Stochastic analysis of fractal-generated turbulence

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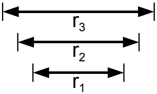
Outline

- Introduction: stochastic description of the turbulent (velocity increment) cascade by a Fokker-Planck equation
- Analysis of fractal-generated turbulence for different Re_λ:
- Markov properties
- Estimation of drift and diffusion functions
- Comparison to jet turbulence

The turbulent cascade as a stochastic process

- velocity increment:

$$u_r(x) = v(x + r/2) - v(x - r/2)$$



$$r_1 < r_2 < r_3 ...$$

- Markov properties:

$$p(u_1, r_1 | u_2, r_2) = p(u_1, r_1 | u_2, r_2; u_3, r_3; ...; u_n, r_n)$$

- n-scale joint pdfs factorize:

$$p(u_1, r_1; u_2, r_2; ...; u_n, r_n) =$$

$$p(u_1, r_1 | u_2, r_2) \cdot p(u_2, r_2 | u_3, r_3) \cdot ... \cdot p(u_{n-1}, r_{n-1} | u_n, r_n) \cdot p(u_n, r_n)$$

- stochastic process:

$$\frac{\partial}{\partial r} p(u, r | u_0, r_0) = ...?$$

Kramers-Moyal expansion – Fokker-Planck equation

Kramers-Moyal expansion:

$$-\frac{\partial}{\partial r} p(u, r|u_0, r_0) = \sum_{k=1}^{\infty} \left(-\frac{\partial}{\partial u}\right)^k D^{(k)}(u, r)\ p(u, r|u_0, r_0)$$

$$D^{(k)}(u, r) = \lim_{\Delta r \to 0} \frac{1}{k!\,\Delta r} \underbrace{\int (u_{r+\Delta r} - u_r)^k\, p(u_{r+\Delta r}, r + \Delta r|u_r, r)\ du_{r+\Delta r}}_{\langle (u_{r+\Delta r} - u_r|u_r)^k \rangle}$$

Kramers-Moyal expansion – Fokker-Planck equation

Kramers-Moyal expansion:

$$-\frac{\partial}{\partial r} p(u, r | u_0, r_0) = \sum_{k=1}^{\infty} \left(-\frac{\partial}{\partial u} \right)^k D^{(k)}(u, r) \; p(u, r | u_0, r_0)$$

$$D^{(k)}(u, r) = \lim_{\Delta r \to 0} \frac{1}{k! \, \Delta r} \underbrace{\int (u_{r+\Delta r} - u_r)^k \, p(u_{r+\Delta r}, r + \Delta r | u_r, r) \; du_{r+\Delta r}}_{\langle (u_{r+\Delta r} - u_r | u_r)^k \rangle}$$

If (a) Markov properties, and (b) $D^{(4)} = 0 \quad \Rightarrow$

Fokker-Planck equation:

$$-\frac{\partial}{\partial r} p(u, r | u_0, r_0) = \left[-\frac{\partial}{\partial u} D^{(1)}(u, r) + \frac{\partial^2}{\partial u^2} D^{(2)}(u, r) \right] \; p(u, r | u_0, r_0)$$

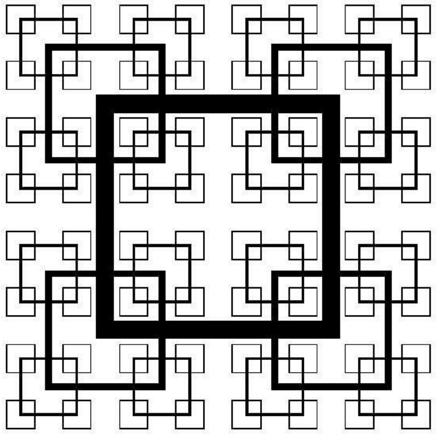
Langevin equation:

$$-\frac{\partial}{\partial r} u(r) = \frac{1}{r} D^{(1)}(u, r) + \sqrt{\frac{1}{r} D^{(2)}(u, r)} \cdot \Gamma(r)$$

Data:

fractal square grid data
(Seoud, Vassilicos):

data	$\bar{v}[m/s]$	$x[cm]$	Re_λ
1	8.6	120	354
2	17.4	300	425
3	19.1	300	554



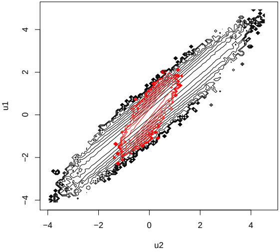
helium free jet data
(Chanal, Chabaud):

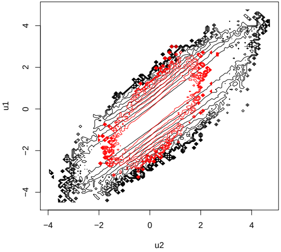
data	Re_λ
1	325
2	458
3	680

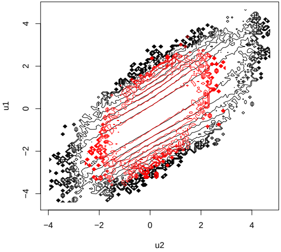
Markov properties

Fractal square grid, $R_\lambda = 425$:

Markov properties: $p(u_1, r_1 | u_2, r_2) = p(u_1, r_1 | u_2, r_2; u_3, r_3)$







$\Delta r =$ 0.5λ 1.0λ 1.5λ

$r_1 = L/2$
$r_2 = r_1 + \Delta r$
$r_3 = r_1 + 2\Delta r$

The Einstein-Markov coherence length (1)

Estimation of the Einstein-Markov length with the **Mann-Whitney-Wilcoxon test**:

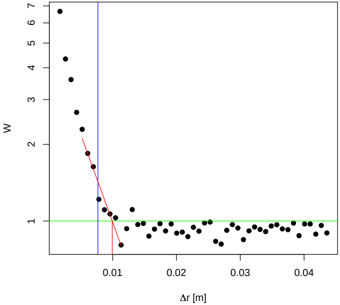
$$
\begin{aligned}
x &= u_1(r_1)|_{u_2(r_2)} \\
y &= u_1(r_1)|_{u_2(r_2),u_3(r_3)}
\end{aligned}
$$

Hypothesis: $p(x) = \tilde{p}(y)$

Define Quantities:

$$
\begin{aligned}
q_i &= \text{Number of values } x_j < y_i. \\
Q &= \sum_i q_i \\
W(r, \Delta r) &= \frac{|Q - \langle Q \rangle_{p=\tilde{p}}|}{\sigma_Q(m,n)_{p=\tilde{p}}\sqrt{2/\pi}} \\
\langle W \rangle_{p=\tilde{p}} &= 1
\end{aligned}
$$

fractal square grid, $Re_\lambda = 425$

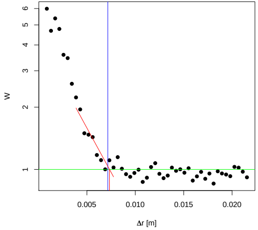


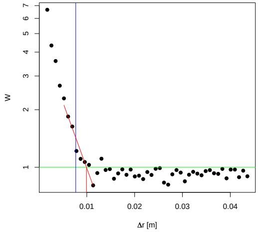
$l_{EM} \approx 1.3\lambda$

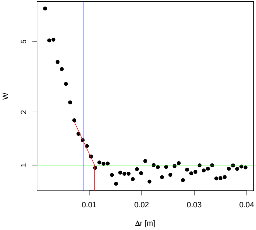
The Einstein-Markov coherence length (2)

Fractal square grid:

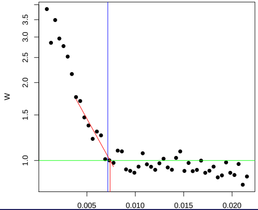
$Re_\lambda = 354$ $Re_\lambda = 425$ $Re_\lambda = 554$ $(u_3 = 0)$

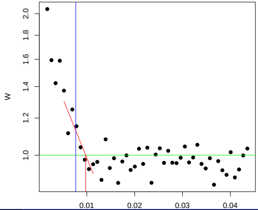


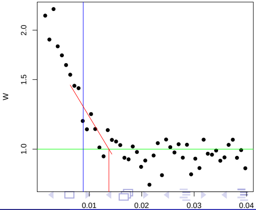




$l_{EM} \approx 1.0\lambda$ $l_{EM} \approx 1.3\lambda$ $l_{EM} \approx 1.2\lambda$ $(u_3 = 1; 2; 2\ m/s)$







Conclusions

- Cascade process for fractal-generated turbulence has Markov properties, $l_{EM} \approx \lambda$.
- Drift and diffusion show the 'usual' dependence on u_r.
- Fractal-generated turbulence can be described by a Fokker-Planck equation.

Estimation of drift and diffusion functions

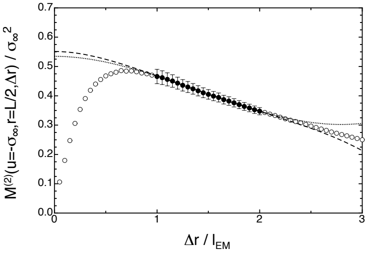
Estimation of $D^{(1)}$ and $D^{(2)}$ directly from the data:

(1) calculate:

$$M^{(k)}(u,r,\Delta r) = \frac{1}{k!\,\Delta r}\langle (u_{r+\Delta r} - u_r | u_r)^k \rangle$$

(2) estimate:

$$D^{(k)}(u,r) = \lim_{\Delta r \to 0} M^{(k)}(u,r,\Delta r)$$



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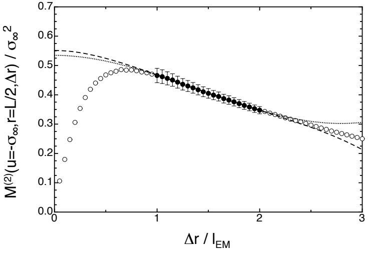
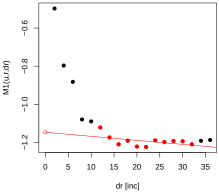
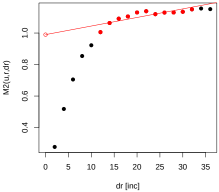


fractal square grid
$Re_\lambda = 425$



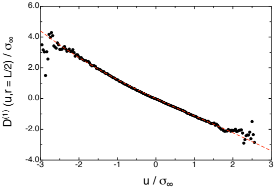
$M^{(1)}$

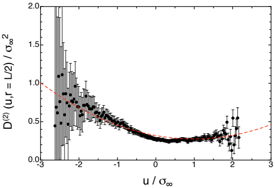


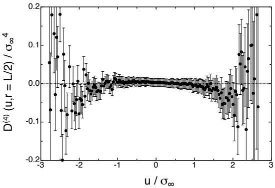
$M^{(2)}$

Drift and diffusion functions (jet)

Example: free jet, $R_\lambda = 190$ *(Ch. Renner, 2002)*:







$D^{(1)} = -d_{11}\,u + d_{12}\,u^2$

$D^{(2)} = d_{20} + d_{21}\,u + d_{22}\,u^2$

$D^{(4)} = 0$

Drift and diffusion functions (jet)

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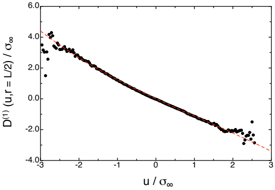
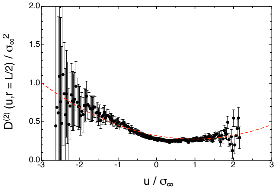
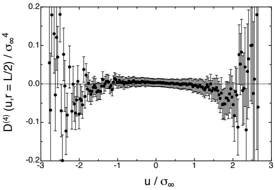
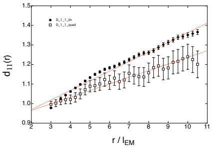
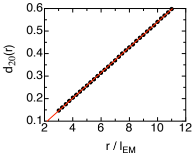


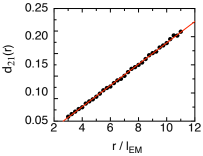


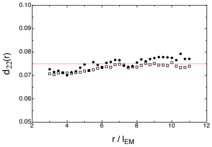


$D^{(1)} = -d_{11}u + d_{12}u^2$ $\quad$ $D^{(2)} = d_{20} + d_{21}u + d_{22}u^2$ $\quad$ $D^{(4)} = 0$





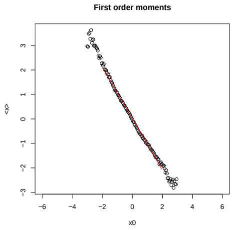


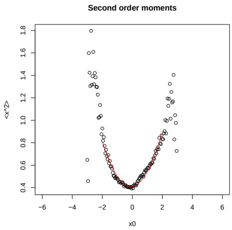


$d_{11}(r)$ $\quad$ $d_{20}(r)$ $\quad$ $d_{21}(r)$ $\quad$ $d_{22}(r)$

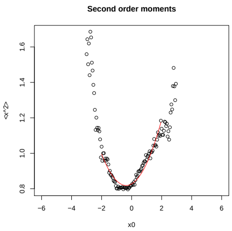
Dependence of $D^{(k)}$ on u

$r = L/2$





$r = L$



$$D^{(1)} = -d_{11}\,u$$

$$D^{(2)} = d_{20} + d_{21}\,u + d_{22}\,u^2$$

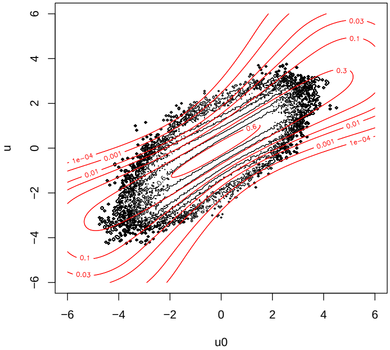
Conclusions

- Cascade process for fractal-generated turbulence has Markov properties, $l_{EM} \approx \lambda$.
- Drift and diffusion show the 'usual' dependence on u_r.
- Fractal-generated turbulence can be described by a Fokker-Planck equation.

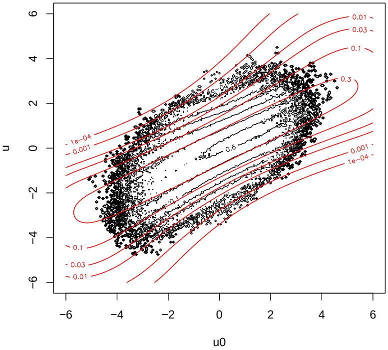
Conditional increment pdfs can be reconstructed by numerical integration of the Fokker-Planck equation.

reconstructed conditional pdfs (fractal grid, $Re_\lambda = 425$)

$p(u, r = 2l_{EM} | u_0, r = 3l_{EM})$



$p(u, r = L - 2l_{EM} | u_0, r = L)$



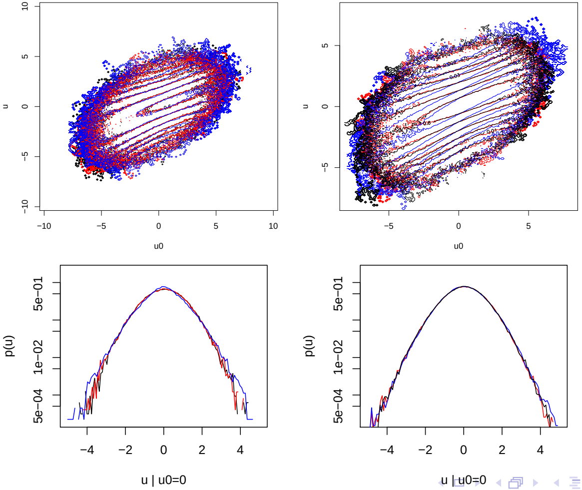
Conclusions

- Cascade process for fractal-generated turbulence has Markov properties, $l_{EM} \approx \lambda$.
- Drift and diffusion show the 'usual' dependence on u_r (and r).
- Fractal-generated turbulence can be described by a Fokker-Planck equation.

Conditional pdfs for different *Re* (jet & fractal grid) (1)

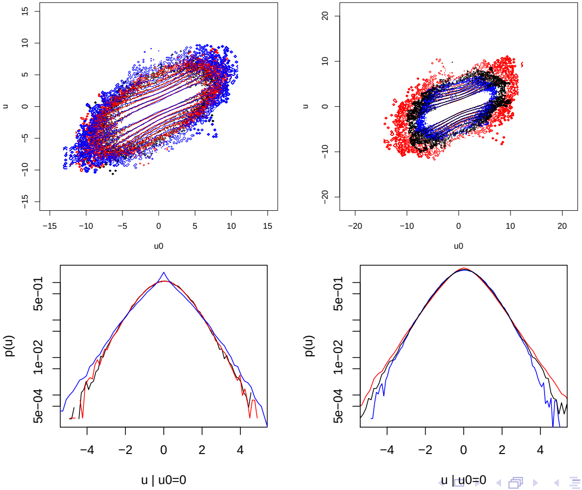
fractal grid | jet

$$p(u, r = L/2 | u_0, r_0 = L)$$



fractal grid   jet

$$p(u, r = l_{EM} | u_0, r_0 = 2l_{EM})$$



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Outlook:

- More data of fractal-generated turbulence (different Re_λ).
- Compare to other flows.
- Better estimation of stochastic process (optimisation).

End

Thanks to R.E. Seoud and J.C. Vassilicos
for fractal grid data and discussions.

Thank you for your attention!