EXAMINATION

MSc IN QUANTUM FIELDS AND FUNDAMENTAL FORCES

TP.7 — Supersymmetry

Friday, June 3rd **2005**

14:00 — 17:00

Answer **Three** of the following **Four** Questions

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title and the number of the question attempted.

---

1. Consider a superspace for supersymmetric quantum mechanics with a Grassman even real co-ordinate t and a Grassman odd *real* co-ordinate θ. Define $Q = \partial_\theta + i\theta\partial_t$ and $D = \partial_\theta - i\theta\partial_t$.

a) Determine $\{D, D\}$, $\{Q, Q\}$ and $\{D, Q\}$.

b) A superfield $F(t, \theta)$ is either Grassman even, $[F] = 0$, or Grassman odd $[F] = 1$, and transforms under supersymmetry via $\delta F = \epsilon Q F$. It has the component expansion $F(t, \theta) = F_1(t) + \theta F_2(t)$.

i) Determine $[F_1]$ and $[F_2]$.

ii) Determine how the components F_i transform under supersymmetry. Also determine the commutator of two supersymmetry transformations, $[\delta_1, \delta_2]$, acting on both F_1 and F_2.

iii) Explain why

$$\int dt d\theta F(t, \theta)$$

is a supersymmetric invariant.

c) Consider N real superfields $X^i(t, \theta) = x^i(t) + i\theta\lambda^i(t)$, $i = 1, ..., N$, and the action

$$S = i \int dt d\theta D X^i \dot{X}^j g_{ij}(X)$$

where g_{ij} are the components of a metric. Derive the component form of the action and write it in a way that makes its covariance properties manifest.

2. The non-trivial bracket of the N extended supersymmetry algebra with vanishing central charges is given by

$$\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} = 2P_m(\sigma^m)_{\alpha\dot{\beta}}\delta^I_J$$

where $(\sigma^m)_{\alpha\dot{\alpha}} = (-1, \sigma^i)$, $(\bar{\sigma}^m)^{\dot{\alpha}\alpha} = (-1, -\sigma^i)$ and σ^i are the Pauli matrices. The super-Poincaré algebra has the additional generators M_{mn} and P_m of the Poincaré group. The commutator of the supercharge Q^I_α with the generators of the Lorentz group M^{mn} is given by

$$[M_{mn}, Q^I_\alpha] = i(\sigma_{mn})_\alpha{}^\beta Q^I_\beta$$

where $\sigma_{mn} = (1/4)(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)$, while $[Q^I_\alpha, P^m] = 0$.

Consider representations where the generators act on positive norm Hilbert spaces.

a) Show that an irreducible representation of the superPoincaré algebra contains states with equal mass.

b) Show that states in a representation have $P^0 \geq 0$ and that states can have $P^0 = 0$ if and only if they are annihilated by all supercharges.

c) Analyse the massless representations of the supersymmetry algebra. Work in a frame where $P_m = (-E, 0, 0, E)$ and use the fact that in this frame the helicity of the states are given by the eigenvalues of M_{12}.

   i) First show that Q^I_2 is realised trivially (i.e. as zero). Hence show that the problem boils down to finding representations of the algebra

   $$\{a^I, a^\dagger_J\} = \delta^I_J$$

   with $\{a^I, a^J\} = 0$, $a^\dagger_I = (a^I)^\dagger$ and where a^I lowers the helicity by 1/2 while $a^\dagger_I$ raises the helicity by 1/2.

   ii) Show that the irreducible massless multiplets have 2^{N-1} bosons and 2^{N-1} fermions.

   iii) For the N=1 case, why is it that reducible massless multiplets appear in physical theories? Construct the multiplets corresponding to a massless chiral superfield. Repeat the construction for a vector multiplet.

3. Consider $D = 4$ superspace with coordinates $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. The supercharges and superspace derivatives are given by $Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - \mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m$, $\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m$, $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m$, $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m$,

a) Prove that $\theta^\alpha\theta^\beta = -(1/2)\epsilon^{\alpha\beta}\theta\theta$ where $\theta\theta \equiv \theta^\alpha\theta_\alpha$.

b) Solve the chiral superfield constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$, showing thus that $\Phi(x, \theta, \bar{\theta})$ may be represented as

$$\Phi(y,\theta) = \varphi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \qquad y^m = x^m + \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} \ ,$$

Chiral fields transform under supersymmetry via $\delta\Phi = (\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})\Phi$. Determine the supersymmetry transformation of the component field F.

c) Consider the chiral superfield action

$$I_{\mathrm{WZ}} = \int d^4x d^2\theta d^2\bar{\theta}\Phi^{i\dagger}\Phi^i + \left(\int d^4x d^2\theta \mathcal{W}(\Phi^i) + \mathrm{h.c.}\right) \ ,$$

What is the potential for the scalar components φ^i of the chiral superfields Φ^i? What are the conditions for this theory to have a supersymmetric vacuum?

d) Consider a model with three chiral superfields X, Y and Z with superpotential $\mathcal{W} = gXYZ$. For $g \neq 0$ determine the supersymmetric vacua. Does this model have an R-symmetry?

4. a) Write down the fields in the standard model, including the Higgs sector, with their $SU(3) \times SU(2) \times U(1)_Y$ quantum numbers, using a normalisation such that $Q_{em} = I_3 + Y$.

b) Write down the fields of the minimal supersymmetric standard model (MSSM) with their $SU(3) \times SU(2) \times U(1)_Y$ quantum numbers, using the same normalisation. Make clear what kinds of supermultiplets the fields lie in. Briefly explain why supersymmetry requires the introduction of additional Higgs doublets.

c) Discuss the presence of the soft supersymmetry breaking terms of the MSSM. You should mention the hierarchy problem, supersymmetry breaking and describe the form of the soft supersymmetry breaking terms in the Lagrangian and provide one concrete example.