a structure that leads, when combined with the locality requirements for counterterms expressed in Weinberg's theorem, to the non-renormalisation theorem of part c).

      ii) The solution $Q = \bar{D}^2X$ actually introduces a superspace "prepotential gauge symmetry" for X, *i.e.* a transformation δX that leaves Q invariant. Find this δX transformation, with a parameter superfield $\bar{\Lambda}_{\dot\alpha}$. (This prepotential gauge symmetry requires gauge fixing, similar to that in super Maxwell theory, but this does not disturb the validity of the non-renormalisation theorem.)