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A Lorentz Invariant Discrete d'Alembertian

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Abstract

We construct non-local retarded d'Alembertians, B_k on fixed causal set backgrounds that are well approximated by a Minkowski spacetimes of 2 and 4 dimensions. We show that the expectation value of the 2-d non-local d'Alembertian with respect to a Poisson distribution, $\Box_k$ reproduces the exact continuum d'Alembertian, $\Box$ in the limit $l \to 0$. We perform the same check on a modified mean of the 4-d non-local d'Alembertian. We also show that if one fixes the non-locality scale k then one must impose a large scale cutoff R_0 in order to avoid IR divergences. Constraints on the type of functions that $\Box_k$ works well for are given. In the conclusion we give a brief argument as to how these results can be generalised to other dimensions; discuss the possible use of these non-local d'Alembertians to propagate fields on causal sets; and outline a possible approach to causal set dynamics which makes use of these d'Alembertians.

Acknowledgments

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Contents

1 Introduction

1.1 Causal Sets

The causal set approach to quantum gravity [1] is based on the founding principle that spacetime is fundamentally discrete. There are a number of reasons why fundamental discreteness is appealing, starting from the infinities of general relativity and quantum field theory, which with the introduction of a short distance cutoff would seem to be cured; to the finiteness of black hole entropy where, with no short distance cutoff, the entanglement entropy of quantum fields seems to be infinite [2]. Other suggestions of a fundamental discreteness also come from other approaches to quantum gravity like loop quantum gravity [3].

The causal set programme is based on a theorem by David Malament [4] which states that if there is a bijective map between two past and future distinguishing spacetimes which preserves their chronological structure then the map is a conformal isomorphism. Assuming that we live in a future and past distinguishing Universe, we can then use the causal structure of our spacetime to determine most of its geometric properties. The conformal factor that is left undetermined can be determined from knowledge of the volume of regions in the spacetime. As we shall see later, volume has a very simple interpretation in the causal set, it being simply the number of elements of the causal set. This argument has been neatly characterised by R. Sorkin in the slogan "Order+Number=Geometry".

Given the above reasons for fundamental discreteness how should one proceed? The clue is in the theorem mentioned above, which shows the remarkable amount of information stored in the causal order of spacetime. Given that the causal order encodes most of the geometrical structure of a future/past distinguishing Lorentzian manifold $(\mathcal{M}, g)$ where g is the metric, it is possible to reformulate $(\mathcal{M}, g)$ as a partially ordered set, also known

as a *poset*. A poset is a set C together with an order relation $\prec$, i.e. a pair $(C, \prec)$, which obeys the following axioms:

1. transitivity: if $x \prec y$ and $y \prec z$ then $x \prec z$, $\forall x, y, z \in C$;
2. reflexivity: $x \prec x$, $\forall x \in C$;
3. acyclicity: if $x \prec y$ and $y \prec x$ then $x = y$, $\forall x, y \in C$.

The above mathematical structure is simply re-expressing the notion of a manifold with a causal Lorentzian metric $(\mathcal{M}, g)$ in terms of a partial ordered set, so there is nothing discrete about it. In order to implement discreteness one must add the following axiom

4. Local finiteness: $\forall x, z \in C$ the set $\{y \mid x \prec y \prec z\}$ of elements is finite.

where $\mathbf{card}X$ denotes the cardinality of a set X. This axiom ensures that there only exist a finite number of elements between any two elements in the poset; this implies that the poset is discrete. A poset that satisfies axiom (4) is called a causal set or causet. One can represent a causal set pictorially via a Hasse diagram, an example of such a diagram is illustrated in figure (1.1).

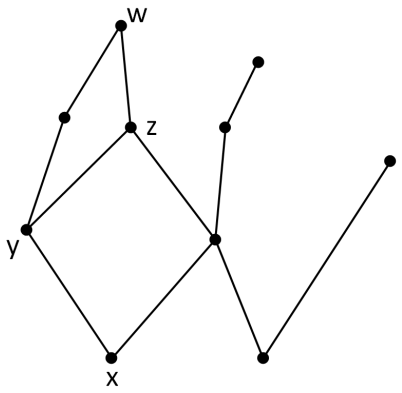

Figure 1.1: A Hasse diagram of a causal set. In such a diagram the elements of the causal set are represented by vertices, the relations between the elements by edges and the direction of the order correspond to the vertical direction. Relations implied by transitivity are not added to the diagram, for example $x \prec z$ is implied by $x \prec y$ and $y \prec z$ so the edge from x to z is omitted.

1.2 Recovering the Continuum

Given that spacetime is fundamentally a causal set, or a superposition of causal sets, how do we determine which causal sets actually resemble the continuum manifold we wish to study? Ultimately it is hoped that in a full dynamical theory the causets which are manifold-like will, on large scales, emerge naturally from the space of all possible causets (when the sum over histories is taken). But even in this case one would need some sort of "test" to check whether such causal sets actually do look like the large scale Lorentzian manifolds. So we must find some other way of determining whether a given causet C resembles a given manifold $\mathcal{M}$. To this end we make the following definitions:

1. An *embedding* is a map $i : C \to \mathcal{M}$ such that

$$x \prec y \;\Leftrightarrow\; i(x) \in J^-\left(i(y)\right) \quad \forall\, x, y \in C$$

2. A *sprinkling* is a random selection of points from a spacetime via a Poisson process with density ρ. This means that the probability of sprinkling n points in a region of d-volume V is given by

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!} \tag{1.1}$$

   where ρ is usually taken to be the fundamental (or Planckian) density.

The sprinkling above defines an embedded causal set. A Lorentzian manifold $(\mathcal{M}, g)$ is said to approximate a causal set C if C could have come from a sprinkling of $(\mathcal{M}, g)$ with relatively high probability; in which case C is said to be faithfully embeddable in $\mathcal{M}$.

1.3 Discreteness, Lorentz invariance and Non-locality

In this section we are interested in the propagation of a scalar field ϕ on a fixed background causal set C that is well approximated by a Minkowski spacetime $\mathbb{M}^d$. To describe such propagation one needs some sort of discrete version of the d'Alembert operator $\Box$ defined on the causal set. The definition of such an operator requires a notion of "nearest-neighbours" which

is fully Lorentz invariant. The identification of these nearest neighbours is not difficult, however the consequences are immense: if Lorentz invariance is to hold at all energies then the theory must be non-local! Let us see why.

For simplicity we will work in 2-dimensions, although similar arguments follow in higher dimensions. Let C be a causet that is well approximated by a Minkowski spacetime $\mathbb{M}^2$, i.e. C can be faithfully embedded in $\mathbb{M}^2$. This means that we can give the elements of C coordinates. We will use light-cone coordinates (u, v) defined by

$$u = \frac{1}{\sqrt{2}}(t - x) \ , \ v = \frac{1}{\sqrt{2}}(t + x) \tag{1.2}$$

where (t, x) are Cartesian coordinates on $\mathbb{M}^2$.

Choose a point $p \in C$ with light-cone coordinates $(u, v) = (0, 0)$. We may identify the past nearest neighbours to p to be those points $q \in C$ such that $q \prec p$ and $n(p, q) := \mathbf{card}\{r \in C : q \prec r \prec p\} = 0$. Let $N_p = \{q \in C : q \prec p \text{ and } n(p, q) = 0\}$, then Lorentz symmetry implies that $\mathbf{card} N_p = \infty$ (almost surely).

To see why this is true consider an element $q \in N_p$. We say that q is *linked* to p and denote the link relation by $q \prec\!\!* p$. Let q have coordinates (u_q, v_q). Now consider the region $R := \{r \in C : \ -\infty < u < u_q, \quad v_q < v < 0\}$ where (u, v) are coordinates of r (see figure (2.2)); this region is obviously infinite in extent. Since R has infinite volume the probability that there is at least one element sprinkled in it is 1. It is also true that with probability 1 there exists at least one element in R which is linked to p. We will prove this by contradiction. Suppose there exists no point $r \in R$ that is linked to p. This implies that given any $r \in R$ there exists at least 1 element $r' \in R'$, where R' is the causal interval $R' = \langle r, p\rangle - \langle r, p\rangle \bigcap \langle q, p\rangle$ and where angled brackets denote the order interval, e.g. $\langle r, p\rangle := \{s \in C : r \prec s \prec p\}$. However, we started with the assumption that there exist no elements in R which are linked to p so there must exist at least another element $s \in R'$ such that $r' \prec s \prec p$. One can imagine carrying this process to infinity therefore showing that there must exist an infinite number of causet elements lying in any region R' defined by some $r \in R$. But the probability of sprinkling an infinite number of points in a finite region is zero: contradiction. We therefore have that there exists at least one element $w \in R$ such that $w \prec\!\!* p$. One can then imagine repeating this procedure, this time starting from

w, and then repeating again ad infinitum showing that with probability 1 there are an infinite number of elements linked to p. In any given frame, these elements are mostly remote from p and lie inside and close to the interior of the past light cone of p[1]. Equivalently one can say that the probability of any given element $p \in C$ having a finite number of nearest neighbours is zero. We thus conclude that if one wants to preserve Lorentz invariance in a discrete theory of spacetime then one has to accept the consequence that the theory will be non-local. This whole discussion can be succinctly encapsulated in Sorkin's maxim: "discreteness plus Lorentz invariance entails non-locality"[5].

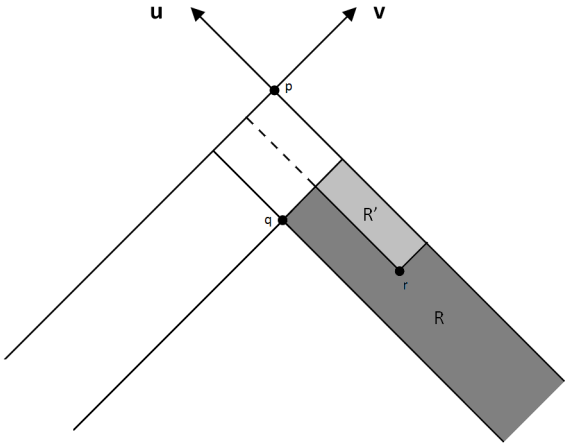

Figure 1.2: A spacetime diagram of a sprinkling of $\mathbb{M}^2$, with only 3 points shown. The shaded region represents R. It is obvious that R is infinite in extent, so given the Poisson distribution we know that with probability 1 there must be element sprinkled in R.

Given these non-local nearest neighbour relations how should one go about building a d'Alembert operator on a causal set? First we will assume that our discrete d'Alembertian acts linearly on ϕ, where ϕ is a scalar field defined on the causal set C. This means that we are looking for a matrix B_{xy} to play the role of $\Box$, where the indices x, y range over the elements of the causal set; thus B_{xy} is the causet analogue of the 2d d'Alembertian $\Box$. Because of the non-locality discussed above the only way that such a matrix

[1]This proof can obviously be repeated in the exact same fashion for the other half of the light-cone

can reproduce a local expression like the exact continuum d'Alembertian is if the majority of terms (the non-local ones) cancel in the sum $\sum_y B_{xy}\phi_y$. In the next chapter we will outline how this can be done by closely following [5]. In chapter 4 we then generalize the above expressions for B_{xy} and its average $\bar{B}_k$ to 4 dimensions. For a full account the reader is referred to [5].

2 Scalar fields on 2d Causal Sets

2.1 Constructing a d'Alembert operator on a 2-d Causal set

Consider a causet C which is well approximated by a 2-dimensional Minkowski spacetime $\mathbb{M}^2$. Let ϕ be a scalar field defined as a mapping of C into the real or complex numbers, $\phi : C \to \mathbb{R}$ or $\mathbb{C}$. Let x be the element of C where we wish to evaluate $B\phi$. We seek to build a B such that in the sum $\sum_y B_{xy}\phi_y$ most of the non-local terms cancel. We gain a clue as to how we should proceed by noticing the following fact about a lattice discretisation. Let x be a point in $\mathbb{M}^2$ and let y and z be points lying on the left and right halves of the past light cone of x. Choose another point w in the past light cone of x so as to close off a rectangle, see figure (2.1). Now

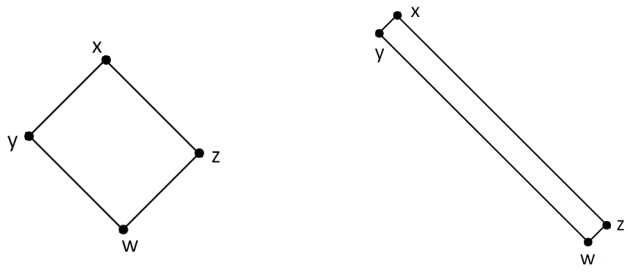

Figure 2.1: The diagram on the left shows the nearest neighbour couplings in some frame R. The diagram on the right shows the same couplings in a highly boosted frame R'.

the discretization of the d'Alembertian on this lattice would look something like $\Box\phi \approx \frac{1}{l^2}(\phi(x) - \phi(y) - \phi(z) + \phi(w))$, where l is the lattice length scale. If we now perform a large boost on this frame we end up with a "squashed" version of the previous rectangle (see figure 2.1), where x is close to y and z close to w. If we also assume that the field varies slowly on scales comparable to l then we find that in the highly boosted frame

$\phi(x) - \phi(y) - \phi(z) + \phi(w) \approx 0$. This mean that we will get a tiny contribution to B from this rectangle, which is precisely the kind of cancellation we were looking for. It seems possible at this point that by summing over all possible rectangles, one will not only achieve the cancellations required, but also find an expression which is fully Lorentz invariant. Before going to the next example it is important to note how the oscillating signs in the definition of $\Box$ made it possible for the highly boosted frame to give a small contribution to B. This notion of oscillating signs will turn out to be essential in our definition of B_{xy}[1]; the next example will confirm this expectation.

Consider the 1D d'Alembertian acting on ϕ which, up to a sign, is given by $\partial^2\phi/\partial t^2$. Its discretization is simply given by $\phi(p) - 2\phi(q) + \phi(r)$ where p,q and r are 3 evenly spaced points on the line. The nice thing about this example is that its configuration of points has a causet analog: n-chains, which in this particular example are 3-chains (a *chain* in a causal set C is a subset in which any two elements are related, an n-chain is a chain with $n+1$ elements). Just as in the 2D example we have that any single one of these chains will determine a frame and hence break Lorentz invariance, however the collection of all of them does not.

Given the above arguments [5] arrives at a simple ansatz for the operator B

$$B\phi(x) = \frac{4}{l^2}\left(-\frac{1}{2}\phi(x) + \left(\sum_{y\in L_1} -2\sum_{y\in L_2} + \sum_{y\in L_3}\right)\phi(y)\right) \tag{2.1}$$

where the sums are over the elements preceding x which have been subdivided into layers as follows

$$\begin{aligned} L_1 &:= \{y \in C : \ y \prec x \quad \text{s.t. } n(x,y) = 0\} \\ L_2 &:= \{y \in C : \ y \prec x \quad \text{s.t. } n(x,y) = 1\} \\ L_3 &:= \{y \in C : \ y \prec x \quad \text{s.t. } n(x,y) = 2\} \end{aligned}$$

[1]It is possible, in 2 dimensions, to determine a discrete d'Alembertian B' (which plays the role of the B defined here) by inverting the discretized version of the Green's function of $\Box$. Computer simulations started by Alan Daughton [6] and continued by Roberto Salgado [7] use this method to determine B' and find that for a typical sprinkling, the individual elements B'_{xy} seem to be evenly distributed between positive and negative values and, furthermore the magnitude of these elements are small unless the proper distance between x and y is near zero.

The matrix B_{xy} can then easily be read-off from (2.1)

$$\frac{l^2}{4}B_{xy} = \begin{cases} -1/2 & \text{for } x = y \\ 1, -2, 1 & \text{for } n(x,y) = 0, 1, 2 \text{ respectively and } x \neq y \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

It is clear, given the random nature of the sprinkling of C into $\mathbb{M}^2$, that the value of $B\phi(x)$ will depend on the particular sprinkling one is considering. However (as we will show in the next section), the expectation value of this quantity with respect to the Poisson distribution that generates the sprinkling, converges to the actual continuum d'Alembertian $\Box\phi(x)$ in the limit where the fundamental length scale goes to zero

$$\langle B\phi(x)\rangle \longrightarrow \Box\phi(x) \quad \text{as } l \to 0$$

where the angled brackets denote the expectation value of the function $B\phi(x)$. Using the expressions for the Poisson distribution it is easy to show that the average over sprinklings gives

$$\langle B\phi(x)\rangle := \int \bar{B}(x-y)\phi(y)d^2y \tag{2.3}$$

$$= -\frac{2}{l^2}\phi(x) + \frac{4}{l^4}\int_{y\in J^-(x)} d^2y\, e^{-\xi}(1 - 2\xi + \frac{1}{2}\xi^2)\phi(y) \tag{2.4}$$

where we have defined $\xi := V(x,y)/l^2$ and $V(x,y)$ is the volume of the causal interval between x and y. In §2.2 we verify that this expression does indeed $\to \Box\phi$ as $l \to 0$.

It turns out that (2.1) is not adequate for building a field theory on a causal set. This is because the fluctuations in (2.1) grow with the number of sprinkled points N rather than die away. To fix this problem one must introduce a new scale k to replace the scale $1/l^2$ in (2.4), and take it to be much larger than the Planck scale, i.e. $1/\sqrt{k} \gg l$. This leads to a new continuum approximation to $\Box$:

$$\bar{B}_k(x-y) := -2k\delta^{(2)}(x-y) + 4k^2 p(\xi')e^{-\xi'} \tag{2.5}$$

where $p(\xi') = 1 - 2\xi' + \frac{1}{2}\xi'^2$ and $\xi' = kV(x,y)$. The introduction of this new scale k implies that the non-locality scale associated with the above

expression is no longer l but rather $k^{-1/2}$. What we are really interested in though is the analogous expression in the causal set. Retracing the steps that took us from (2.1) to (2.5) brings us to the following proposition.

Proposition. The causet expression whose sprinkling average gives (2.5) is given by

$$B_k\phi(x) = \frac{4\epsilon}{l^2}\left(-\frac{1}{2}\phi(x) + \epsilon\sum_{y\prec x} f(n(x,y),\epsilon)\phi(y)\right), \tag{2.6}$$

where $\epsilon = l^2k$ and

$$f(n,\epsilon) = (1-\epsilon)^n\left(1 - \frac{2\epsilon n}{1-\epsilon} + \frac{\epsilon^2 n(n-1)}{2(1-\epsilon)^2}\right). \tag{2.7}$$

Proof. The first term in (2.6) is trivial. The rest of the proof rests on the fact that the average of a sum of a random variable is the sum of the individual averages; and that the average of a product of random variables is the product of the averages, provided the two variables are independent. Consider discretising the spacetime into discrete cells labelled by $i = 1, \ldots \infty$. Let n_i denote the number of elements sprinkled in the volume V_i, where V_i is the volume of the causal interval between the point x, where we wish to evaluate B_k, and the cell i. Let

$$\chi_i = \begin{cases} 1 & \text{if cell } i \text{ is filled} \\ 0 & \text{otherwise} \end{cases}$$

We are interested in taking the average of the second term in (2.6)

$$\langle\sum_i \chi_i f(n_i,\epsilon)\rangle = \sum_i \langle\chi_i f(n_i,\epsilon)\rangle = \sum_i \langle\chi_i\rangle\langle f(n_i,\epsilon)\rangle$$

where the third step was possible since χ_i and n_i are independent. But $\langle\chi_i\rangle = \rho\Delta V_i$, where ΔV_i is the volume of the cell i and ρ is the fundamental sprinkling density which we will take to be $\rho = 1/l^2$, l being the Planck length. So we find that

$$\langle\sum_i \chi_i f(n_i,\epsilon)\rangle = \sum_i \rho\Delta V_i\langle f(n_i,\epsilon)\rangle$$

Now consider the average of the first term in (4.2)

$$\begin{aligned}\langle(1-\epsilon)^{n_i}\rangle &= \sum_{n=0}^{\infty}(1-\epsilon)^n \times \text{Prob}(n_i = n) \\ &= \sum_{n=0}^{\infty}(1-\epsilon)^n \frac{(\rho V_i)^n e^{-\rho V_i}}{n!} \\ &= e^{-\rho V_i}\sum_{n=0}^{\infty}\frac{(\rho V_i(1-\epsilon))^n}{n!} \\ &= e^{-\rho V_i}e^{\rho V_i(1-\epsilon)} \\ &= e^{-\epsilon\rho V_i}\end{aligned}$$

Then the first term in (2.8) becomes, in the continuum limit (substituting in for the fundamental density $\rho = 1/l^2$, so $\epsilon\rho = k$)

$$\sum_i \langle\chi_i(1-\epsilon)^n\rangle\,\phi_i = \sum_i \rho\Delta V_i e^{-\epsilon\rho V_i} \;\longrightarrow\; \frac{1}{l^2}\int dy\, e^{-kV(x,y)}\phi(y)$$

Similarly, taking the average over sprinklings of the second term in (4.2)

$$\begin{aligned}\langle n(1-\epsilon)^{n-1}\rangle &= \sum_{n=0}^{\infty} n(1-\epsilon)^{n-1}\frac{(\rho V_i)^n e^{-\rho V_i}}{n!} \\ &= \rho V_i \sum_{n=1}^{\infty}\frac{(\rho V_i(1-\epsilon))^{n-1}e^{-\rho V_i}}{(n-1)!} \\ &= \rho V_i e^{-kV_i}\end{aligned}$$

Thus

$$-2\epsilon\sum_i \langle\chi_i n_i(1-\epsilon)^{n_i-1}\rangle\,\phi_i \;\longrightarrow\; -\frac{2}{l^2}\int dy\; kV(x,y)e^{-kV(x,y)}\phi(y)$$

And finally, the average of the third term in (4.2) gives

$$\begin{aligned}\langle n(n-1)(1-\epsilon)^{n-2}\rangle &= \sum_{n=0}^{\infty} n(n-1)(1-\epsilon)^{n-2}\frac{(\rho V_i)^n e^{-\rho V_i}}{n!} \\ &= \rho V_i \sum_{n=2}^{\infty}\frac{(\rho V_i(1-\epsilon))^{n-2}e^{-\rho V_i}}{(n-2)!} \\ &= \rho^2 V_i^2 e^{-kV_i}\end{aligned}$$

Thus

$$\frac{1}{2}\epsilon^2 \sum_i \langle \chi_i n_i (n_i - 1)(1-\epsilon)^{n_i - 2} \rangle \, \phi_i \;\longrightarrow\; \frac{1}{2l^2} \int dy \; k^2 V(x,y)^2 e^{-kV(x,y)} \phi(y)$$

□

The way to interpret (2.6) and (4.2) is as follows. The weighting function $f(n, \epsilon)$ plays the role of the sum over layers in (2.1). However instead of summing over individual layers defined by their "distance" from x in terms of number of intervening elements, it sums "thickened" layers. This means that each layer will now not just be made up of those elements $y \in C$ such that $n(x, y) = i$, for $i = 1, 2, 3$, but will comprise a much wider range of layers. The thickness of these layers will depend on the magnitude of ϵ.

We thus now have a one parameter family of non-local d'Alembertians on the causal set, where the parameter ϵ determines the scale of non-locality. Simulations by Sorkin [5] show that in order for the fluctuations of $B_k\phi$ to be small and go to zero as $N \to \infty$ one needs $\epsilon \ll 1$. This means that we need a very large separation between the Planck scale l and $\lambda_0 := k^{-1/2} = \sqrt{\epsilon}$. This does not confute the model since there are many orders of magnitude between the Planck energy and energies which have been explored so far experimentally, say 1TeV. This means that this new non-locality scale λ_0 can lie anywhere in the range $10^{16} l > \lambda_0 >> l$. This 1-parameter family of d'Alembertians on the causet can then be averaged over all sprinklings to give a retarded non-local continuum modification of the exact continuum d'Alembertian. In the next section we will show how this modified 2-d d'Alembertian, $\bar{B}_k$ reproduces the true d'Alembertian up to corrections of order $1/k$.

2.2 The mean of the 2d non-local d'Alembertian

In this section we compute the non-local continuum d'Alembertian (2.5) defined in §2.2 for arbitrary scalar fields ϕ by introducing a cutoff at large scales. We then show that for fields of compact support the non-local d'Alembertian converges to the exact continuum d'Alembertian in the limit $k \to \infty$. We also show that for some of functions it is possible to take the cutoff R to infinity, while keeping the non-locality scale k fixed, and $\bar{B}_k\phi$

will remain finite.

To compute (2.5) we use the light-cone coordinates defined in §1.3. In these coordinates the metric is given by $ds^2 = -2dudv$ and the exact continuum d'Alembertian is $\Box = -2\partial^2/\partial u\partial v$.

Let ϕ be a scalar field on $\mathbb{M}^2$, denote the origin by 0 and let x be a point in $\mathbb{M}^2$ with coordinates (u, v). The non-local d'Alembertian evaluated on ϕ at the origin is[2]

$$\Box_k\phi(0) = -2k\phi(0) + 4k^2 \int_{x\in J^-(0)} dudv\ \phi(x)\ e^{-k\sigma(x)}(1 - 2k\sigma(x) + \frac{1}{2}k^2\sigma(x)^2) \tag{2.8}$$

where $\sigma(x) = uv$ is Synge's world function [8] in $\mathbb{M}^2$. The Synge world function is defined to be 1/2 of the square of the geodesic distance between two points: $\sigma(x) = \frac{1}{2}\tau^2 (= uv)$ where τ is the proper time separating the two points[3]. (2.8) can be rewritten in the following way

$$\begin{aligned}\Box_k\phi(0) &= -2k\phi(0) + 4k^2(1 + 2k\frac{\partial}{\partial k} + \frac{1}{2}k^2\frac{\partial^2}{\partial k^2}) \int_{x\in J^-(O)} dudv\ \phi(x)\ e^{-k\sigma(x)} \\ &=: -2k\phi(0) + 4k^2 J_2(k)\end{aligned} \tag{2.9}$$

Where we have defined

$$J_2(k) := \hat{O}I(k) = \hat{O}\int_{x\in J^-(O)} dudv\ \phi(x)\ e^{-k\sigma(x)}, \tag{2.10}$$

and

$$\hat{O} = \frac{1}{2}\left(k\frac{\partial}{\partial k} + 1\right)\left(k\frac{\partial}{\partial k} + 2\right) = 1 + 2k\frac{\partial}{\partial k} + \frac{1}{2}k^2\frac{\partial^2}{\partial k^2}. \tag{2.11}$$

[2]For notational simplicity we will change notation in this section and denote the non-local continuum d'Alembertian by $\Box_k$ instead of $\bar{B}_k$

[3]In $\mathbb{M}^2$ the Synge world function is equal to the volume of the order interval. This equivalence will break down in higher dimensions and and one can imagine using either the world function (which is some power of the volume) or any other power of the volume. Whichever one chooses the real task is then to find the appropriate $\hat{O}$.

Note that the operator $\hat{O}$ has the following properties

$$\hat{O}\left(\frac{1}{k}\right) = 0\,, \quad \hat{O}\left(\frac{1}{k^2}\right) = 0.$$

We are now left with the task of computing the integral $I(k)$ for suitable test functions ϕ which are polynomials of the coordinates.

We Taylor expand ϕ around the origin

$$\begin{aligned}\phi(u,v) =& \phi(0,0) + v\cdot\left.\frac{\partial\phi}{\partial u}\right|_{(0,0)} + u\cdot\left.\frac{\partial\phi}{\partial v}\right|_{(0,0)} + uv\cdot\left.\frac{\partial^2\phi}{\partial u\partial v}\right|_{(0,0)} \\ &+ \frac{1}{2}u^2\cdot\left.\frac{\partial^2\phi}{\partial u^2}\right|_{(0,0)} + \frac{1}{2}v^2\cdot\left.\frac{\partial^2\phi}{\partial v^2}\right|_{(0,0)} + \dots \end{aligned} \tag{2.12}$$

Notation: we will denote derivatives using the standard notation $\partial\phi/\partial u = \phi_{,u}$ and similarly for other derivatives. We will also always assume that ϕ and its derivatives are evaluated at the origin, e.g $\phi(0) = \phi$. Plugging the above into (2.10), and integrating using the (u,v) coordinate system defined above we find

$$\begin{aligned}J_2(k) =& \hat{O}\int_{-\infty}^{0} du \int_{-\infty}^{0} dv\ (\phi + v\cdot\phi_{,u} + u\cdot\phi_{,v} + uv\cdot\phi_{,uv} \\ &+ \frac{1}{2}u^2\cdot\phi_{,uu} + \frac{1}{2}v^2\cdot\phi_{,vv} + \dots)e^{-kuv}\end{aligned} \tag{2.13}$$

Now the integrals in (2.13) which we need to compute are given by

$$I^{nm}(k) = \int_{-\infty}^{0} du \int_{-\infty}^{0} dv\ u^n v^m e^{-kuv} \tag{2.14}$$

for any $n, m \in \mathbb{N}$. In order to make the computation of these integrals possible analytically we will regularise them by introducing a cutoff R. We then show that for certain polynomials of the coordinates, which need not be of compact support, the results hold even if one takes $R \to \infty$. Introducing the cutoff R in (2.14) we get

$$I_R^{nm}(k) = \int_{-R}^{0} du \int_{-R}^{0} dv\ u^n v^m e^{-kuv} \tag{2.15}$$

A general solution to (2.15) is given in appendix A. The particular solutions we need in this case are: (A.2), (A.3) with $n = 1$ and (A.4) with $n = 1, 2$

and $m = 0$. Substituting these in (2.13) we find

$$\begin{aligned} J_2(k) =& \hat{O}\left[\phi\,\frac{1}{k}\mathrm{Ein}(kR^2) + (\phi_{,v}+\phi_{,u})\left(\frac{-R}{k}+\frac{1-e^{-kR^2}}{Rk^2}\right)\right. \\ &+\phi_{,uv}\left(\frac{1}{k^2}\mathrm{Ein}(kR^2)-\frac{1}{k^2}+\frac{e^{-kR^2}}{k^2}\right) \\ &\left.+\frac{1}{2}\left(\phi_{,uu}+\phi_{,vv}\right)\left(\frac{R^2}{2k}-\frac{1}{R^2k^3}+\left(1+\frac{1}{kR^2}\right)\frac{e^{-kR^2}}{k^2}\right)+\dots\right] \end{aligned} \tag{2.16}$$

where $\mathrm{Ein}(z)$ is an entire function and is defined by[4]

$$\mathrm{Ein}(z) = \int_0^z \frac{1-e^{-t}}{t}dt = \sum_{n=1}^{\infty}\frac{(-)^{n+1}z^n}{nn!}$$

Equation (2.16) looks fairly complicated, but this is where the operator $\hat{O}$ comes to our rescue. Recalling that $\hat{O}$ kills $1/k$ and $1/k^2$, and ignoring exponentially small terms we finally get

$$J_2(k) = \phi\cdot\left(\frac{1}{2k}\right)+\phi_{,uv}\cdot\left(\frac{-1}{2k^2}\right)+\frac{1}{2}\left(\phi_{,uu}+\phi_{,vv}\right)\cdot\left(\frac{-1}{k^3R^2}\right)+\dots \tag{2.17}$$

Plugging this back into (2.9) we find

$$\Box_k\phi(0) = -2\frac{\partial^2\phi}{\partial u\partial v}\bigg|_{(0,0)} - \frac{2}{kR^2}\left(\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right)\bigg|_{(0,0)}+\dots \tag{2.18}$$

which is the actual continuum d'Alembertian (plus corrections).

We have thus shown that our non-local expression for the 2d d'Alembertian (2.8) does indeed reproduce the standard continuum version of the operator up to corrections of order $1/k$.

Our discussion so far has concerned general fields ϕ, and a cutoff R which allowed us to compute the integrals explicitly. We mentioned in §2.2 that $\Box_k\phi$ goes to the exact continuum d'Alembertian $\Box\phi$ in the limit $l \to 0$. If we suppose ϕ is of compact support, so that we can keep the cutoff R finite (this ensures that potential divergences due to higher order polynomials

[4]For properties of this function the reader is referred to appendix A.

(2.15) are avoided), then we can see that is indeed the case:

$$\Box_k \phi(0) \to \Box \phi(0) \quad \text{as} \quad l \to 0 \text{ or } k \to \infty.$$

Let us go back to the more general picture where the fields ϕ we consider are not of compact support. How can one determine what kind of fields our non-local continuum d'Alembertian $\Box_k$ is valid for in the limit $R \to \infty$? To answer this question we must study in detail the integral (2.15):

$$I_R^{nm}(k) = \int_{-R}^{0} du \int_{-R}^{0} dv \; u^n v^m e^{-kuv} \tag{2.19}$$

We already know from our computation of $\Box_k \phi$ that terms which are up to quadratic order in the coordinates are well behaved in the $R \to \infty$ limit (while also keeping the non-locality scale k fixed). So we want to study (2.19) for higher order terms. We will consider the cases $n = m$ and $n \neq m$ separately:

1) $n = m \neq 0$. The general solution for this integral is

$$\tilde{I}_R^{nn}(k) = (-)^n \sum_{i=0}^{n} \binom{n}{i} p^{(n-i)}(k) q^{(i)}(k) \tag{2.20}$$

where $p(k) := 1/k$, $q(k) := \mathrm{Ein}(kR^2)$ and $f^{(n)}(k)$ denotes the nth partial derivative of $f(k)$ with respect to k, as is proved in appendix A. Acting on (2.20) with the operator $\hat{O}$ and multiplying by k^2 we get

$$\begin{aligned} k^2 J^{nn}(k) &:= k^2 \left(1 + k\frac{\partial}{\partial k} + \frac{1}{2}k^2 \frac{\partial^2}{\partial k^2}\right) \tilde{I}_R^{nn}(k) \\ &= k^2 (-)^n \sum_{i=0}^{n} \binom{n}{i} \Big[p^{(n-i)}(k) q^{(i)}(k) + 2k \Big(p^{(n-i+1)}(k) q^{(i)}(k) \\ &\quad + p^{(n-i)}(k) q^{(i+1)}(k) \Big) + \frac{1}{2}k^2 \Big(p^{(n-i+2)}(k) q^{(i)}(k) \\ &\quad + 2p^{(n-i+1)}(k) q^{(i+1)}(k) + p^{(n-i)}(k) q^{(i+2)}(k) \Big) \Big] \end{aligned} \tag{2.21}$$

Looking at the (2.21) we can see that it only depends on the functions $p(k) = 1/k$ and $q(k) = \mathrm{Ein}(kR^2)$ and their derivatives. The derivatives of $p(k)$ won't give any bad behaviour in the $R \to \infty$ limit since these are

independent of R. The potential problems can therefore arise only from $q(k)$ and its derivatives. We know however that $q^{(1)}(k) = (1 - e^{-kR^2})/k$ so the only potential divergences come from the $q(k) = \mathrm{Ein}(kR^2)$ terms surviving in (2.21). In the large R limit we have

$$\mathrm{Ein}(kR^2) \to \ln(kR^2) \ \text{ as } \ R \to \infty$$

i.e. a logarithmic (infrared, I.R.) divergence. For $n = 0, 1, 2$ no Ein terms survive in $J(k)$, so these potential I.R. divergences can only arise for $n \geq 3$. But the $q(k) = \mathrm{Ein}(kR^2)$ terms only survive if no derivative acts on them, so only those $q^{(i)}(k)$ with $i = 0$ can give problems. These terms however are all multiplied by the n-th derivative of $p(k) = 1/k$ so their behaviour in the limit $R \to \infty$ is (after multiplying by k^2)

$$\frac{\ln(kR^2)}{k^{n-1}} \ \text{ as } \ R \to \infty \quad \text{for } \ n \geq 3$$

These terms, for k fixed, are clearly an issue if one sends R to infinity. To avoid this one must introduce an effective I.R. cutoff, $R = R_0$, which we will take to be the size of the observable universe, i.e. $R_0 = 10^{60} l$ where l is the Planck length. Has the divergence been cured by the introduction of this large but finite cutoff? To answer this question we need to know what the magnitude of the non-locality scale k is. We already mentioned in §2.2 that this must lie somewhere between the TeV scale and the Planck scale, if we say that we know physics is local at all energy scales lower than a TeV. Thus we can still set this non-locality scale to be very large, e.g. $\lambda_0 \sim 10^{16} l$. Then $k = k_0 \sim 10^{-32} l^{-2}$. Finally we need to recall that these terms, which arose from the integration of polynomials of the form $u^n v^n$, will be multiplied by $(2n)$-th order derivatives of ϕ. If we impose the condition that these derivatives must be small at the length scale $\lambda_0 = 1/\sqrt{k_0}$ then

$$\frac{1}{\phi} \frac{\partial^{2n} \phi}{\partial u^n \partial v^n} \ll \frac{1}{\lambda_0^{2n}} = k_0^n$$

Putting these results together we thus find that the behaviour of $\Box_k \phi$ for functions $\phi = u^n v^n$ for $n \geq 3$ is

$$\Box_k \phi \sim k_0 \ln(k_0 R_0^2) \sim k_0 = 10^{-32} l_p^{-2}$$

which is independent of n: this is a remarkable result! It is telling us that for any function of the proper time squared, i.e. $\tau^2 = uv$, our non-local continuum d'Alembertian $\Box_k$ is well defined and is equal to the exact continuum d'Alembertian up to corrections that are small order by order. We will explore this in more detail later on. For now let us go back to (2.19) and study its behaviour in the 2nd case.

2) $n > m$, $m \geq 0$. The general solution for this integral is (ignoring exponentially small terms)

$$\tilde{I}_R^{nm}(k) = \frac{m!(-)^{n+m}R^{n-m}}{(n-m)k^{m+1}} + \frac{(-)^{n-1}(n-m-1)!}{R^{n-m}}\frac{\partial^m}{\partial k^m}\left(\frac{1}{k^{n-m+1}}\right) \quad (2.22)$$

Acting on (2.22) with $\hat{O}$ and multiplying by k^2 we find

$$\begin{aligned} k^2 J^{nm}(k) = {} & \frac{m!(-)^{n+m}R^{n-m}}{(n-m)k^{m-1}} + \frac{k^2(-)^{n-1}(n-m-1)!}{R^{n-m}}\frac{\partial^m}{\partial k^m}\left(\frac{1}{k^{n-m+1}}\right) \\ & + 2\left[\frac{(m+1)!(-)^{n+m+1}R^{n-m}}{(n-m)k^{m-1}} + \frac{k^3(-)^{n-1}(n-m-1)!}{R^{n-m}}\frac{\partial^{m+1}}{\partial k^{m+1}}\left(\frac{1}{k^{n-m+1}}\right)\right] \\ & + \frac{1}{2}\left[\frac{(m+2)!(-)^{n+m}R^{n-m}}{(n-m)k^{m-1}} + \frac{k^4(-)^{n-1}(n-m-1)!}{R^{n-m}}\frac{\partial^{m+2}}{\partial k^{m+2}}\left(\frac{1}{k^{n-m+1}}\right)\right] \end{aligned} \quad (2.23)$$

This term will be multiplied by derivatives of $\phi = u^n v^m$ which, just as we did in (1), we impose to be small at the length scale $\lambda_0 = 1/\sqrt{k_0}$:

$$\frac{1}{\phi}\frac{\partial^{n+m}\phi}{\partial u^n \partial v^m} \ll \frac{1}{\lambda_0^{n+m}} = k_0^{\frac{n+m}{2}}$$

It is immediately obvious that, because of the factors of R appearing in the numerator of some of the terms in (2.23), $\Box_k\phi$ will not give good results for functions of the form $\phi = u^n v^m$ for $n - m > 0$ and $m > 2$.

We thus conclude that our non-local continuum d'Alembertian $\Box_k$ is works well only for functions of the proper time squared. For other functions ϕ it will not give reliable results unless one supposes that these are of compact support, this ensures that the divergences due to terms of order R vanish.

3 The 4d non-local d'Alembertian

What is the most natural generalisation of (2.1) and its continuum counterpart (2.5) in 4 dimensions? It turns out [5] that the correct generalisation of (2.1) is

$$B\phi(x) = \frac{1}{l^2}\left(a\phi(x) + b\left(\sum_{y\in L_1} -3\sum_{y\in L_2} +3\sum_{y\in L_3} -\sum_{y\in L_4}\right)\phi(y)\right) \tag{3.1}$$

where a, b are constants to be determined by acting with (3.1) on suitable test functions, and L_i, $i = 1, \ldots, 4$ are the layers defined in §2.2. Note that what has happened is that the coefficient pattern 1 -2 1 has been replaced by 1 -3 3 -1. This appearance of the binomial coefficients (with alternating signs) is related to an identity which expresses

$$(H+1)(H+2)\ldots(H+n)e^{-k}$$

where $H = k\partial/\partial k$, in terms of binomial coefficients. The natural generalisation to (2.5), i.e. the mean of (3.1) is then simply:

$$\Box_k\phi(0) = \frac{\alpha}{l^2}\phi(0) + \frac{\beta}{l^6}\int_{x\in J^-(0)} d^4x\ \phi(x)e^{-\xi(x)}\left(1 - 3\xi(x) + \frac{3}{2}\xi(x)^2 - \frac{1}{6}\xi(x)^3\right) \tag{3.2}$$

where $\xi(x) = V(x)/l^4$ and once again the constants α and β are to be chosen so that (3.2) gives the correct results for suitable test functions.

The integrals in (3.2) are very hard to compute. To simplify the problem we note that in 4-d the Synge world function, $\sigma(x)$ is given by

$$\sigma(x) = \frac{1}{2}\tau^2 = \frac{1}{2}(t^2 - r^2)$$

where τ is the proper time. If we use light-cone coordinates (u, v, θ, φ) defined by $u = \frac{1}{\sqrt{2}}(t-r)$ and $v = \frac{1}{\sqrt{2}}(t+r)$, where $r \geq 0$ is the usual radial

coordinate so that $v \geq u$, then we have $\sigma(x) = uv$. The actual 4-volume of the causal interval in $\mathbb{M}^4$ is given by $\text{Vol} = (\pi/24)\tau^4$ so the relation to sigma is

$$\text{Vol} = \frac{\pi}{6}\sigma(x)^2$$

This means that if we replace $\xi(x) = V(x)/l^4$ by $\sqrt{\xi(x)} = (V(x)/l^4)^{1/2} = (\pi/6)^{1/2}\sigma(x)/l^2$ everywhere in (3.2) then the integrals become of the same form as those computed in §2 and appendix A. If we also replace the scale $(\pi/6l^4)^{1/2}$ by the non locality scale k (in a similar fashion to what we did in §2.1 for the 2-d case) then $\sqrt{\xi(x)} = k\sigma(x)$.[1] With $\xi(x)$ replaced everywhere by $\xi(x)^{1/2}$ and with the introduction of the non-locality scale k (3.2) becomes

$$\begin{aligned}\Box_k\phi(0) &= \alpha k\phi(0) + \beta k^3 \int_{x\in J^-(0)} d^4x\ \phi(x)e^{-k\sigma(x)}\left(1 - 3k\sigma(x) + \frac{3}{2}k^2\sigma(x)^2 - \frac{1}{6}k^3\sigma(x)^3\right)\\ &= \alpha k\phi(0) + \frac{\beta}{6}k^3\,(H+3)\,(H+2)\,(H+1)\int_{x\in J^-(0)} d^4x\ \phi(x)e^{-k\sigma(x)}\end{aligned} \tag{3.3}$$

The integrals in (3.3) are precisely those computed in §2 for the 2-d case. We may therefore use the results from appendix A to evaluate them explicitly.

As we will now show, the correct expression for the non-local continuum d'Alembertian in $\mathbb{M}^4$, including constants, is

$$\begin{aligned}\Box_k\phi(0) = -2k\phi(0) + \frac{6}{\pi}k^3 \int_{x\in J^-(0)} du dv d\theta d\varphi\ r^2 \sin\theta\,\phi(x)e^{-k\sigma(x)}(1 - 3k\sigma(x)\\ + \frac{3}{2}k^2\sigma(x)^2 - \frac{1}{6}k^3\sigma(x)^3)\end{aligned} \tag{3.4}$$

where the integral is in (u, v, θ, φ) coordinates, with u, v defined above and θ, φ being the usual angular coordinates: $\theta \in [0, \pi)$ and $\varphi = [0, 2\pi)$.

To show that (3.4) does indeed approximate the exact continuum d'Alembertian in 4d we proceed as for the 2d case. Recall that in Cartesian coordinates

[1] Although it remains to be checked, the introduction of the scale k should ensure that fluctuations of the causal set expression get smaller as the number of sprinkled points N increases (if one sets $1/\sqrt{k} \gg l$)

$x^\mu = (t, x, y, z)$ the d'Alembert operator takes the form

$$\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \vec{x}^2}$$

We first re-write (3.4) in terms of the operator $\hat{O}$ acting on the integral $I(k)$

$$\Box_k \phi(0) = -2k\phi(0) + \frac{6}{\pi} k^3 \hat{O} I(k) \tag{3.5}$$

where we have defined

$$I(k) = \int_{x \in J^-(0)} du dv d\theta d\varphi \; r^2 \sin\theta \phi(x) e^{-k\sigma(x)} \tag{3.6}$$

and

$$\begin{aligned} \hat{O} &\equiv \frac{1}{6}(H+3)(H+2)(H+1) \\ &= 1 + 3k\frac{\partial}{\partial k} + \frac{3}{2}k^2\frac{\partial^2}{\partial k^2} + \frac{1}{6}k^3\frac{\partial^3}{\partial k^3} \end{aligned} \tag{3.7}$$

Just as for the 2D case it is important to note that

$$\hat{O}\left(\frac{1}{k}\right) = 0, \quad \hat{O}\left(\frac{1}{k^2}\right) = 0, \quad \hat{O}\left(\frac{1}{k^3}\right) = 0 \tag{3.8}$$

We must now compute $I(k)$ for various test functions which are polynomials in the coordinates of degree two or less. In the (u, v, θ, φ) coordinates defined above we have

$$I(k) = \frac{1}{2}\int_{-\infty}^{0} du \int_{-\infty}^{0} dv \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; \frac{(v-u)^2}{2} \sin\theta \; \phi(x) e^{-kuv} \tag{3.9}$$

Next we proceed to Taylor expand our field ϕ around the origin in Cartesian coordinates

$$\phi(x) = \phi(0) + x^\mu \partial_\mu \phi\big|_0 + \frac{1}{2} x^\mu x^\nu \partial_\mu \partial_\nu \phi\big|_0 + \ldots \tag{3.10}$$

where $x^\mu = (t, x, y, z)$. We then perform the integration over the angles. By plugging the Taylor expanded field into (3.9) we find that the integrals over the angular coordinates are given by

$$\int_0^\pi d\varphi \int_0^{2\pi} d\theta \ \sin\theta = 4\pi \tag{3.11}$$

$$\int_0^\pi d\varphi \int_0^{2\pi} d\theta \ x^\mu \sin\theta = \begin{cases} 4\pi t & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases} \tag{3.12}$$

$$\int_0^\pi d\varphi \int_0^{2\pi} d\theta \ x^\mu x^\nu \sin\theta = \begin{cases} 4\pi t^2 & \mu = 0, \nu = 0 \\ \frac{4}{3}\pi r^2 \delta^{ij} & \mu = i, \nu = j \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

substituting the above results together with ϕ's Taylor expansion into (3.9)we find

$$I(k) = \frac{1}{4}\int_{-\infty}^0 du \int_{-\infty}^0 dv \ (v-u)^2(4\pi\phi(0) + 2\pi(u+v)\phi_{,t}\big|_0 + \frac{\pi}{2}(u+v)^2\phi_{,tt}\big|_0 + \frac{\pi}{6}(v-u)^2\phi_{,ii}\big|_0 + \dots)e^{-kuv} \tag{3.14}$$

Once again, as for the 2D case, we will regularize these integrals by introducing a cutoff R. For fields of compact support this cutoff simply represents the boundary of the region where the fields are zero. For more general fields however we will see that this cutoff must be physical since the limit $R \to \infty$ cannot be taken without running into infinities. With the cutoff R the integrals (3.14) can be computed using the expressions found in appendix A. A straightforward computation gives

$$
\begin{aligned}
L_1(k) :&= \int_{-R}^{0} du \int_{-R}^{0} dv\ (v-u)^2 e^{-kuv} \\
&= \frac{R^2}{k} + \frac{2}{k^2}(1 - \mathrm{Ein}(kR^2)) - \frac{2}{k^3R^2}(1 - e^{-kR^2}) \qquad (3.15) \\
L_2(k) :&= \int_{-R}^{0} du \int_{-R}^{0} dv\ (v-u)^2 (v+u) e^{-kuv} \\
&= -\frac{2R^3}{3k} + \frac{2R}{k^2} - \frac{4}{Rk^3} + \frac{4}{R^3k^4}(1 - e^{-kR^2}) \qquad (3.16) \\
L_3(k) :&= \int_{-R}^{0} du \int_{-R}^{0} dv\ (v-u)^2 (v+u)^2 e^{-kuv} \\
&= \frac{R^4}{2k} + \frac{2}{k^3}(3 - 2\,\mathrm{Ein}(kR^2)) + \frac{12e^{-kR^2}}{k^4R^2} \\
&\quad - \frac{12}{k^5R^4}(1 - e^{-kR^2}) \qquad (3.17) \\
L_4(k) :&= \int_{-R}^{0} du \int_{-R}^{0} dv\ (v-u)^4 e^{-kuv} \\
&= \frac{R^4}{2k} - \frac{4R^2}{k^2} + \frac{6}{k^3}(2\,\mathrm{Ein}(kR^2) - 3) \\
&\quad + \frac{12}{k^4R^2}(2 - e^{-kR^2}) - \frac{12}{k^5R^4}(1 - e^{-kR^2}) \qquad (3.18)
\end{aligned}
$$

Acting with $\hat{O}$ on the above expressions, and ignoring exponentially small terms we find

$$
\begin{aligned}
j_1(k) &:= \hat{O}L_1(k) = \frac{1}{3k^2} \qquad (3.19) \\
j_2(k) &:= \hat{O}L_2(k) = -\frac{4}{k^4R^3} \qquad (3.20) \\
j_3(k) &:= \hat{O}L_3(k) = -\frac{4}{3k^3} + \frac{48}{k^5R^4} \qquad (3.21) \\
j_4(k) &:= \hat{O}L_4(k) = \frac{4}{k^3} - \frac{24}{k^4R^2} + \frac{48}{k^5R^4} \qquad (3.22)
\end{aligned}
$$

Finally, putting everything together

$$
\begin{aligned}
J_4(k) &:= \hat{O}I(k) \\
&= \frac{1}{4}\left(4\pi j_1(k) + 2\pi\phi_{,t}\big|_0 j_2(k) + \frac{\pi}{2}\phi_{,tt}\big|_0 j_3(k) + \frac{\pi}{6}\phi_{,ii}\big|_0 j_4(k) + \dots\right) \qquad (3.23)
\end{aligned}
$$

and substituting this expression in (3.4) we get

$$\begin{aligned}\Box_k\phi(0) &= -2k\phi(0) + \frac{6}{\pi}k^3 J_4(k) \\ &= -\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial \vec{x}^2} - \frac{6}{kR^2}\left(\frac{2}{R}\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial \vec{x}^2}\right) + \frac{12}{k^2R^4}\left(3\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial \vec{x}^2}\right) + \dots\end{aligned} \tag{3.24}$$

For ϕ of compact support, where divergences due to higher order terms are avoided thanks to the cutoff R, we can see that this non-local d'Alembertian converges to the exact continuum d'Alembertian in the limit $k \to \infty$:

$$\Box_k\phi \to \Box\phi \ \text{ as } \ k \to \infty$$

Once again it will be interesting to study the behaviour of $\Box_k\phi$ for general fields and for $k = k_0$ fixed. The general integral we will be interested in is of the form

$$\bar{I}_R^{nm}(k) = \int_{-R}^{0} du \int_{-R}^{0} dv\ (v-u)^2 u^n v^m e^{-kuv} \tag{3.25}$$

where $g(u,v) = u^n v^m$. This is identical to the integral studied in 2 dimensions, (2.15), with the exception that there is now a factor of $(v-u)^2$ in the measure. We can re-write (3.25) as a sum of three integrals:

$$\begin{aligned}\bar{I}_R^{nm}(k) &= \int_{-R}^{0} du \int_{-R}^{0} dv\ \left(u^n v^{m+2} + u^{n+2} v^m - 2u^{n+1}v^{m+1}\right) e^{-kuv} \\ &= I_R^{n(m+2)}(k) + I_R^{(n+2)m}(k) - 2I_R^{(n+1)(m+1)}(k)\end{aligned} \tag{3.26}$$

It is clear from looking at this integral that $\Box_k$ it will not benefit from the nice properties possessed in 2 dimensions. This is because even for functions of the proper time squared, i.e. $\phi = (uv)^n$ one will get integrands in (3.26) which are not symmetric in u, v and thus end up having factors of order R in $\Box_k\phi$. The reason for this difference in behaviour bewtween the d'Alembertian in 2-d and 4-d when acting on functions of the proper time squared, is that considering a function of the proper time squared in 2-d effectively reduces the problem to a 1-d one. In 4-d however this is not the case. To see why this is so consider a function $f := f(\tau^2) = f(uv)$. Then in 2-d:

$$\Box f(uv) = -2\frac{\partial^2 f(uv)}{\partial u \partial v} = -2\frac{df}{d\tau} - 2uv\frac{d^2 f}{d\tau^2}$$

where everything is a function of uv only. So the problem is effectively 1-d now. In 4-d the d'Alembertian is (ignoring angular derivatives)

$$\Box = -\frac{1}{2}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) + \frac{1}{2r^2}\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\left[\frac{1}{r^2}\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\right]$$

If we act on f with this operator, the resulting expression will not only involve the pair uv but also u and v separately. So the problem has not been reduced to a 1-d one. We thus conclude that in order for $\Box_k$ to work well for functions which are polynomials in the coordinates of cubic or higher order, one must impose the condition that they are of compact support: this condition will ensure that any potential divergences are removed.

4 Conclusion and Outlook

The expressions derived in this paper for the non-local retarded d'Alembertians in 2 and 4 dimensions ((3.4) and (2.9)), have been shown to reproduce the exact continuum d'Alembertians in the limit $l \to 0$. The 2-d non-local d'Alembertian in particular was shown to be a good approximation when acting on all functions of the proper time squared. The 4-d non-local d'Alembertian instead was shown to work well only when acting on functions which are polynomials of the coordinates of degree two or less.

4.1 The 4-d discrete non-local d'Alembertian

In §2.1 we noted how a modification of (2.1) was required in order to dampen the fluctuations of B_k in the large N limit. We did not repeat this procedure for the 4-d case because we restricted our discussion to a $\Box_k$ with the world function σ as the "volume" in the exponent. This meant that there was no straightforward discretisation[1]. We give here the discretised version of $\Box_k$; with $\Box_k$ defined with the actual 4-volume $V(x)$ (as in (3.2) but with the non-locality scale k replacing $1/l^4$ everywhere). This means that our non-locality scale k now has to have dimensions of $1/(\text{length})^4$. We thus now define k to be a quantity of dimension $1/L^4$ such that $1/k^{1/4} \gg l_p$. The most natural generalisation of (2.6) then is

$$B_k\phi(x) = \frac{\sqrt{\epsilon}}{l^2}\left(\alpha\phi(x) + \beta\epsilon\sum_{y\prec x} f(n(x,y),\epsilon)\phi(y)\right), \tag{4.1}$$

[1]The difficulty lies in coming up with a discrete expression whose expectation value with respect to a Poisson distribution gives the square root of the volume in the exponent. For this to be possible one needs to somehow "invert" the Poisson distribution in order to find a random variable whose mean is the square root of the volume and functions thereof.

where $\epsilon = l^4 k$, α and β are constants to be determined by acting with B_k on suitable test functions, and

$$f(n,\epsilon) = (1-\epsilon)^n \left(1 - \frac{3\epsilon n}{1-\epsilon} + \frac{3\epsilon^2 n(n-1)}{2(1-\epsilon)^2} - \frac{\epsilon^3 n(n-1)(n-2)}{6(1-\epsilon)^3}\right). \quad (4.2)$$

Just as in 2-d it is possible to show that the expectation value of this quantity with respect to the Poisson distribution gives

$$\Box_k \phi(0) = \alpha\sqrt{k}\phi(0) + \beta k^{3/2} \int_{x \in J^-(0)} d^4x \; \phi(x) e^{-kV(x)} (1 - 3kV(x) + \frac{3}{2}k^2 V(x)^2 - \frac{1}{6}k^3 V(x)^3) \quad (4.3)$$

which is precisely (3.2) with $1/l^4$ replaced by k. This confirms that (4.1) is the correct discrete version of the non-local retarded continuum d'Alembertian in 4-d.

The issue of checking that the (4.3) does indeed approximate the exact continuum d'Alembertian is currently being addressed by the author and will appear in future work. This would be of great importance since it would provide us with a well defined discrete causet expression for the d'Alembertian, i.e. (4.1).

4.2 Propagating fields

The 2-d discrete non-local retarded operator (2.6) can be used to propagate scalar fields on fixed causal sets C that are well approximated by a Minkowski spacetime $\mathbb{M}^2$.[2] This can be done in the following way: let Σ be a *maximal anti-chain* in C. An *anti-chain* is a set of causally unrelated elements; a *maximal* anti-chain is an anti-chain such that any element of C not in it is related to it. Anti-chains are the natural analogues of spatial hypersurfaces in the causal set. Suppose we know the value of ϕ in the whole of the causal past of Σ, i.e. we know $\phi(y)$ for all $y \in J^-(\Sigma)$. We may then compute ϕ at any point $x \in J^+(\Sigma)$ such that x is linked to at least one $y \in \Sigma$ by imposing the condition: $B_k\phi(x) = 0$. This gives us the value

[2] The equivalent expression in 4-d, i.e. (4.1), could be used to propagate scalar fields on causets which are well approximated by $\mathbb{M}^4$. Before this can be done however one must check that its mean, (4.3), does indeed approximate the exact d'Alembertian and determine the unknown coefficients α and β.

of the field at x in terms of a sum over elements in $J^-(\Sigma)$:

$$\Box_k\phi(x) = -\frac{2\epsilon}{l^2}\phi(x) + \frac{4\epsilon^2}{l^2}\sum_{y\in J^-(\Sigma)} f(n(x,y),\epsilon)\phi(y) = 0$$
$$\Rightarrow \quad \phi(x) = 2\epsilon \sum_{y\in J^-(\Sigma)} f(n(x,y),\epsilon)\phi(y) \tag{4.4}$$

This procedure can then be iterated to give the propagation of the field on the causet.

4.3 Other Dimensions

4.3.1 Even Dimensions

In this dissertation we only worked in even dimensions, namely 2-d and 4-d. Generalising to higher (even) dimensions is straight forward: one simply needs to redefine k to be of the right dimensionality, redefine $\hat{O}$ in such a way that it will kill unwanted terms in the integral $I(k)$ (generalised to higher dimensions), and find the right coefficient in front of the delta function term. The general expression for $\hat{O}$ in $2n$-dimensions is

$$\hat{O}_{2n} = \frac{1}{n+1}(H+(n+1))(H+n)\dots(H+1) \tag{4.5}$$

where $H = k\partial/\partial k$ is the usual homogeneity operator. It remains to be checked however that the higher dimensional non-local retarded d'Alembertians enjoy all the advantages of the two- and four-dimensional operators discussed above.

4.3.2 Odd Dimensions

In odd dimensions, say 3-d, there is no natural choice of $\hat{O}$ in between the 2-d and 4-d case (similarly for any other odd dimension lying between $\hat{O}_{2n}$ and $\hat{O}_{2n+2}$). To see what the the right choice might be one needs to go back to the 1-dimensional scenario. As noted in §2.1 the discretisation of the 1-d d'Alembertian (in the causet) will be given by something like

$$B_k\phi(x) = \frac{1}{l^2}\left(\phi(w) - 2\phi(y) + \phi(z)\right) \tag{4.6}$$

where $x, w, y, z \in C$ such that $z \prec * y \prec * w \prec * x$. The obvious difference between this expression and the 2-d one (apart from the sum over layers) is the fact that this expression is missing the "delta function" term. This means that the field cannot be propagated forward iteratively. Note that the coefficient pattern is exactly the same as that in the 2-d case. Given this result we put forward the following conjecture: the $(2n+1)$-dimensional d'Alembertian is the same as the $(2n+2)$-dimensional one except that it will not possess the delta function term. If this conjecture is true then one cannot propagate scalar fields in odd dimensions forward in time using the procedure defined in §4.2.

4.4 Outlook: Curved manifolds and the Ricci Scalar

In this thesis we have concentrated solely on the d'Alembertian in flat space-times or causets which approximate flat space-times. The next step would be to generalise these operators to spacetimes with curvature (but not curvature that is large compared to the non-locality scale k one is working with). If these approximate d'Alembertians hold good for curved spacetimes also then they form the beginning for a new approach to causal set dynamics. To see why consider the field $\Omega(x) := \Box\sigma(0, x)$. It is shown in [8] that the d'Alembertian of this field at the origin gives the scalar curvature there:

$$R(0) = \Box\Omega(x)\big|_{x=0} \tag{4.7}$$

The geodesic distance between two timelike points in a causet can be estimated, it has been conjectured, independently of curvature. So if we have a d'Alembertian which works for fields in curved spacetimes, then we also have a way of computing the scalar curvature. This would then provide the first step in the construction of an action for causal sets.

A Useful Integrals

We compute integrals of the form

$$I_R^{nm}(k) = \int_{-R}^{0} du \int_{-R}^{0} dv \; u^n v^m e^{-kuv} \tag{A.1}$$

for any non-negative $n, m \in \mathbb{Z}$ and $R \in \mathbb{R}^+$. We will consider three separate cases.

Case 1: $n = m = 0$.

$$\begin{aligned} \tilde{I}_R^0(k) &= \int_{-R}^{0} du \int_{-R}^{0} dv \; e^{-kuv} \\ &= \int_{-R}^{0} du \left[\frac{-1 + e^{kRu}}{ku} \right] \\ &= \frac{1}{k} \operatorname{Ein}(kR^2) \end{aligned} \tag{A.2}$$

where $\operatorname{Ein}(kR^2)$ is the entire function defined in §2.3.
Case 2: $n = m \neq 0$

$$\begin{aligned} \tilde{I}_R^{nn}(k) &= \int_{-R}^{0} du \int_{-R}^{0} dv \; u^n v^n e^{-kuv} \\ &= (-)^n \frac{\partial^n}{\partial k^n} \int_{-R}^{0} du \int_{-R}^{0} dv \; e^{-kuv} \\ &= (-)^n \frac{\partial^n}{\partial k^n} \left(\frac{1}{k} \operatorname{Ein}(kR^2) \right) \\ &= (-)^n \sum_{m=0}^{n} \binom{n}{m} \left[\left(\frac{\partial^{n-m}}{\partial k^{n-m}} \frac{1}{k} \right) \left(\frac{\partial^m}{\partial k^m} \operatorname{Ein}(kR^2) \right) \right] \\ &=: (-)^n \sum_{m=0}^{n} \binom{n}{m} p^{(n-m)}(k) q^{(m)}(k) \end{aligned} \tag{A.3}$$

where we have defined the two functions $p(k) := 1/k$, $q(k) := \operatorname{Ein}(kR^2)$ and

$f^{(n)}(k)$ denotes the nth partial derivative of $f(k)$ with respect to k:

$$f^{(n)}(k) = \frac{\partial^n f}{\partial k^n}(k)$$

.

Case 3: $n > m$, $m \geq 0$

$$\begin{aligned}
\tilde{I}_R^{nm}(k) &= \int_{-R}^{0} du \int_{-R}^{0} dv\; u^n v^m e^{-kuv} \\
&= (-)^m \frac{\partial^m}{\partial k^m} \int_{-R}^{0} du \int_{-R}^{0} dv\; u^{n-m} e^{-kuv} \\
&= (-)^{m+1} \frac{\partial^m}{\partial k^m} \int_{-R}^{0} du\; \frac{u^{n-m-1}}{k} \left(1 - e^{kRu}\right) \\
&= (-)^{m+1} \frac{\partial^m}{\partial k^m} \left[\frac{(-)^{n-m+1} R^{n-m}}{(n-m)k} - \frac{1}{kR^{n-m-1}} \frac{\partial^{n-m-1}}{\partial k^{n-m-1}} \int_{-R}^{0} du\; e^{kRu} \right] \\
&= \frac{(-)^{n+m} m! R^{n-m}}{(n-m)k^{m+1}} + (-)^m \frac{\partial^m}{\partial k^m} \left[\frac{1}{kR^{n-m}} \frac{\partial^{n-m-1}}{\partial k^{n-m-1}} \left(\frac{1 - e^{-kR^2}}{k} \right) \right] \\
&= \frac{m!(-)^{n+m} R^{n-m}}{(n-m)k^{m+1}} + (-)^m \frac{\partial^m}{\partial k^m} \left[\frac{1}{kR^{n-m}} \frac{(-)^{n-m-1}(n-m-1)!}{k^{n-m}} \right] + O(e^{-k}) \\
&= \frac{m!(-)^{n+m} R^{n-m}}{(n-m)k^{m+1}} + \frac{(-)^{n-1}(n-m-1)!}{R^{n-m}} \frac{\partial^m}{\partial k^m} \left(\frac{1}{k^{n-m+1}} \right) + O(e^{-k})
\end{aligned} \tag{A.4}$$

the case where $m > n$, $n \geq 0$ is equivalent to the above so we will not reproduce it here.

The $\mathrm{Ein}(x)$ function may be re-written in terms of the $\mathrm{E}_1(x)$ function

$$\mathrm{Ein}(x) = \mathrm{E}_1(x) + \ln x + \gamma \tag{A.5}$$

where

$$\mathrm{E}_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$$

and γ is the Euler-Mascheroni constant. It is clear from the definition of $\mathrm{E}_1(x)$ that this function will go to zero as $x \to \infty$. This implies that $\mathrm{Ein}(x) \to \ln x$ for large values of x and hence

$$\frac{\mathrm{Ein}(x)}{x} \to 0 \qquad \text{as } x \to \infty$$

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