Morphology and dynamics of strongly stratified flows

Toroidal cascade

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Claude Cambon and Fabien S. Godeferd

Laboratoire de Mécanique des Fluides et d'Acoustique
CNRS – ECL – UCB Lyon I

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Context: conceptual approach, not geophysics, but ...

- General theme: rotating stratified flows with two external N and f parameters
- *pure* rotation ($N = 0$): phase-mixing and *weak* wave-turbulence, simple instances
- *Pure* stratification : simple scaling laws, exact Lin-type equations.
- The toroidal cascade as a toy-model : a very complex anisotropic behaviour for *strong* turbulence
- The QG model revisited. Open problems, conclusions

Rotating stratified (unbounded) flows : governing equations

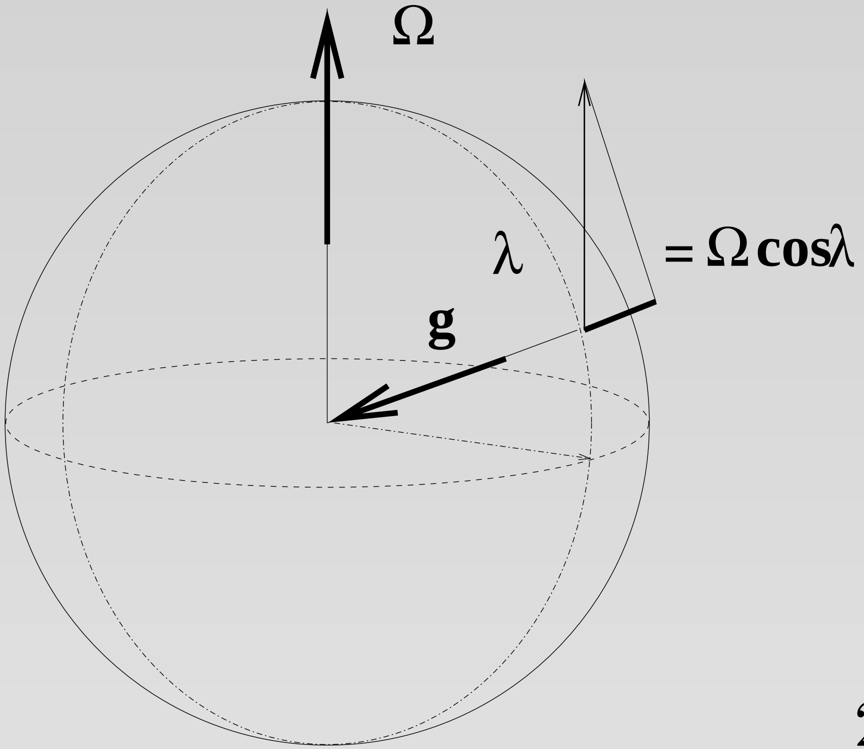
$$\frac{\partial u_i}{\partial t} + \underbrace{f \epsilon_{i3j} u_j}_{\text{Coriolis}} - \underbrace{b \delta_{i3}}_{\text{buoyancy}} + \frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}, \quad \frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial b}{\partial t} + \underbrace{N^2 u_3}_{\text{stratification}} = P_r \nu \nabla^2 b - u_j \frac{\partial b}{\partial x_j}$$

2 external parameters N and f (frequencies)

Valid for a liquid or a gas. P_r characterizes the diffusivity of the stratifying agent (temperature, salt)

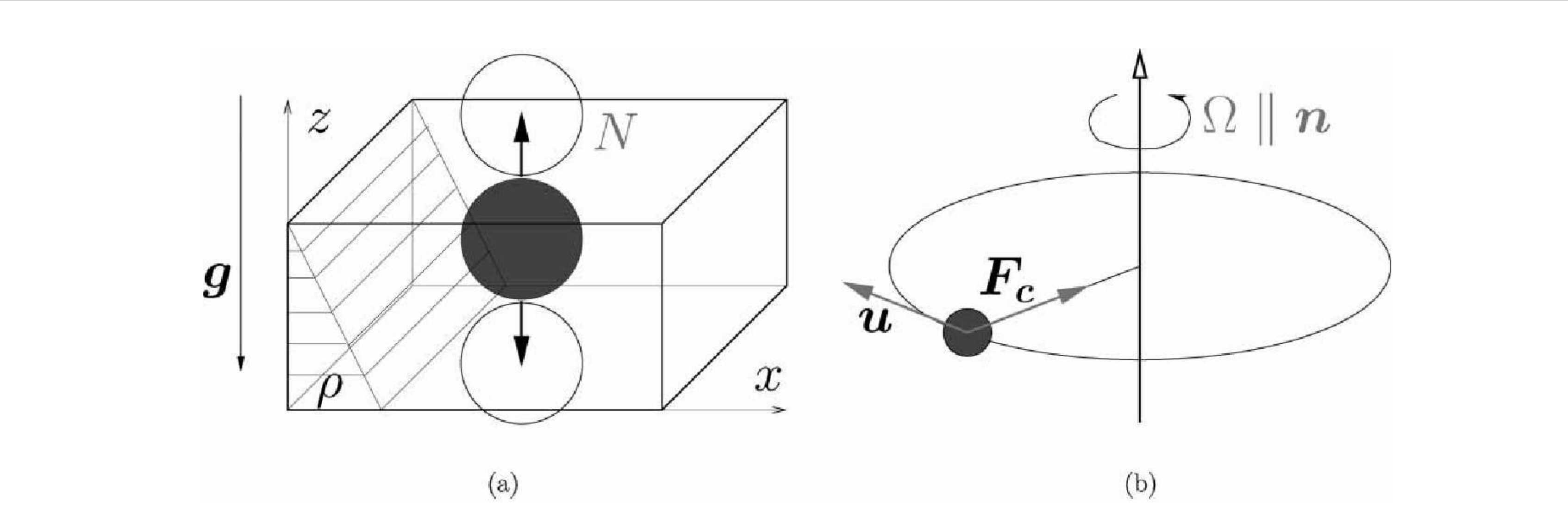
Geophysical context



$$2\Omega \rightarrow f = 2\Omega\cos\lambda$$

Stable stratification in ocean (under the mixed zone) and in atmosphere (temporary inversion in troposphere, stratosphere)

A first cartoon of linear effects



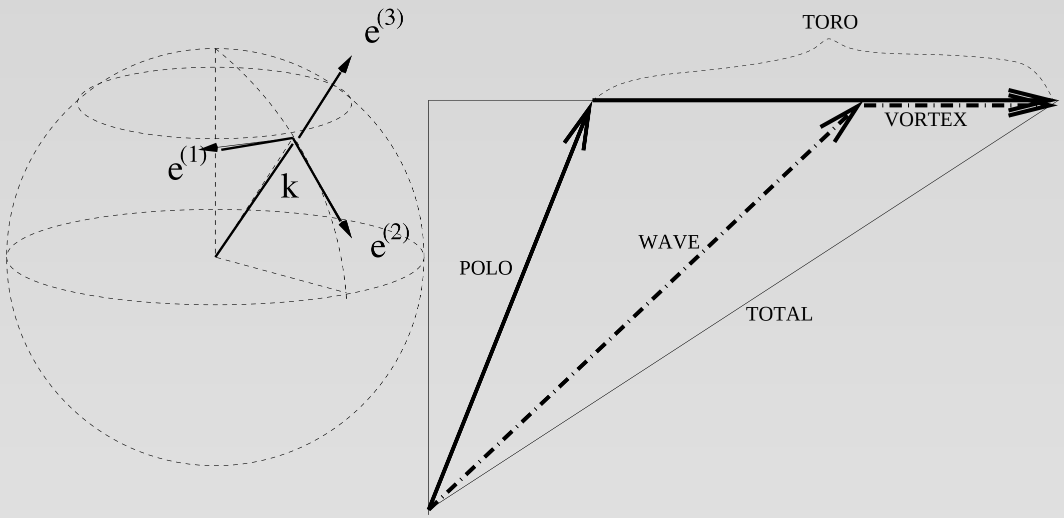
Fluctuating pressure ? incompressibility ?

Eigenmodes decomposition

- incompressibility and pressure $\longrightarrow$ 3D Fourier space
$$(\boldsymbol{u}, b)(\boldsymbol{x}, t) =$$
$$\sum e^{\imath \boldsymbol{k}\cdot\boldsymbol{x}} \left(\underbrace{a_0 \boldsymbol{N}^{(0)}}_{\text{vortex (QG)}} + \underbrace{a_{+1}\boldsymbol{N}^{(1)} e^{\imath\sigma_k t} + a_{-1}\boldsymbol{N}^{(-1)} e^{-\imath\sigma_k t}}_{\text{wave (AG)}} \right)$$
- Dispersion law $\sigma_k = \sqrt{N^2 \sin^2\theta + f^2 \cos^2\theta}$
- Linear dynamics: slow amplitudes $a_{0,\pm 1}$ are constant. Nonlinear case
- Advantages $\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$, five $(u_1, u_2, u_3, b, p) \longrightarrow$ three (a_0, a_{+1}, a_{-1}).

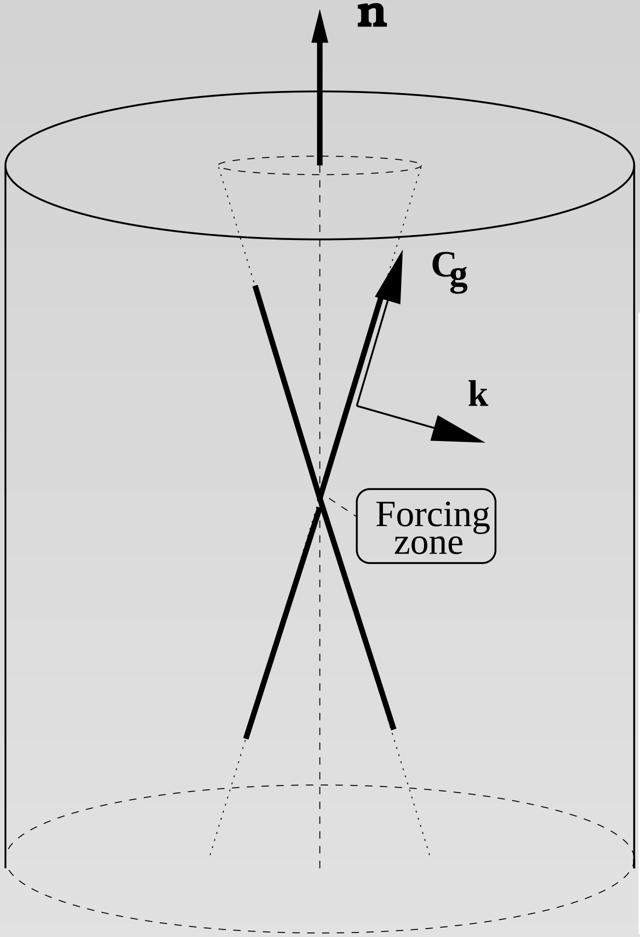
ref. Cambon et al., Bartello, Smith & Waleffe, Morinishi, Kaneda ...etc

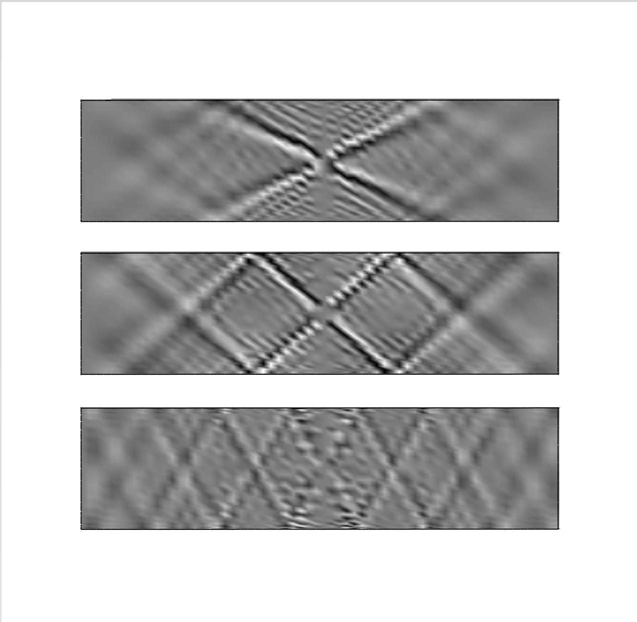
Geometric aspects of eigenmodes



Toroidal/poloidal (Craya-Herring, standard) and “Vortex/wave” ($f/N-$ depending)

Wave aspects





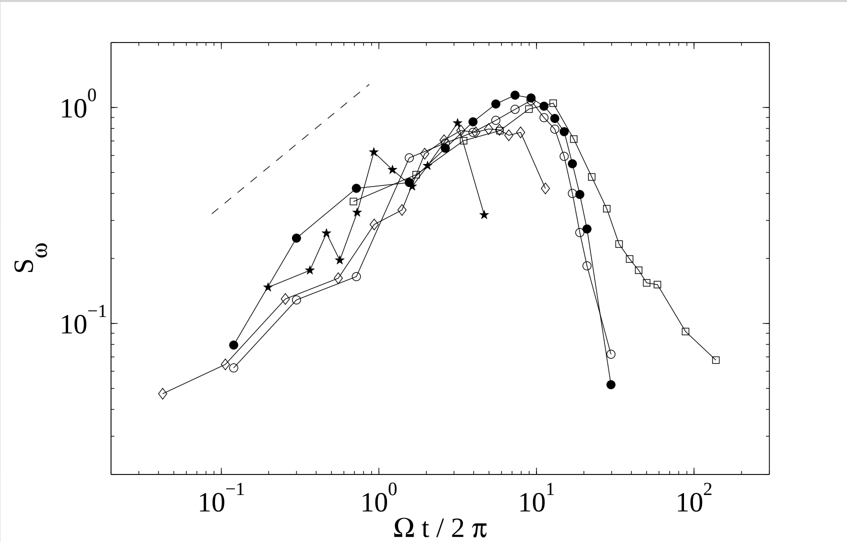
Mc Ewan (1967), Mowbray & Rarity (1967), Godeferd & Lollini, JFM (1999)

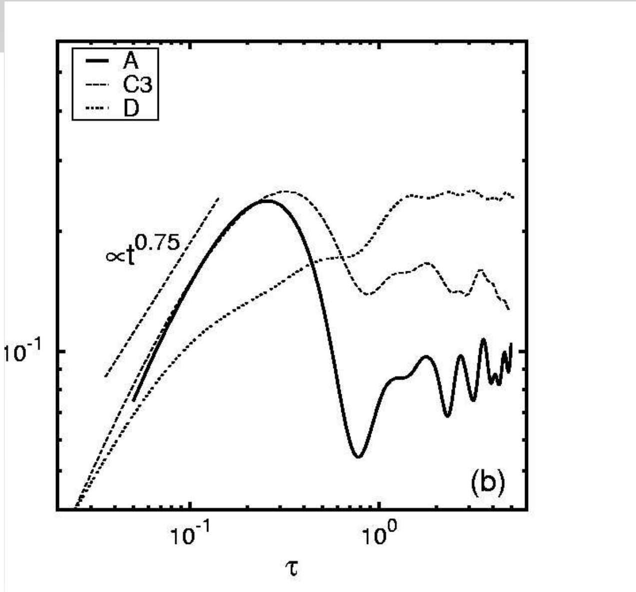
Rotation, phase mixing: Linear and nonlinear

- Relevance of linear solution depends on the *order and type* of statistical correlations
  -) doubles: 2 point 1 time: $e^{\imath\sigma_k t}$, $e^{-\imath\sigma_k t}$
  -) doubles: 2 point 2 time: $e^{\imath\sigma_k (t\pm t')}$
  -) triples: 3 point 1 time: $e^{\imath(\pm\sigma_k \pm\sigma_p \pm\sigma_q)t} \longrightarrow$ nonlinear $\cdots$

$$< \omega_3^3 >= \sum \int \exp[\imath f t (\frac{k_3}{k} + s' \frac{p_3}{p} + s'' \frac{q_3}{q})] S(\boldsymbol{k}, \boldsymbol{p}, \epsilon t) d^3\boldsymbol{p} d^3\boldsymbol{k}$$

needs for initial triple correlations at THREE point. Many other correlations.

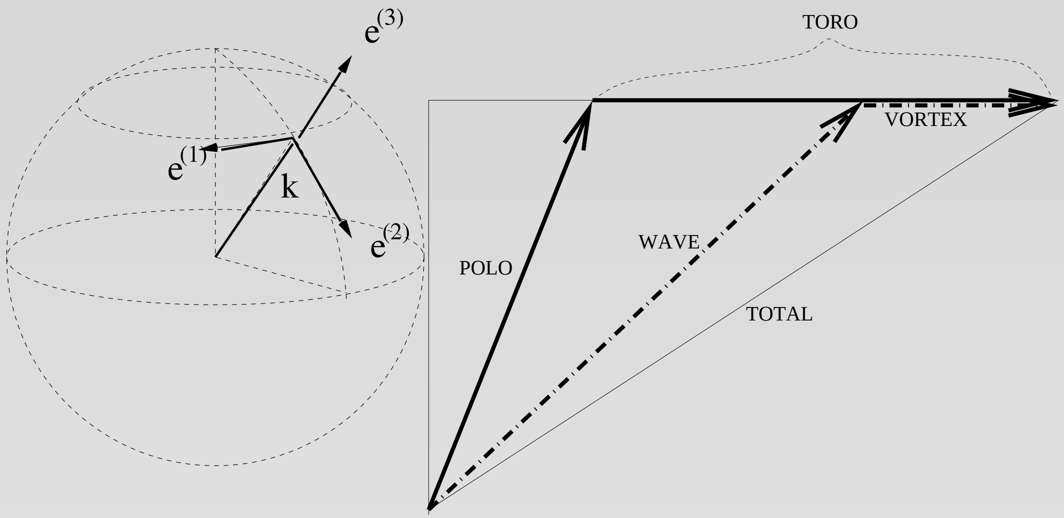




Cyclonic / anticyclonic asymmetry: Bartello et al. (1994), Morize et al. (2005), Gence & Frick (2001), van Bokhoven et al. (2006). No need for centrifugal inst.

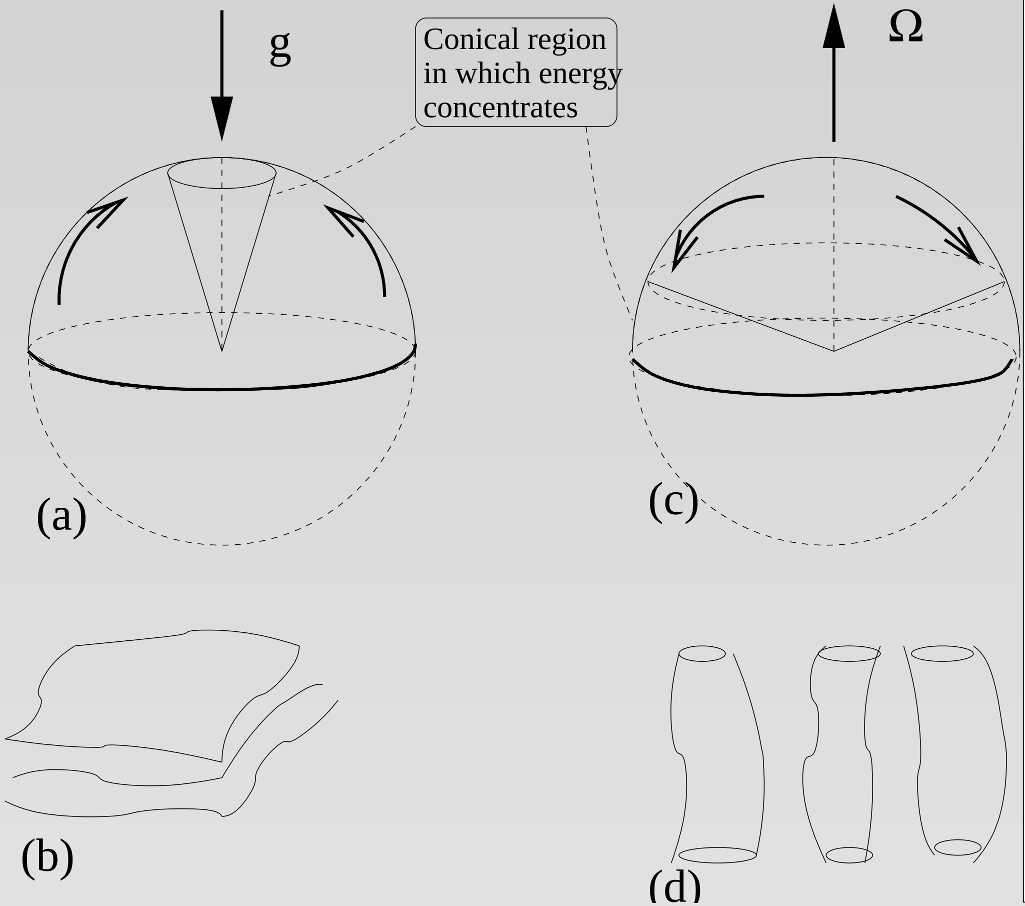
Reintroducing stratification : ANTI 2D and toroidal (strong) cascade

Geometric aspects of eigenmodes

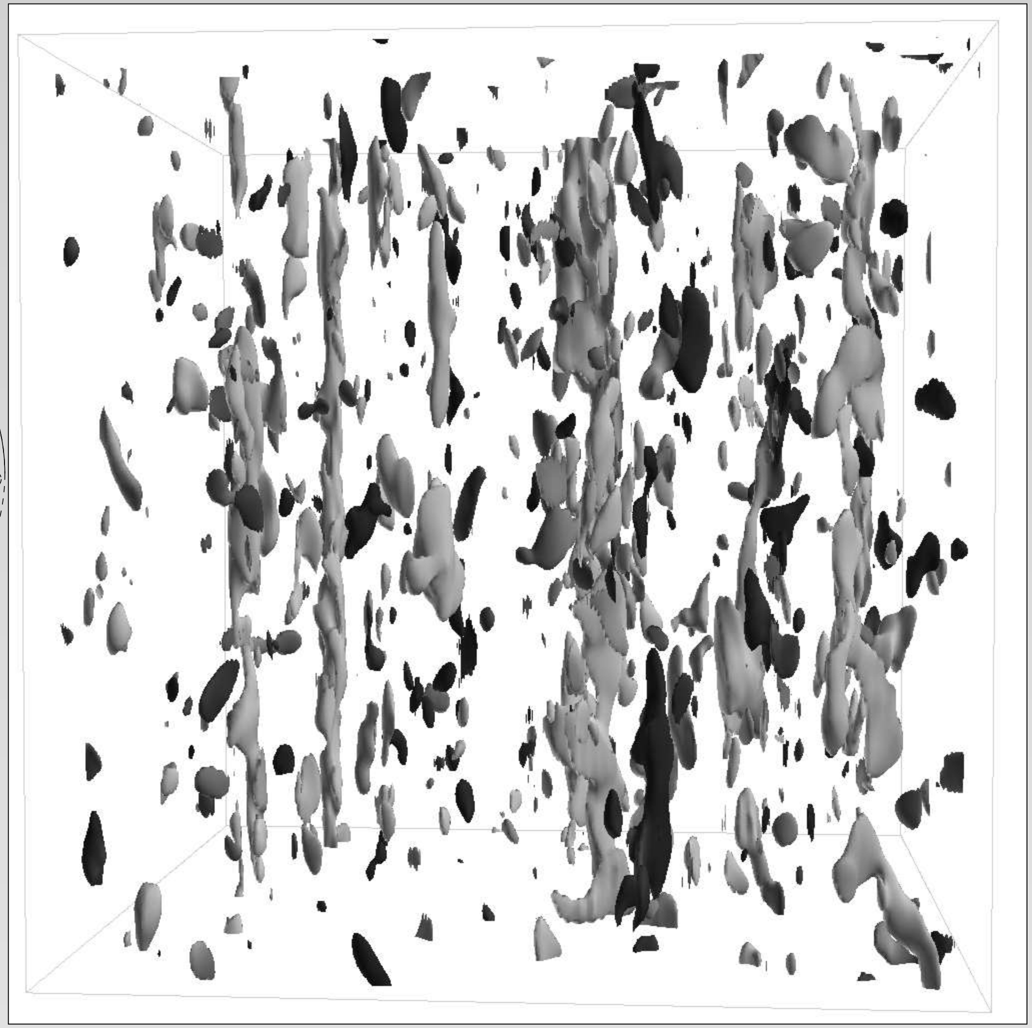


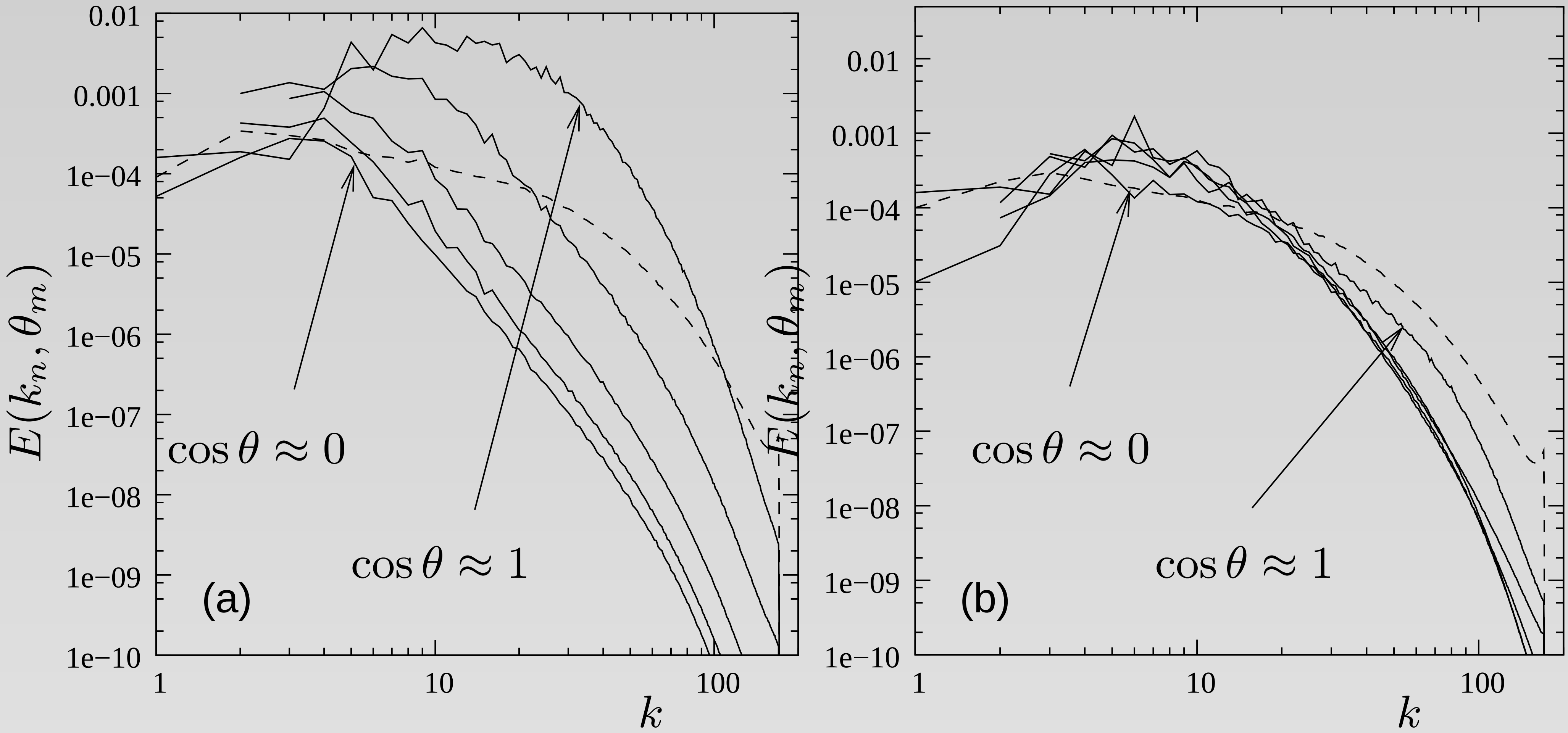
Toroidal/poloidal

(Craya-Herring, standard) and “Vortex/wave” ($f/N-$ depending)

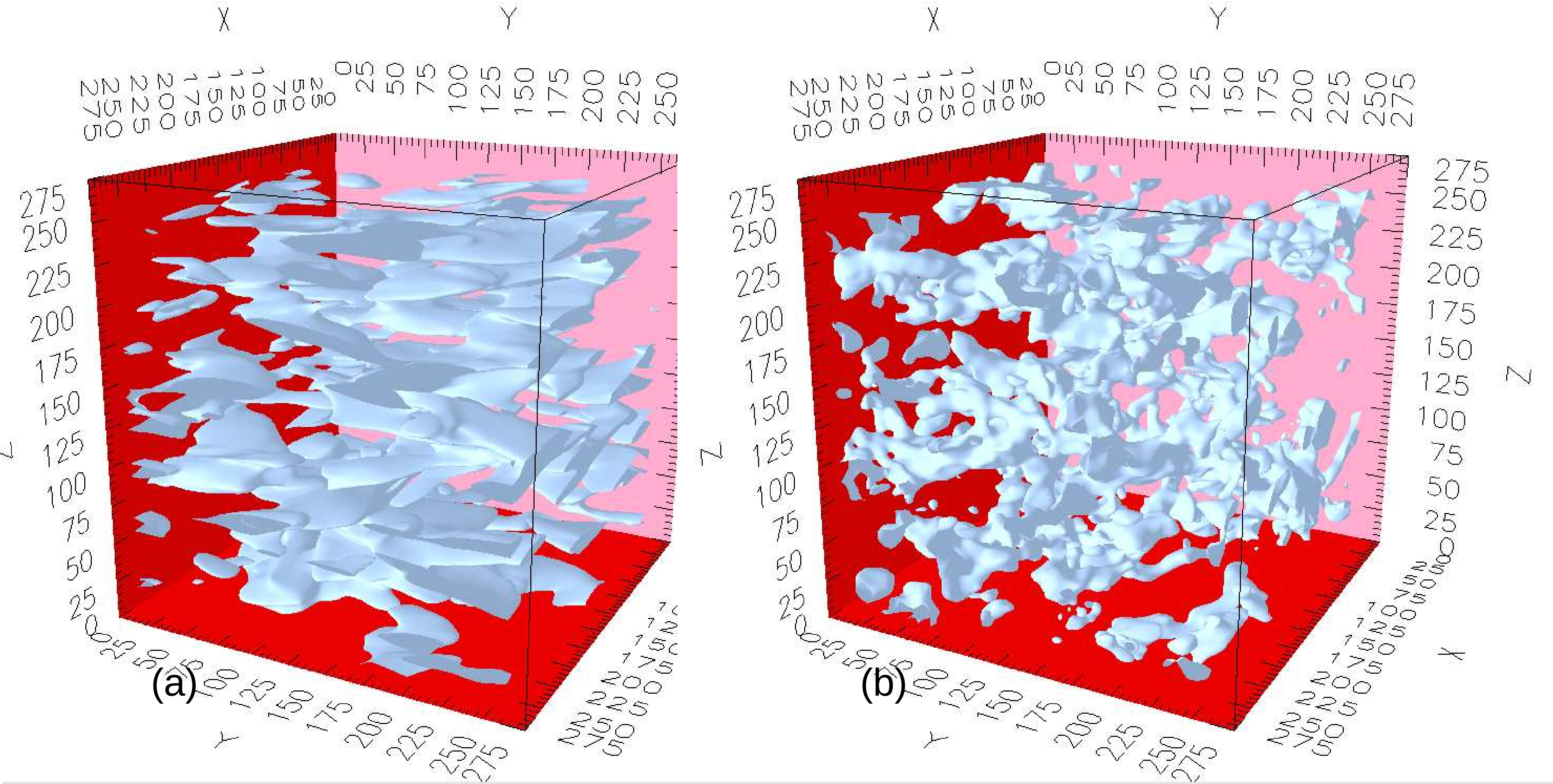


g
Conical region in which energy concentrates
Ω
(a)
(c)
(b)
(d)





Angle-dependent toroidal and poloidal modes (Liechtenstein, 2006)


(a)
(b)

Pure stratification: scaling arguments

- 2D or not 2D ? Charney (1971), Lilly (1983), Lindborg (1999) using third order structure functions from observations.
- Froude numbers, horizontal and vertical, $Fr_h = U/(NL_h)$, $Fr_v = U/(NL_v)$, $L_h \gg L_v$.
- $L_v \sim U/N$ ("zig-zag" instability ? Billant & Chomaz 1999, 2002) $\longrightarrow Fr_v \sim 1$ (to contrast with Riley et al. 1981 ?)
- Proposed scaling (Linborg 2006) $E_{hh}(k_h) = C_1 \varepsilon_k^{2/3} k_h^{-5/3}$, $E_p(k_h) \sim C_2 \epsilon_p \epsilon_K^{-1/3} k_h^{-5/3}$, $E_{vv} \sim N^2 k_v^{-3}$.
- $Ri \sim 1/4$ threshold vs. $Fr^2 Re$ scaling ?, rediscovery of polo-toro decomposition (Brethouwer, Linborg, Billant, Chomaz.)

Pure stratification: Generalized Lin equations

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e^{(tor)} = T^{(tor)} \tag{1}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) e^{(w)} = T^{(w)} \tag{2}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2 + 2\imath N\frac{k_\perp}{k}\right) Z' = T^{(z')} \tag{3}$$

Energy spectra $e^{(tor),(pol),(pot)}(k_\perp, k_\parallel)$, imbalance deviator Z',
$\Re Z' = e^{(pol)} - e^{(pot)}$, $e^{(w)} = e^{(pol)} + e^{(pot)}$.
A lot of information can be generated, vs. second and third-order structure functions.

The toroidal cascade. Why not 2D ?

- Why the toroidal component only ? $\boldsymbol{e}^{(1)} \boldsymbol{\cdot} \ldots \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} + \boldsymbol{\nabla}\left(p + \frac{u^2}{2}\right) - b\boldsymbol{n}$,
  $\dot{u}^{(1)} + \boldsymbol{e}^{(1)} \boldsymbol{\cdot} \sum_{p+q=k} (\hat{\boldsymbol{\omega}}(\boldsymbol{p}) \times \hat{\boldsymbol{u}}(\boldsymbol{q})) = 0$,
  $\hat{\boldsymbol{u}} = u^{(1)}\boldsymbol{e}^{(1)} + u^{(2)}\boldsymbol{e}^{(2)}$, $\hat{\boldsymbol{\omega}} = \imath k \left(u^{(1)}\boldsymbol{e}^{(2)} - u^{(2)}\boldsymbol{e}^{(1)}\right)$

- Following Kraichnan and Waleffe (1992, 1993): stability of a single triad

$$\dot{u}_k^{(1)} = (p_\perp^2 - q_\perp^2) G u_p^{(1)*} u_q^{(1)*}, \tag{4}$$

$$\dot{u}_p^{(1)} = (q_\perp^2 - k_\perp^2) G u_q^{(1)*} u_k^{(1)*}, \tag{5}$$

$$\dot{u}_q^{(1)} = (k_\perp^2 - p_\perp^2) G u_k^{(1)*} u_p^{(1)*}, \tag{6}$$

- quasi 2D or not, reverse or direct cascade ? cylinder to cylinder, shell to shell, angle to angle : very rich and various morphology ...

Analogy with an 'Euler problem'

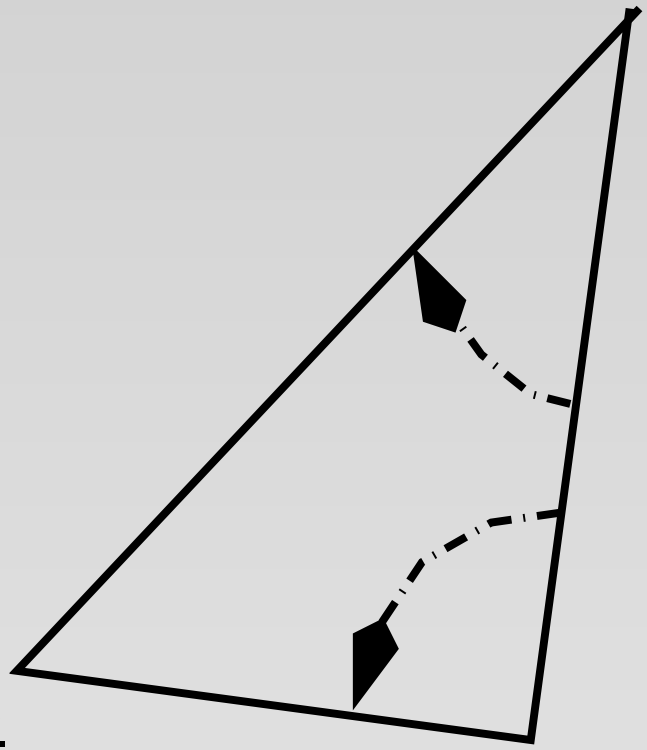
- The solid in its principle axes of inertia

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3, \tag{7}$$

$$I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_3\Omega_1 \tag{8}$$

$$I_3\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2, \tag{9}$$

- Conservations laws, rot. kin. energy $(I_1\Omega_1^2 + \ldots) \longrightarrow$ kin. energy (triad), norm of the angular momentum $(I_1\Omega_1)^2 + \ldots \longrightarrow$ vertical enstrophy (triad)



- Instabilities, reverse interactions only.

Revisiting the QG cascade

- Detailed conservation of QG energy and potential enstrophy (linear ?)

$$\frac{k^2\sigma_k^2}{N^2}\xi^{(0)}\xi^{(0)*} = k_\perp^2 u^{(1)}u^{(1)*} + \left(\frac{f}{N}k_\parallel\right)^2 u^{(3)}u^{(3)*}$$

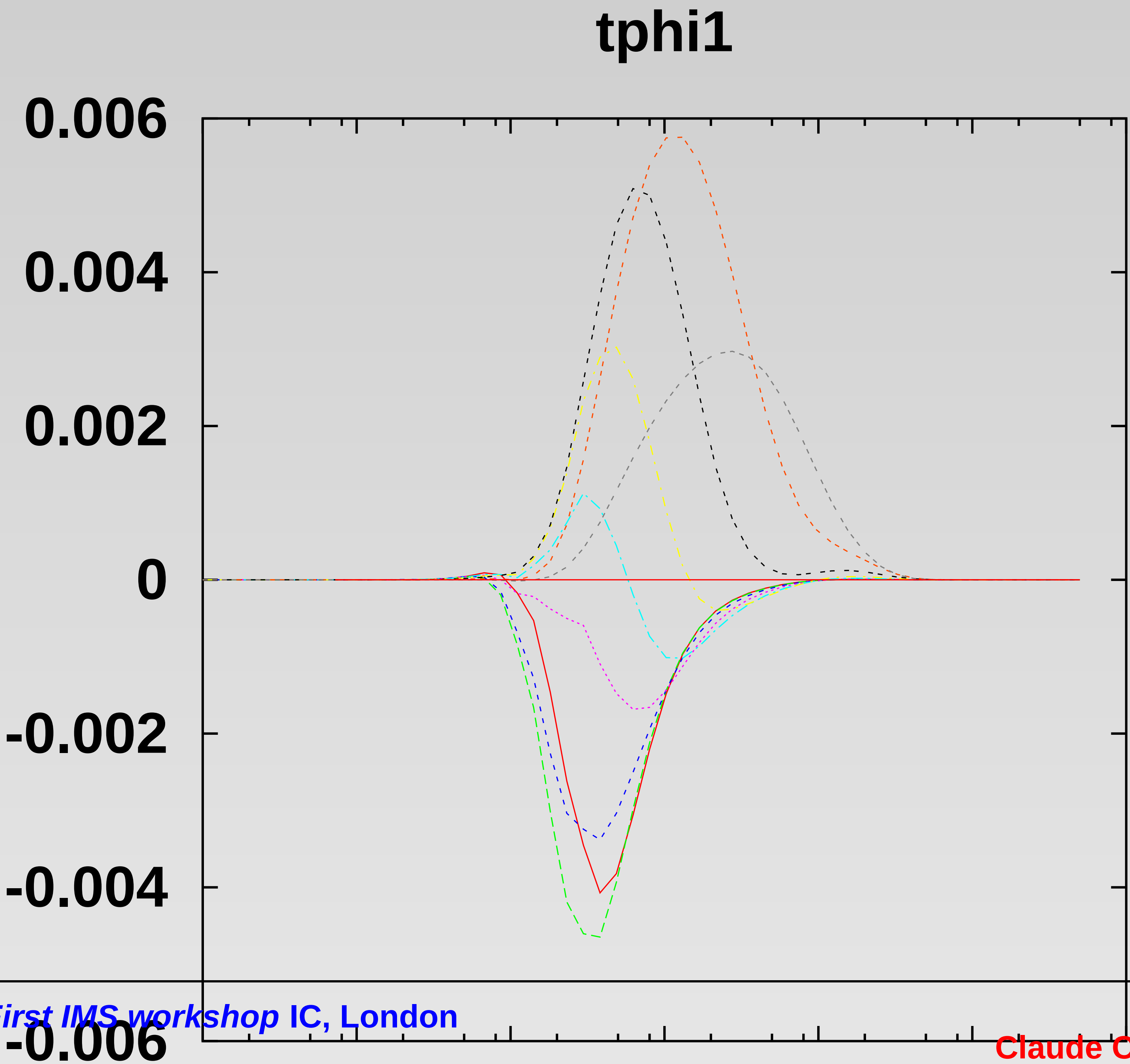
- Reworking on N^{000} (Bartello 1995)

$$\dot{\xi}_k^{(0)} = (p^2\sigma_p^2 - q^2\sigma_q^2)G'\xi_p^{(0)*}\xi_q^{(0)*}, \quad (10)$$

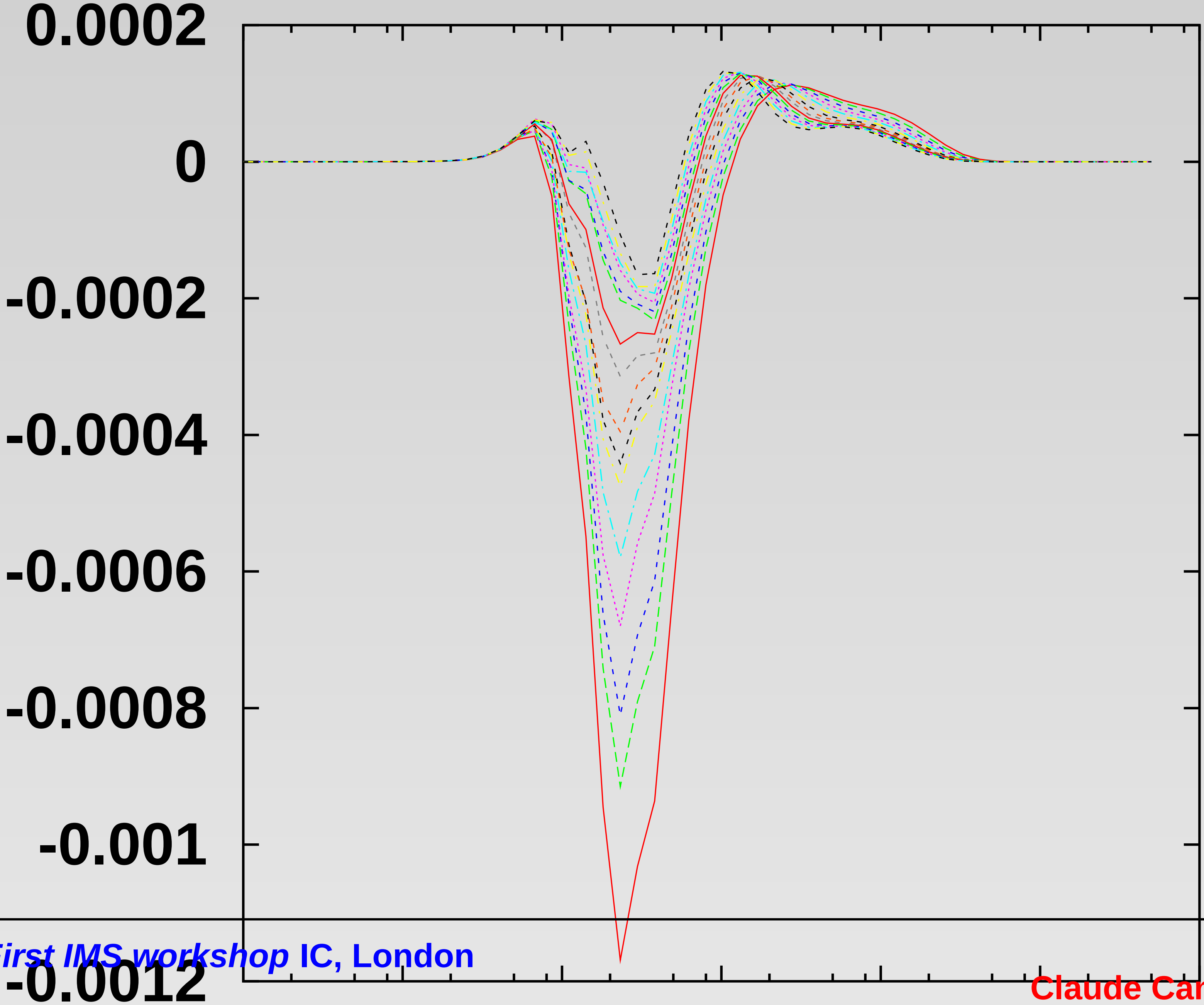
$$\dot{\xi}_p^{(0)} = (q^2\sigma_q^2 - k^2\sigma_k^2)G'\xi_q^{(0)*}\xi_k^{(0)*}, \quad (11)$$

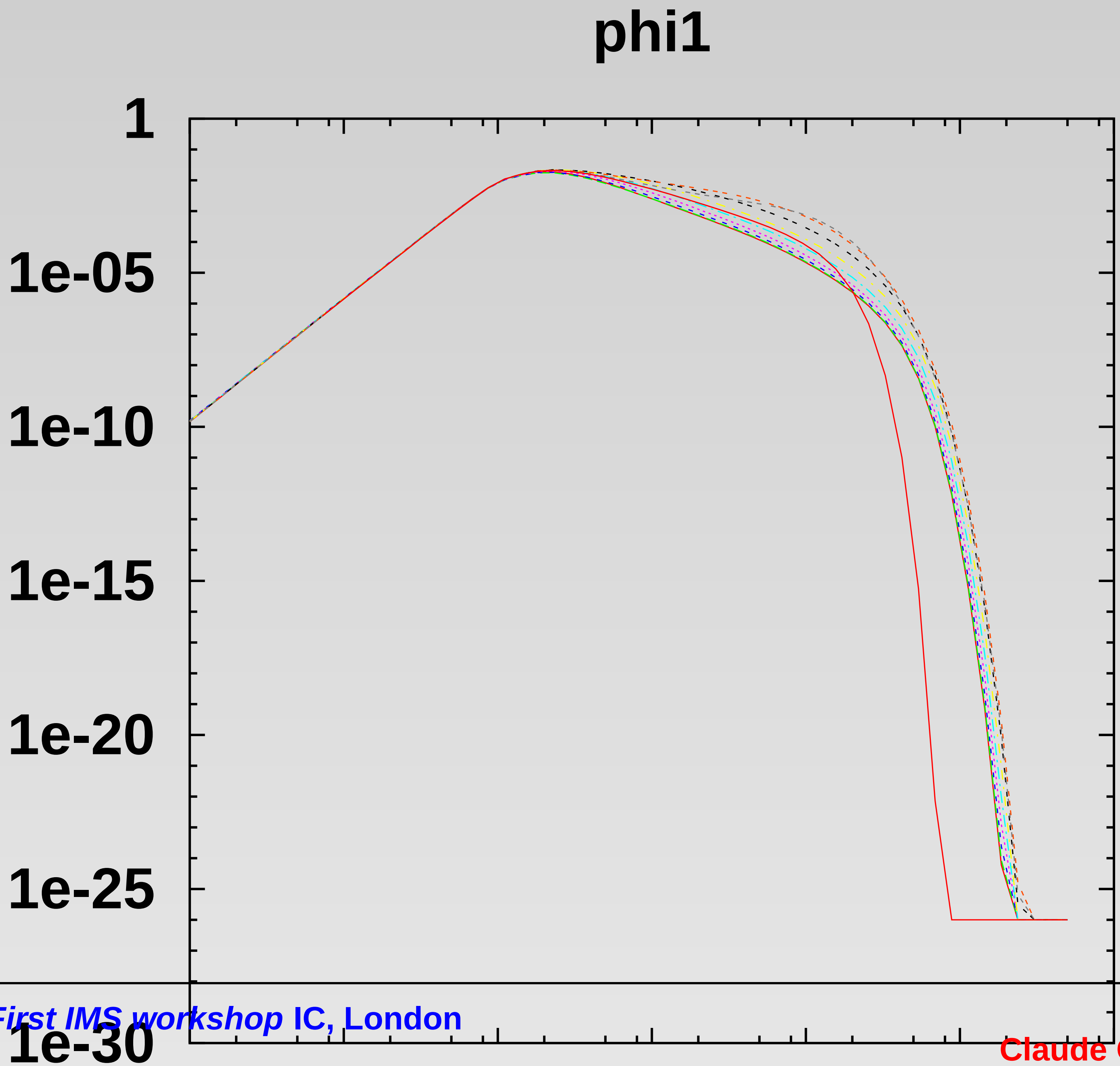
$$\dot{\xi}_q^{(0)} = (k^2\sigma_k^2 - p^2\sigma_p^2)G'\xi_k^{(0)*}\xi_p^{(0)*}, \quad (12)$$

- ‘linear’ (quadratic) limit ? Ertel theorem, flat isopycnes. Dual cascade or not, why ?

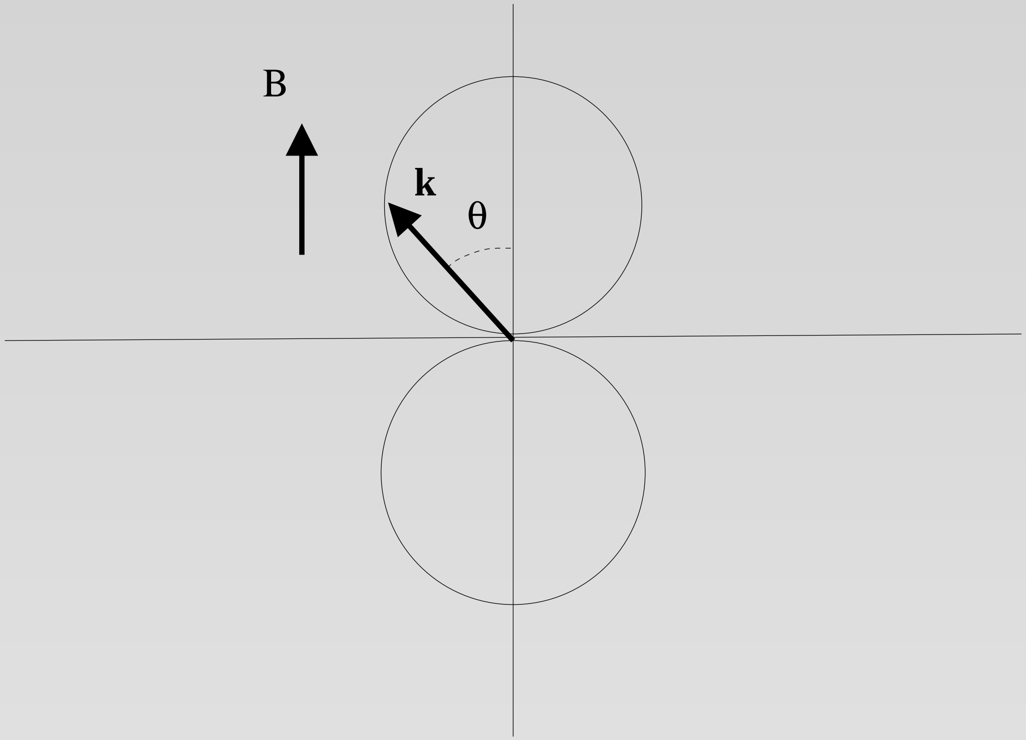

tphi1
0.006
0.004
0.002
0
-0.002
-0.004
-0.006

tspe1




phi1
1
1e-05
1e-10
1e-15
1e-20
1e-25
1e-30

Coexistence of weak and strong turbulence: MHD, aeroacoustics



Alfven-wave turbulence competing with strong turbulence with additional Joule dissipation effect (Moffat 1967, etc)

Concluding comments

1. Stability analysis vs. anisotropic statistical theory
   -) Layering in stratified flows (zig-zag, Kelvin-Helmholtz)
   - Cyclonic/anticyclonic asymmetry in rotating flows (centrifugal)

2. Anisotropic multimodal EDQNM incorporating RDT vs. DNS/LES
   -) A systematic way to construct triadic correlations
   -) Need for improved ED for strong turbulence (TFM, LRA ?)

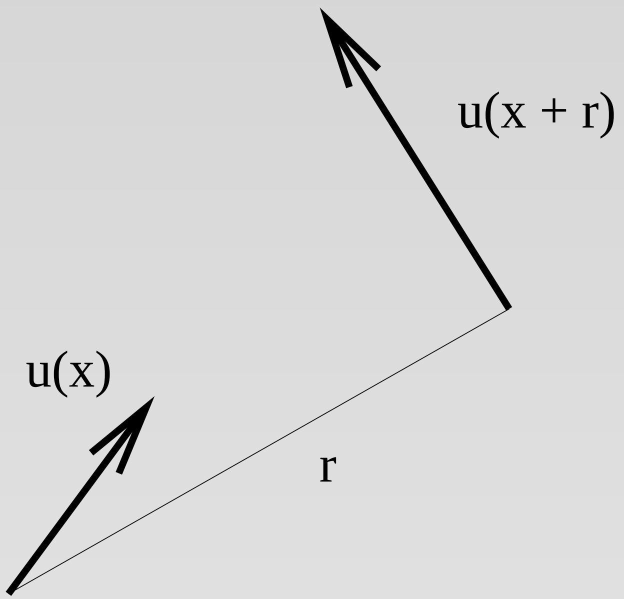
3. Competition between waves and vortices to organise Lagrangian diffusion (also plasmas ?)

4. A better link between physical and spectral space ?

More on strong anisotropy

Anisotropic description

- ANISOTROPY/ inhomogeneity/ Intermitency
- structure functions or correlations, two-point : $R_{ij}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x}+\boldsymbol{r}) \rangle$



-) Single-point: componentality only

-) Two-point: directional anisotropy

- Low dimension parameterization, SO(3) symmetry group (Arad *et al.*,PRE,1999)

Anisotropic description. 3D Fourier space

- Anisotropic scalar (e. g. spherical harmonics) for both 'physical' and 'spectral'

$$\frac{1}{2}R_{ii}(\boldsymbol{r}) \quad \rightarrow \quad \frac{1}{2}\hat{R}_{ii}(\boldsymbol{k}) = e(\boldsymbol{k})$$

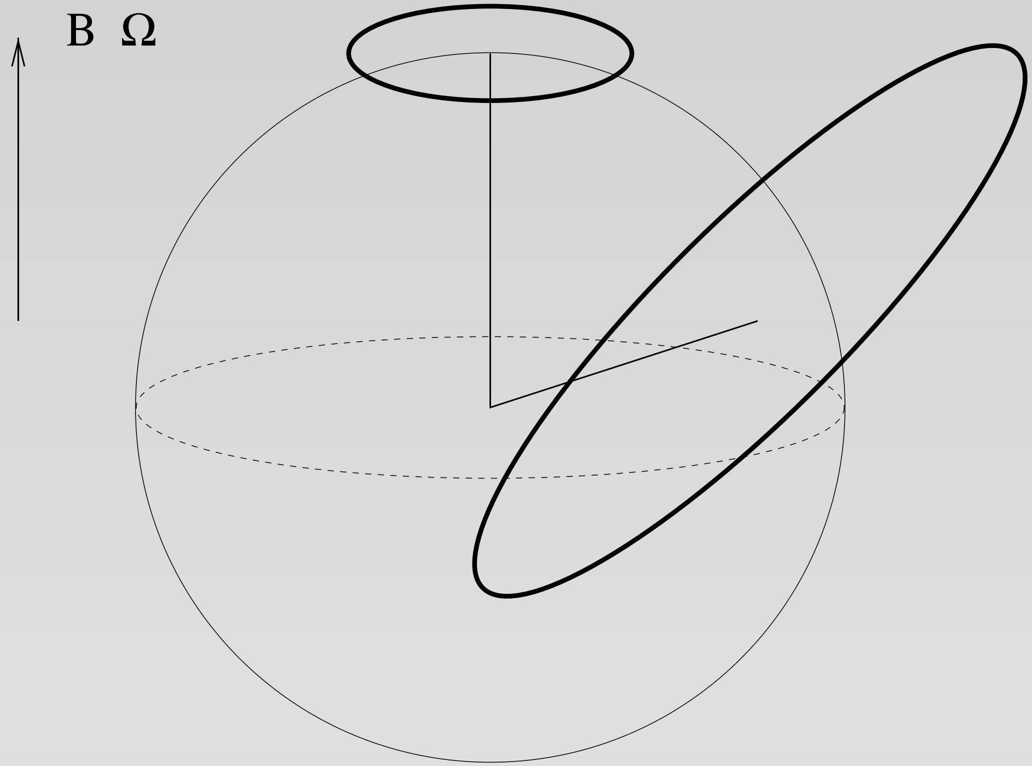
$$\sum r_n^m(r)Y_n^m(\theta_r, \phi_r) \quad \rightarrow \quad \sum \varphi_n^m(k)Y_n^m(\theta_k, \phi_k)$$

Avoiding a 'schizophrenic' viewpoint ! (Cambon & Teissèdre 1985, CRAS Paris)

- A trace-deviator decomposition restricted to solenoidal space

$$\hat{R}_{ij} = \underbrace{\underbrace{U(k)P_{ij}}_{\text{isotropic}} + \underbrace{\mathcal{E}(\boldsymbol{k})P_{ij}}_{\text{directional}}}_{eP} + \underbrace{\Re\left(Z(\boldsymbol{k})N_iN_j\right)}_{\text{polarization}}.$$

(Cambon & Jacquin, JFM, 1989), $P_{ij} = \delta_{ij} - \frac{k_ik_i}{k^2}$, $\boldsymbol{N}$ 'helical mode'. Helicity ?

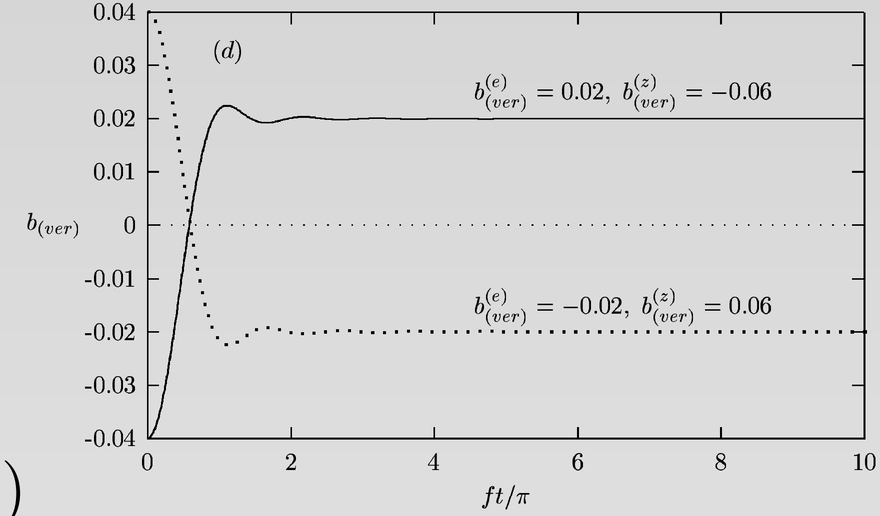


Rotating turbulence, MHD simplified case

More on pure rotation

Pure rotation, phase-mixing

- Basic linear operator $\hat{\boldsymbol{u}}(\boldsymbol{k}, t) = \Re[N_i N_j^* e^{\sigma_k (t-t')}]$, $\sigma_k = f \frac{k_\parallel}{k}$
- Second-order statistics :



$$e(\boldsymbol{k}, t) = e(\boldsymbol{k}, 0) \qquad Z(\boldsymbol{k}, t) = e^{2\imath\sigma t} Z(\boldsymbol{k}, 0)$$

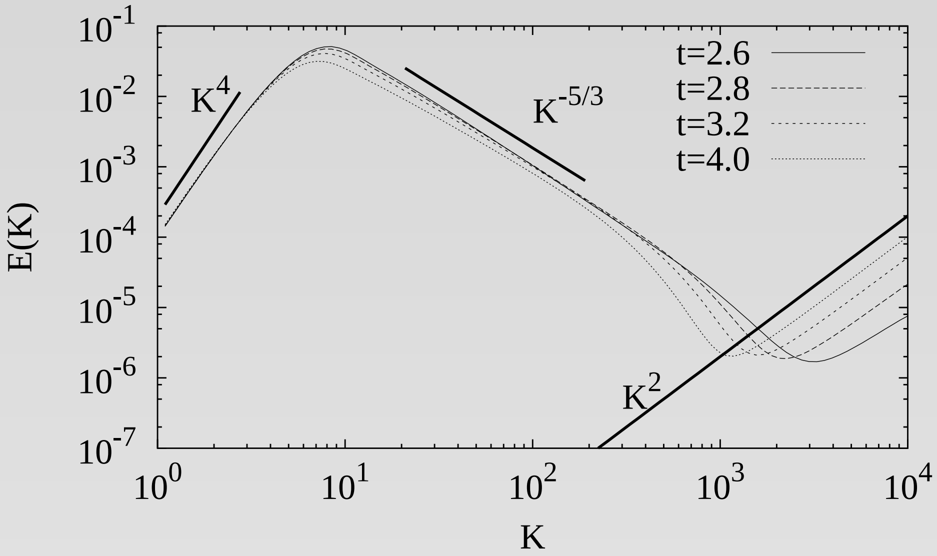
$$\int_0^1 Z(x) \exp(\imath \underbrace{f x}_{\sigma_k} t) dx \to 0; \quad x = \cos\widehat{(\mathbf{k}, \mathbf{n})}$$

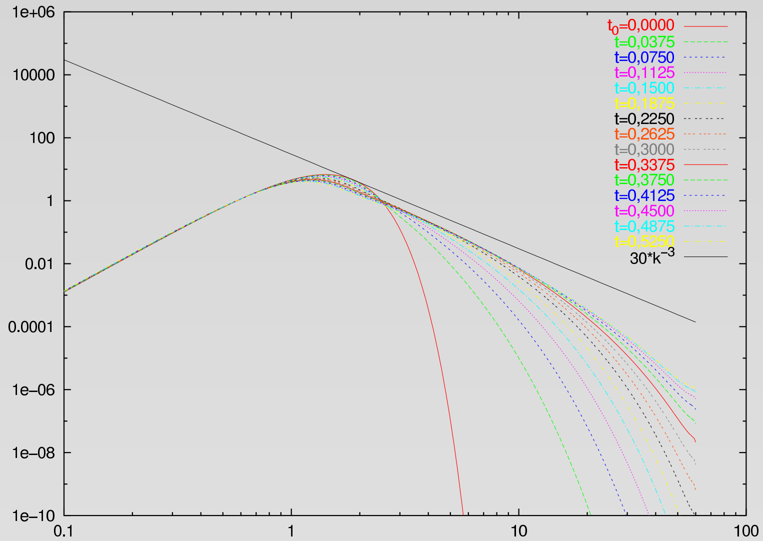
Phase-mixing for two-time second order statistics

Applications to single-particle Lagrangian diffusion (at the end)

Results. NONLINEAR statistical theory

- From classical EDQNM (isotropic, no rotation, Orszag 1970, Bos & Bertoglio, 2006) ...

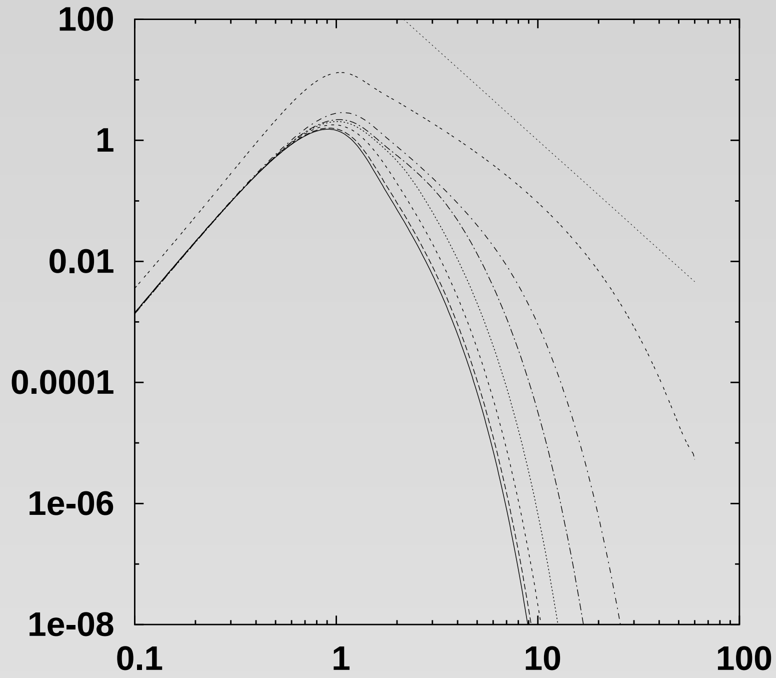




- ... to EDQNM3 $\longrightarrow$ (A) QNM energy equation (Bellet *et al.*, JFM, 2006)

Angle-dependent spectrum

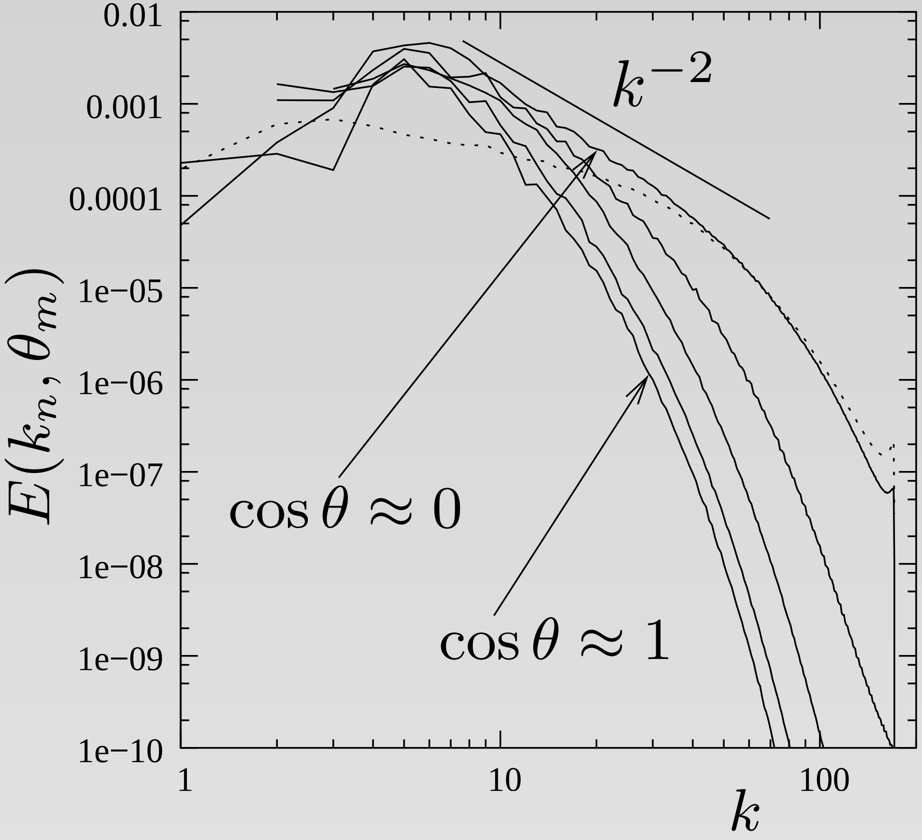
- Isotropy breaking by spectral transfer $T^{(e)}(\boldsymbol{k})$: directional anisotropy:

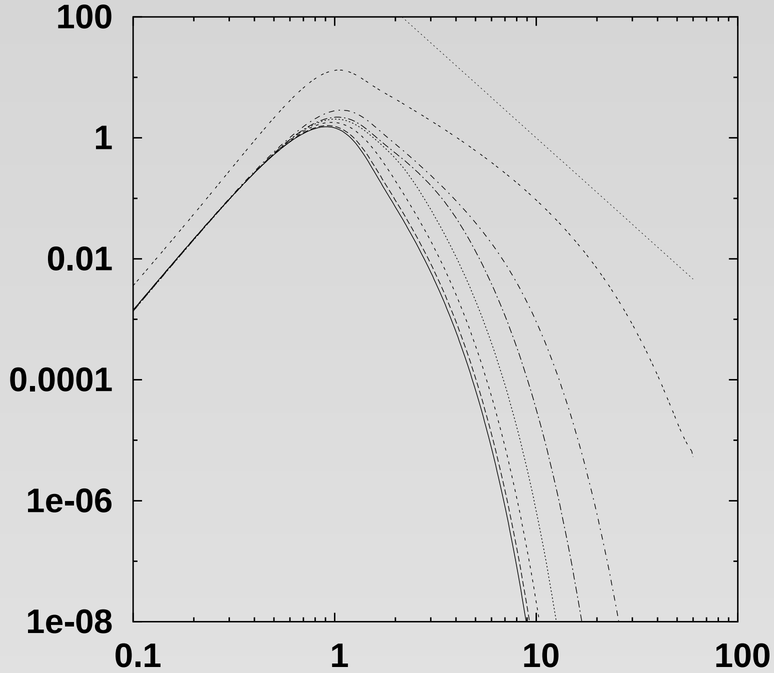


$$4\pi k^2 e(\boldsymbol{k}, t_f) = 4\pi k^2 e(k, \underbrace{\cos\theta}_{k_{\parallel}/k}, t_f)$$

- Spherical averaging $\longrightarrow E(k, t_f)$, prefactor $E \sim \frac{\Omega}{t} k^{-3}$, not 2D !

AQNM and DNS





512^3 DNS by Liechtenstein *et al.*, JOT, 2005

Inertial wave-turbulence, resonant interactions, 2D or not 2D

- *Low dimension* of active manifolds : overestimated in forced ? DNS/LES ?
  'TRUE' 2D embedded in 3D : a DIRAC singularity

$$E(k) \sim f^2 k^{-3}, \qquad e(k_\perp, k_\parallel) = \underbrace{\frac{E(k_\perp)}{2\pi k_\perp}}_{\sim f^2 k_\perp^{-4}} \delta(k_\parallel)$$

- integral singularity from theoretical wave-turbulence

$$e(k_\perp k_\parallel) \sim k_\parallel^{-1/2} k_\perp^{-7/2} = k^{-4} x^{-1/2} \quad \text{Galtier 2002}$$

$$e(k_\perp k_\parallel) \sim k_0^{-1/2} k_\parallel^{-1/2} k_\perp^{-3} = k_0^{-1/2} k^{-7/2} x^{-1/2} \quad \text{CRG 2004}$$

$$E(k) \sim \frac{f}{t} k^{-3}, \quad e(k, x \sim 0) \sim k^{-4} \quad \text{BGSC-2006}$$