THE MULTISPIRAL MODEL OF TURBULENCE AND INTERMITTENCY.

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ABSTRACT. A spiral vortex sheet modelled as in Moffatt (1992) yields similarity exponents ξ_p for the velocity field's statistics – $\langle(\delta u_{\|}(\mathbf{r}))^p\rangle \sim r^{\xi_p}$ – which have the same properties as the ξ_p derived in the β - model (linear in p plus a constant). A superposition of such spiral vortex sheets with different Kolmogorov capacities D_K yields results for ξ_p that resemble closely the results of the multifractal model of turbulence (*e.g.* ξ_{2p} is the Legendre transform of a function $D_K(\sigma)$ characteristic of the multispiral structure of the velocity field). Unlike the multifractal and the β - models, the multispiral and the spiral β - models of turbulence carry an important difference between even and odd statistics; a breaking of symmetry (left/right) is needed for the odd p statistics not to vanish and for the properties of intermittency and non-linear energy dissipation to be incorporated in the model.

The method used here for the solution of these two models of homogeneous and isotropic turbulence (the spiral β - model and the multispiral model) is a generalisation of the method used in Vassilicos & Hunt (1991) to derive the relation between the power spectrum ($p = 2$ statistics) and the fractal dimension (Kolmogorov capacity D_K) of interfaces. It is more general than is needed here (*e.g.* it does not depend on the specific *spiral* structure of the velocity field); it is of use, in particular, when a field's structure is not fractal in the Hausdorff sense, but has nevertheless non-trivial Kolmogorov capacities.

1. Introduction

In 1941 Kolmogorov published two papers dealing, respectively, with the second and the third order statistics of the small scale relative velocities $\delta\mathbf{u}(r) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ between two points of space $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$ in a homogeneous and isotropic turbulent flow ($r = |\mathbf{r}|$). (These papers can be found most easily in a special issue of the Proceedings of the Royal Society of London, series A, vol. 434 (1991), celebrating Kolmogorov's ideas 50 years on). Under the assumptions that the Reynolds number is large enough, that the small scale velocity field is homogeneous, isotropic and

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universal–independence from large scale forcing, and from boundary and initial conditions–and that the mean rate of energy dissipation ϵ of the turbulent fluctuations is the *same everywhere* in space (sufficiently far from the boundaries), Kolmogorov recognised that the relative velocity statistics within the inertial range of length scales can only depend on ϵ and r (the second hypothesis of similarity), and used dimensional analysis to show that

$$\langle(\delta\mathbf{u}(\mathbf{r}))^2\rangle \approx C_2\epsilon^{2/3}r^{2/3} \tag{1a}$$

when $\eta \ll r \ll L$ (L is an integral length scale and η is the Kolmogorov viscous length scale). The value of the constant C_2 in (1a) is not specified by Kolmogorov's theory, but it is expected to be universal.

From the Navier-Stokes equations he was able to deduce that, for homogeneous and isotropic turbulence,

$$\langle(\hat{\mathbf{r}}\cdot\delta\mathbf{u}(\mathbf{r}))^3\rangle \approx -\frac{4}{5}\epsilon r, \tag{1b}$$

($\hat{\mathbf{r}}$ is a unit vector along $\mathbf{r}$), in that same inertial range of length scales where his dimensional arguments lead, in fact, to the conclusion that

$$\langle(\delta u_{\parallel}(\mathbf{r}))^p\rangle \approx C_p(\epsilon r)^{p/3} \tag{1c}$$

for $p = 2, 3, 4, \ldots$ ($\delta u_{\parallel} = \hat{\mathbf{r}}\cdot\delta\mathbf{u}$). According to the theory, the dimensionless constants C_p are all universal (note: C_2 in (1a) is not the same as in (1c)).

A consequence of this theory–specifically of (1c)–is that the small scale turbulent fluctuations are statistically self-similar in space; this is because the statistics of $\lambda^{1/3}\delta u_{\parallel}(\mathbf{r}')$ are the same as these of $\delta u_{\parallel}(\mathbf{r})$ when $\mathbf{r}' = \lambda^{-1}\mathbf{r}$ for any positive real number λ which represents, therefore, a dilation of space.

The assumption of statistical independence of the small scales from the large energy containing scales has recently been questioned (*e.g.* see Hunt et al (1988), Brasseur & Yeung (1991)); this raises the related question of whether the statistics of the small scale turbulence are indeed universal or not. Frisch (1991) argues that all C_p–except for $p = 3$–cannot be universal. Nevertheless, he recovers formulae (1c) by turning Kolmogorov's theory on its head and assuming that the small scale turbulence is statistically self-similar in space; it is now an assumption (not a consequence as in Kolmogorov) that there is a single exponent h such that the statistics of $\lambda^h\delta u_{\parallel}(\mathbf{r}')$ are the same as those of $\delta u_{\parallel}(\mathbf{r})$ when $\mathbf{r}' = \lambda^{-1}\mathbf{r}$ ($\lambda \in \mathrm{R}_+$). The value $h = 1/3$ is then deduced from (1b).

The question of the universality of C_p will not be discussed in the present paper. Here we will concentrate on obtaining scaling laws of the type

$$\langle(\delta u_{\parallel}(\mathbf{r}))^p\rangle \sim r^{\xi_p}, \tag{2}$$

for inertial range values of r. The experimental measurements of Anselmet et al (1984) have shown that such scaling laws do indeed exist in ranges of length scales within the inertial range of small scale turbulence, but that the powers ξ_p may not equal $p/3$ as predicted by Kolmogorov's theory. There have been a few attempts to derive ξ_p theoretically by 'correcting' several aspects of Kolmogorov's theory (log-normal model, β-model, multifractal model; none of these approaches provide a universal means of estimating C_p, and neither will the one presented here).

The present paper investigates how, if the properties of self-similarity and space-fillingness (the property that ϵ is the same everywhere in space) are redefined,

(i) the exponents ξ_p may match the experimental findings and

(ii) the structure of small scale turbulence is consistent with the hypotesis that it is dominated by vortex tubes with a spiral internal structure.

She *et al.* (1991) find that, in DNS of homogeneous isotropic turbulence, the fine scales are dominated by vortex tubes; the internal structure of these vortex filaments seems to be spiral with a few turns (She, private communication). In similar numerical calculations, Ruetsch & Maxey (1992) have identified vortex sheets in a turbulent velocity field which undergo Kelvin-Helmholtz instability and lead to isolated, single spiral structures. Spiral structures of vorticity have also been recently seen in similar numerical simulations by Brasseur & Lin (private communication).

The statistical self-similarity in space of the small scale turbulent fluctuations aroused Mandelbrot's interest who suggested that turbulent velocity fields may have a fractal structure with a non-integer Hausdorff dimension (see Mandelbrot (1982) and references therein). A pattern of spirals with smaller spirals on them–and so on to increasingly smaller scales–is a good example of a fractal. Experimental measurements of the Kolmogorov capacity (or box dimension) using the box-counting algorithm were made for various interfaces in various turbulent flows (see Sreenivasan (1991)); the non-integer value which was obtained does not imply that the structure of these interfaces (and by inference, the structure of the velocity field distorting them) is fractal. Some single spirals can have a non-integer Kolmogorov capacity even though they are smooth objects and their Hausdorff dimension equals their topological dimension (see Vassilicos & Hunt (1991)). In fact, the spiral that arises from numerical integration of a vortex sheet's instability driven evolution by Krasny (1986) does not have a cascade of smaller spirals on it–it is not fractal–and it is such that its intersections with the x-axis (corresponding to the sheet's initial configuration) are at a distance $x_n \sim n^{-2}$ from the centre of the spiral (Moffatt (1992)) (where n numbers the successive coils of the spiral and its intersections with the x-axis from large x inwards–towards $x \to 0$). The Kolmogorov capacity of these intersections is $D_K = 1/3$ (if $x_n \sim n^{-a}$, $D_K = \frac{1}{1+a}$, see Vassilicos &

Hunt (1991)), which is non-integer. If, as Ruetsch & Maxey (1992) seem to observe, Kelvin-Helmholtz instability of vortex sheets is an important mechanism in the generation of small scale turbulence, then it may be safer to assume that the small scale turbulence has a spiral structure characterised by non-integer Kolmogorov capacities D_K, than to assume it to have a fractal structure characterised by non-integer Hausdorff dimensions D_H.

The assumption that the mean rate of dissipation is the same everywhere in space (the space-filling property of Kolmogorov turbulence) was criticised very early on by Landau (*e.g.* see Frisch (1991)). The intermittent character of small scale turbulence, *i.e.* the fact that the turbulent activity, and therefore the energy dissipation, do not occur equally everywhere in space, contradicts Kolmogorov's assumption of space-fillingness.

The first attempt to remedy this shortcoming of Kolmogorov's 1941 theory came in 1961 from Obukhov and Kolmogorov themselves (see Frisch (1991) for a discussion and references) who proposed to replace the original theory with the so-called log-normal model. A central role in this model is played by ϵ_r, the spatial average of the mean rate of dissipation over a ball of radius r; the logarithm of ϵ_r is assumed to have a gaussian (normal) distribution with variance $\sigma_r^2 = A + \mu \ln(r_0/r)$ where μ is a positive parameter. This assumption replaces the assumption of space-fillingness under which ϵ_r would equal ϵ for all r, and both A and μ would vanish. Clearly, what is done in the log-normal model, is to broaden the distribution of $\ln \epsilon_r$ from a delta function (implicit in Kolmogorov's assumption of space-fillingness) to a gaussian with a finite variance that is a function of $\ln r$.

It is precisely the dependence of σ_r^2 on $\ln r$ which leads to divergent results from Kolmogorov's original theory; in the log-normal model,

$$\xi_p = p/3 - \frac{\mu}{18} p(p-3). \tag{3}$$

The intermittency is incorporated in the theory because μ is positive, and therefore extreme events carry more weight than otherwise.

A particular consequence of (3) is that ξ_p decreases with p for large enough values of p; this is shown by Frisch (1991) to be in contradiction with the basic physics of incompressible flow. Frisch (1991) lists and discusses a few more problems of the lognormal model.

A second attempt to replace the assumption of space-fillingness by something that would allow the property of intermittency to tie in with inertial range scaling laws has been made with the β-model (see, again, Frisch (1991) for references). The β-model addreses the assumption of space-fillingness more directly than the log-normal model does; rather than a change in the probability distribution of ϵ_r, it involves a change in the spatial distribution of turbulent activity. Specifically, one starts (as Frisch (1991) does in order to recover Kolmogorov's 1941 results without having to assume universality) from an assumption of self-similarity in space, but modified as follows: there exists a single exponent h and a *fractal* sub-set S_h of $\mathbf{x}$-space, such that if $\mathbf{x} \in S_h$, then $\delta u_{\|}(\mathbf{x}, \mathbf{r}) \sim r^h$ as $r \to 0$; the Hausdorff dimension of S_h is $2 + D_H$ $(0 < D_H \leq 1)$. It appears easy to estimate the statistics (averages over space) of such $\delta u_{\|}$:

$$\langle (\delta u_{\|}(\mathbf{r}))^p \rangle \sim r^{ph} r^{3-(2+D_H)} = r^{ph+1-D_H} \qquad (4)$$

as $r \to 0$ ($\mathbf{x}$ is omitted because of homogeneity). Kolmogorov's assumption of space-fillingness corresponds to the extreme situation where the Hausdorff dimension of S_h is 3 (*i.e.* $D_H = 1$). In this case one recovers $\xi_p = p/3$ ($h = 1/3$ is, like previously, derived from (1b)). Otherwise, the β-model gives

$$\xi_p = ph + 1 - D_H. \qquad (5)$$

The experimental values of ξ_p obtained by Anselmet *et al.* (1984) may not agree with either the log-normal or the β-model. They find that ξ_p increases with p over the entire accessible range of p, but that $\frac{d\xi_p}{dp}$ seems to decrease with p. This led to the introduction of the multifractal model as a way of keeping with the idea that the turbulent activity is concentrated on fractal sets within the flow, and yet fit the experimental data. Similarly to the β-model, the multifractal model of inertial range turbulence starts from an assumption of self-similarity in space (see Frisch (1991)): there exists a range of exponents $h \in (h_{min}, h_{max})$, and different fractal sets S_h of Hausdorff dimension $2 + D_H(h)$ for each of these exponents h, such that if $\mathbf{x} \in S_h$, then $\delta u_{\|}(\mathbf{x}, \mathbf{r}) \sim r^h$ as $r \to 0$ $(0 < D_H(h) \leq 1$ for all $h)$. Equation (4) is now replaced by

$$\langle (\delta u_{\|}(\mathbf{r}))^p \rangle \sim \int d\mu(h)\, r^{ph+1-D_H(h)} \qquad (6)$$

as $r \to 0$. The measure $\mu(h)$ corresponds to the weight of the different scalings. Using the method of steepest descents it then follows that ξ_p is the Legendre transform of $D_H(h)$, *i.e.*

$$\xi_p = min_h(ph + 1 - D_H(h)). \qquad (7)$$

One can always find a function $D_H(h)$ for which (7) fits the experimental values of ξ_p. Unfortunately, Hausdorff dimensions D_H cannot be measured directly in practice, so that it is difficult to check experimentally whether the multifractal model is just another irrelevant way to fit the data, or whether it really captures some fundamental property of small scale turbulent structure.

On the other hand, Kolmogorov capacities are easily accessible in practice via the box-counting algorithm. But, as we have already pointed out previously, a non-integer value of D_K is by no means a proof that the examined geometry is fractal, and if it is not, $D_K \geq D_H = 0$, and the above multifractal and β-models do neither apply nor work. The exponent h, directly sensitive to the singular (non-differentiable) structure of the turbulence, cannot be used to derive (4), (5) and (6), (7) when S_h are nearly everywhere *smooth*, non-fractal sets characterised by non-integer values of D_K. Here we introduce an exponent σ which characterises the sparse–isolated–singular behaviour of the flow by holding information about the spatial extent over which portions of the velocity field are smooth and non-singular.

The object of the present paper is to derive the structural exponents ξ_p from the assumption that the small scale turbulence is predominantly smooth and spiral structured rather than fractal (or multifractal). The analysis of this paper applies to both 2-d and 3-d turbulence: the picture to hold in mind in the 2-d case is one of weak patches of vorticity wrapping around stronger patches of vorticity (homogeneously and isotropically distributed about the plane), thus producing spiral patterns (see Gilbert (1988)); in the 3-d case, the picture we refer to is one of a homogeneous and isotropic distribution of vortex filaments with a spiral internal vortex sheet structure. The analysis in the sequel is essentially 1-dimensional, but the results are valid in more than 1 dimensions because of homogeneity and isotropy. The results in section 2 are written for 3-d turbulence. To obtain the 2-d turbulence results, replace the velocity u by the vorticity ω in the sequel's formulae.

2. The Spiral β-Model and Multispirals

2.1. THE SPIRAL β-MODEL

The turbulence is assumed to be homogeneous and isotropic as $Re \to \infty$, so that a line in any direction through the flow should cut across a sufficiently large number of spiral vortex sheet structures, some of which very near the centre of the spiral accumulation. The points of intersection of these spiraling sheets with the line are assumed to have a non-trivial Kolmogorov capacity D_K ($D_K < 1$). It is also a consequence of the homogeneity and the isotropy of the small scale turbulence that the value of D_K is independent of the 1-d cut chosen to probe the flow.

The velocity field $u_{||}$ sampled along a 1-d cut is assumed to be *smooth* between the points where the cut intersects the spiral vortex sheet structure of the flow. Across these points of discontinuity, $u_{||}$ undergoes sudden jumps. A simplifying assumption in this paper is made with the replacement of the word 'smooth' by the word 'constant'. We take $u_{||}$ to be effectively constant between the points of discontinuity; $\frac{du_{||}}{dx}$ is indeed very small in the regions between jumps (where $u_{||}$ is smooth) compared to the sudden increase in $\frac{du_{||}}{dx}$ on the points where vortex sheets intersect the cut (x is the coordinate along the 1-d cut).

Between two consecutive such points we assume $u_{\|}$ to be proportional to l^{σ} when the consecutive points of discontinuity are at a distance l from each other; specifically, as $l \to 0$,

$$|u_{\|}(l)| \sim l^{\sigma}. \tag{8}$$

This is the assumption which replaces the $\beta - model$. It is not totally ad hoc; it is a reformulation of Moffatt's (1992) modeling of the velocity field inside a spiraling vortex sheet (see Appendix A). The singularity exponent σ which replaces h, is defined with reference to the regions where the flow is smooth–constant. The exponent h used in the β-model is defined instead *on* the set S_h, *i.e.* at these points where the flow is singular. In order to obtain (4) and (5) it is therefore essential to assume S_h to be fractal. This assumption is not made here.

It is shown in Vassilicos & Hunt (1991) (see also Appendix B) that the probability density function $n_0(l)$ for two consecutive points of discontinuity to be at a distance l from each other is

$$n_0(l) \sim l^{-D_0} \tag{9a}$$

as $l \to 0$, when these points have a non-trivial Kolmogorov capacity D_K, and

$$D_0 = D_K. \tag{9b}$$

We want to calculate $\langle(\hat{\mathbf{r}} \cdot \delta\mathbf{u}(\mathbf{x},\mathbf{r}))^p\rangle$, where the average is either taken over the entire 3-d x-space, or over many realisations of the 3-d turbulent flow, or both. Because of homogeneity and isotropy,

$$\langle(\hat{\mathbf{r}} \cdot \delta\mathbf{u}(\mathbf{x},\mathbf{r}))^p\rangle = \langle(u_{\|}(x+r) - u_{\|}(x))^p\rangle, \tag{10}$$

where x and $x+r$ are points on an arbitrary 1-d cut through the flow at a distance $r = |\mathbf{r}|$ from each other, and where the average on the right hand side of (10) is either taken over the entire 1-d x-space of the cut, or over many realisations of $u_{\|}$ on that cut, or both.

As is well known,

$$\langle(u_{\|}(x+r) - u_{\|}(x))^p\rangle = \langle u_{\|}^p\rangle(1+(-1)^p) + \sum_{j=1}^{p-1}(-1)^{p-j}C_p^j\langle u_{\|}^j(x+r)u_{\|}^{p-j}(x)\rangle. \tag{11}$$

For $1 \leq j \leq p-1$, the contributions to these statistics can be decomposed as follows:

$$\langle u_{\|}^j(x+r)u_{\|}^{p-j}(x)\rangle = \sum_{q=0}^{+\infty} T_q(r;p,j), \tag{12}$$

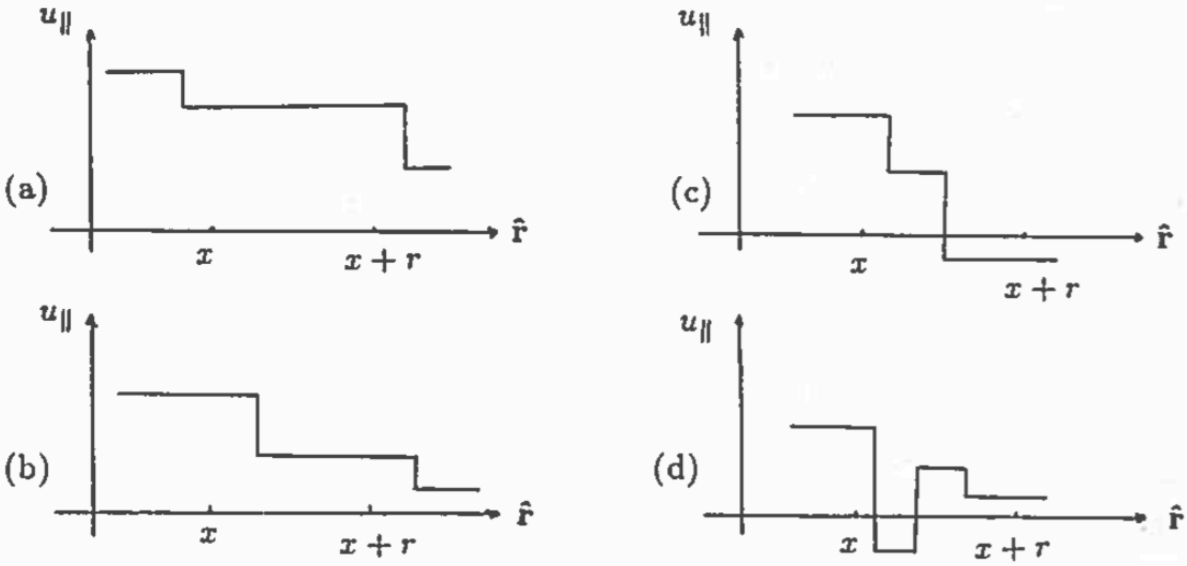

Fig. 1. $T_0(r;p,j)$ carries the contribution from situations like (a). $T_1(r;p,j)$, $T_2(r;p,j)$ and $T_3(r;p,j)$ correspond respectively to (b), (c) and (d).

where $T_q(r;p,j)$ is the average of $u_\parallel^j(x+r)u_\parallel^{p-j}(x)$ over all configurations where there are exactly q points of discontinuity between x and $x+r$ (see figure 1).

The probability density $n_0(l)$ does not hold enough information to enable a calculation of T_q when $q \geq 1$. For $q=0$ though, one can easily see that

$$T_0(r;p,j) = \int_r^{+\infty} n_0(l)\overline{u_\parallel^j(l)u_\parallel^{p-j}(l)}\,dl; \tag{13}$$

the overline indicates an average over the distribution of signs (+ or −) of $u_\parallel$ which has not been specified, and which holds essential information, as will clearly appear in the sequel, as to the difference between odd-p statistics and even-p statistics.

In order to evaluate the integral (13) one needs to introduce an integral length scale. This can be done on the grounds that beyond a certain length scale L (*i.e.* for $l \geq L$), different physics take over, which may be dominated by boundary conditions and which would imply a much faster fall off of $n_0(l)$ than a power law. For simplicity we set $n_0(l)=0$ when $l \geq L$, and therefore obtain, from (8) and (9a), that

$$T_0(r;p,j) \sim \frac{L^{p\sigma+1-D_0}}{p\sigma+1-D_0}(1-(r/L)^{p\sigma+1-D_0}) \tag{14}$$

as $r/L \to 0$. Also, if $p\sigma+1-D_0 \geq 0$,

$$T_0(r;p,j) = T_0(0;p,j)(1-(r/L)^{p\sigma+1-D_0}). \tag{15}$$

Note that $D_0 \leq 1$, and therefore it is sufficient that $\sigma \geq 0$ for (15) to be valid for all positive integers p.

In order to calculate $T_q(r;p,j)$ when $q \geq 1$, we need to introduce the probability density functions $n_q(l)$ which determine the chance for exactly q points of discontinuity to be between two other such points that are at a distance l from each other. One can show (see Appendix B) that for spiral accumulation patterns of the form $x_n \sim n^{-a}$,

$$n_q(l) \sim l^{-D_q} \tag{16}$$

as $l \to 0$, and $D_q = \frac{1}{1+a} < 1$. In fact, it appears that in general (see Appendix B), when $q \geq 2$, $D_{q-1} = D_{Kq}$; D_{Kq} are the generalised dimensions (generalised Kolmogorov capacities in fact) introduced by Hentschel & Procaccia in 1983 ($D_{K0} = D_K = D_0$).

If the generic structure of the turbulence is not only characterised by D_0, but also by D_1, D_2, D_3, *etc...* as defined by (16) (which are not greater than 1, and may not be in general equal to each other–see Appendix B), then $T_1(r;p,j)$ can be estimated as follows:

$$T_1(r;p,j) = \int_r^L n_1(l)\, dl \int_0^l \overline{u_\parallel^j(l-l_1)u_\parallel^{p-j}(l_1)}n_0(l_1)\, dl_1, \tag{17}$$

and a trivial calculation leads to:

$$T_1(0;p,j) - T_1(r;p,j) \sim (r/L)^{p\sigma+2-D_0-D_1} \tag{18}$$

for small r/L.

When $q \geq 2$, $T_q(r;p,j)$ can be shown to be of $O[(r/L)^{p\sigma+3-D_0-D_q-D_{q-2}}]$ because

$$T_q(r;p,j) \leq \int_r^L n_q(l)\, dl \int_0^r n_{q-2}(l_1)\, dl_1 \int_0^{l-l_1} \overline{u_\parallel^j(l_2)u_\parallel^{p-j}(l-l_1-l_2)}n_0(l_2)\, dl_2. \tag{19}$$

It follows, therefore (see (B9)), that to leading order in r/L,

$$\langle u_\parallel^j(x+r)u_\parallel^{p-j}(x)\rangle \approx T_0(r;p,j) + T_1(r;p,j), \tag{20}$$

which implies that if $p\sigma + 1 - D_0 \geq 0$,

$$\langle u_\parallel^p\rangle - \langle u_\parallel^j(x+r)u_\parallel^{p-j}(x)\rangle \sim (r/L)^{p\sigma+1-D_0} \tag{21}$$

as $r/L \to 0$.

This result applies whether p is even or odd. It reflects the fact that at small distances, a space varying quantity like $u_\parallel$ is well autocorrelated with itself if it remains, on average, approximately constant over these small distances. This is the meaning of the approximation in (20).

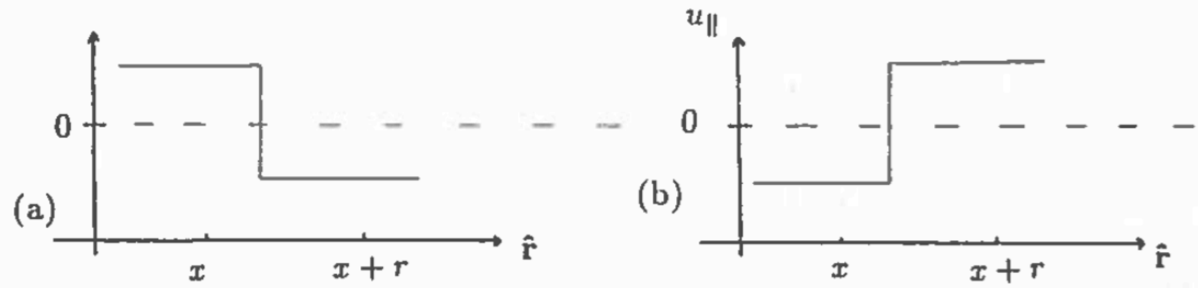

Fig. 2. If (a) is more likely than (b) then the odd $-p$ statistics of the relative velocities are strictly negative.

In order to recover turbulence statistics in terms of relative velocities (which are traditionally linked to the notion of 'turbulent eddies') we need to combine (21) with (11). The result is different for odd and even p; when p is even,

$$\langle (u_\parallel(x+r) - u_\parallel(x))^p \rangle \sim (r/L)^{\xi_p} \tag{22a}$$

as $r/L \to 0$, and

$$\xi_p = p\sigma + 1 - D_0. \tag{22b}$$

When p is odd though, the small-scale relative velocity statistics depend vitally on the distribution of the signs of $u_\parallel$. From (11) it is easily seen, for example, that if this distribution is such that $\langle u_\parallel^j(x+r)u_\parallel^{p-j}(x)\rangle = \langle u_\parallel^{p-j}(x+r)u_\parallel^j(x)\rangle$ for all positive integers $j \le p-1$, then $\langle (u_\parallel(x+r) - u_\parallel(x))^p \rangle = 0$ when p is odd.

One can check that $T_0(r;p,j) = T_0(r;p,p-j)$ irrespective of the distribution of signs of $u_\parallel$ (see (13)). It follows that

$$\langle u_\parallel^j(x+r)u_\parallel^{p-j}(x)\rangle - \langle u_\parallel^{p-j}(x+r)u_\parallel^j(x)\rangle \approx T_1(r;p,j) - T_1(r;p,p-j) \tag{23}$$

at leading order in r/L. For $\langle (u_\parallel(x+r)-u_\parallel(x))^p\rangle$ to be strictly negative when p is odd, it is enough to assume (see (11) and (23)) that the velocity $\hat{\mathbf{r}}\cdot\mathbf{u} = u_\parallel$ is more often negative on the right (towards $\hat{\mathbf{r}}$) of a point of discontinuity and positive on the left (towards $-\hat{\mathbf{r}}$) than positive on the right and negative on the left of such a point (see figure 2). This is an assumption about the skewness of the velocity field.

This additional assumption concerning the spatial distribution of signs of $u_\parallel$ is essential in order to incorporate intermittency in the model. The spiral assumption is not enough in itself, even though it represents a very natural departure from space-fillingness. There can obviously be no intermittency if the odd-p moments of the relative velocity vanish. Furthermore, it is a consequence of the Navier-Stokes equations that $\langle (u(x+r) - u(x))^3\rangle < 0$

(see (1b)). Here all odd statistics are negative. Specifically, when p is odd,

$$\langle (u_{\|}(x+r) - u_{\|}(x))^p \rangle \approx C_p (r/L)^{\xi_p} \tag{24a}$$

as $r/L \to 0$ (from (11), (18) and (23)); $C_p < 0$ (whilst it is obvious that $C_p > 0$ for even integers p), and

$$\xi_p = p\sigma + 2 - D_0 - D_1. \tag{24b}$$

The spiral structure is said to be space-filling when $D_0 = D_1 = 1$. This corresponds to a spiral that is as slow to wind in as possible, and fills in this way the space around its centre of accumulation. In the limit where D_0 and D_1 tend to 1, ξ_p tends to $p\sigma$; in the context of the spiral β-model, the space-fillingness of the energy dissipation, which is assumed in Kolmogorov 1941, is interpreted as being the limit when the spiral sheets of dissipation (where the velocity derivatives are high) has such a slow inwards winding (or accumulating) pattern that $D_0 = D_1 = 1$. One indeed recovers the Kolmogorov expression for ξ_p in that limit, and the spiral β-model allows, in general, for the spirals not to be space-filling, *i.e.* for D_0, $D_1 < 1$. Thus, the turbulent velocity field is intermittent.

It is not clear from the literature why the conventional β-model makes no distinction between odd and even p statistics. In Appendix A we show how Moffatt's (1992) result for $p = 2$ can be recovered within the framework of the spiral β-model. Note, before closing this subsection, that (22) is valid when $p\sigma + 1 - D_0 \geq 0$, and (24) is valid when $p\sigma + 2 - D_0 - D_1 \geq 0$. All the p-statistics of the relative velocities may or may not converge as $r/L \to 0$; there may be an upper bound p_{max}, such that ξ_p is positive when $p \leq p_{max}$, and such that when $p > p_{max}$, these p-statistics diverge.

2.2. MULTISPIRALS

The small scale turbulence is now assumed to be made of different spiral vortex sheet structures of different Kolmogorov capacities D_K ($D_K < 1$). For a spiral of a given D_K, the magnitude of the velocity $u_{\|}$ between two consecutive points of discontinuity is still given by (8), but now σ is a function of D_K. In other words, we assume the existence of a spectrum of exponents σ (as in the multifractal model where a spectrum of exponents h is assumed) which correspond to spiral singularities of Kolmogorov capacity $D_K(\sigma) = D_0(\sigma)$. We also have to assume the existence of a function $D_1(\sigma)$ for the calculation of the odd-p statistics to be feasible. Equations (21), (22a,b) and (24a,b) are now respectively replaced by:

$$\langle u_{\|}^p \rangle - \langle u_{\|}^j(x+r) u_{\|}^{p-j}(x) \rangle \sim \int d\mu(\sigma) (r/L)^{p\sigma + 1 - D_0(\sigma)}, \tag{25}$$

$$\langle (u_{\|}(x+r) - u_{\|}(x))^p \rangle \approx C_p \int d\mu(\sigma)(r/L)^{p\sigma+1-D_0(\sigma)} \tag{26}$$

where $C_p > 0$ when p is even, and

$$\langle (u_{\|}(x+r) - u_{\|}(x))^p \rangle \approx C_p \int d\mu(\sigma)(r/L)^{p\sigma+2-D_0(\sigma)-D_1(\sigma)} \tag{27}$$

where $C_p < 0$ when p is odd. The measure $\mu(\sigma)$ corresponds to the weight of the different local spiral velocity fields. As usual, the three formulas above are valid when $r/L \to 0$.

Using the method of steepest descents we obtain the following results:

$$\xi_p = min_\sigma(p\sigma + 1 - D_0(\sigma)) \tag{28a}$$

when p is even.

$$\xi_p = min_\sigma(p\sigma + 2 - D_0(\sigma) - D_1(\sigma)) \tag{28b}$$

when p is odd. And for all positive integers p, and $1 \leq j \leq p-1$, we have that

$$\langle u_{\|}^p \rangle - \langle u_{\|}^j(x+r) u_{\|}^{p-j}(x) \rangle \sim (r/L)^{min_\sigma(p\sigma+1-D_0(\sigma))}. \tag{29}$$

2.3. SCALAR INTERFACES: THE CASE $\sigma = 0$

When a scalar F is released in the turbulence, and before molecular diffusion has had time to act (we assume the molecular diffusivity of the scalar to be much smaller than the viscosity of the fluid), we may effectively represent F by an extremely contorted interface such that $F = +1$ on one side and $F = -1$ on the other side of the interface. Turbulent interfaces are known by experiment (see Sreenivasan (1991)) to have non-trivial Kolmogorov capacities D_K^i; experimental measurements appear to imply that $D_K^i = 1/3$ above, and $D_K^i = 1$ below the Kolmogorov viscous length scale.

The analysis of subsection 2.1 applies here with $\sigma = 0$ (it would not apply if F did not change signs across the interface). One can indeed write that

$$\langle (F(x+r) - F(x))^p \rangle \sim (r/L)^{\xi_p} \tag{30}$$

as $r/L \to 0$, with $\xi_p = 1 - D_k^i$ when p is even. (The relative p-statistics of F vanish for odd p). As noted by Vassilicos & Hunt (1991), the value $D_K^i = 1/3$ implies $\xi_2 = 2/3$, and $D_K^i = 1$ implies $\xi_2 = 0$, in agreement with theory and the measured statistics of passive scalars. ξ_p does not depend on p for passive interfaces, but only on whether p is even or odd; one may expect $\xi_p = 2/3$ in the inertial range, for all even positive integers p (in the limit of infinite Prandtl number and subject to the right initial conditions where a passive scalar can indeed be regarded as a step function F equal to either +1 or -1). It is not known how D_K^i relates to the generalised Kolmogorov capacities $D_{K0}(\sigma)$ and $D_{K2}(\sigma)$ of the turbulent velocity field.

3. Conclusion

The multispiral model gathers in a single picture the fractal properties of the turbulence, the inertial range scaling laws of the turbulent statistics and the observed smoothness of the velocity field with localised singularities which appear in the form of thin and elongated vortex tubes that have an internal vortex sheet spiral structure.

The property of intermittency can also be incorporated provided that the following assumption is made: as one samples the velocity $\hat{\mathbf{r}} \cdot \mathbf{u} = u_{\|}$ along a linear cut through the flow in the direction $\hat{\mathbf{r}}$, $u_{\|}$ jumps more often from a positive to a negative value than from a negative to a positive one at those points where the cut crosses a vortex sheet. This implies that the odd-p statistics of the turbulence do not vanish, which is a necessary condition for the average turbulent energy dissipation not to vanish either, and for the turbulence to be skewed. The turbulent velocity field is then more or less intermittent as the accumulation patterns of the turbulent structures are more or less space-filling.

The main results of this paper are equations (28) and (29); the singularity exponents σ characterise the spatial extent over which the velocity field is smooth, and $D_0(\sigma)$ and $D_1(\sigma)$ are, respectively, the generalised Kolmogorov capacities $D_{K0}(\sigma)$ and $D_{K2}(\sigma)$ (see Appendix B) characterising the accumulation pattern of the spiral structures corresponding to the exponents σ. Formally, the results of this paper do not depend on the existence of spiral structures, but on the existence of isolated accumulation patterns of some kind. Spiral structures seem to be the most plausible accumulation patterns, and a class of them are known to have the properties assumed here (non-trivial generalised Kolmogorov capacities–see Appendix B).

Formulae (28) can be tested by experiment; ξ_p, $D_{K0}(\sigma)$ and $D_{K2}(\sigma)$ are measurable unlike the Hausdorff dimensions of the multifractal model which are not. It is indeed surprising that multifractals make no distinction between even and odd p statistics. Such a distinction is central if one aims at an understanding of the dynamics (*e.g.* the energy dissipation) of the turbulence.

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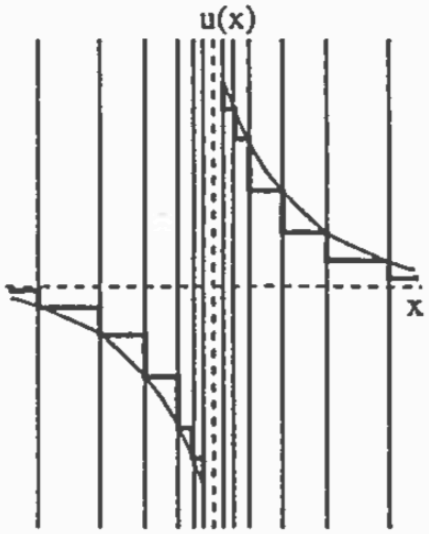

Fig. 3. From Moffatt (1992); the velocity profile on a transversal through the centre of the spiral vortex sheet.

Appendix A: Moffatt's (1992) Spiral Model

Moffatt (1992) assumes the following velocity profile on a transversal through the centre of a spiral vortex sheet (see figure 3): between two intersections of the sheet with the linear cut, the velocity is $u_n \sim n^b$, and these intersections are located at $x_n \sim n^{-a}$ $(n = 1, 2, 3, ...)$. Let $\delta x_n = x_n - x_{n+1} \sim n^{-a-1}$; $u_n \sim \delta x_n^{-\frac{b}{1+a}}$, so that $\sigma = -\frac{b}{1+a}$ (see (8)). It is shown in Vassilicos & Hunt (1991) that $D_0 = \frac{1}{1+a}$, and therefore a straightforward application of the results in subsection 2.1 gives

$$\xi_2 = 2\sigma + 1 - D_0 = 1 - \frac{2b+1}{1+a} \tag{A1}$$

provided that $2\sigma + 1 - D_0 = \frac{a}{2} - b \geq 0$. (A1) has indeed been obtained by Moffatt (1992) using Fourier methods under the condition $b \leq \frac{a}{2}$.

Appendix B: The Generalised Kolmogorov capacities

One can find in Vassilicos & Hunt (1991) the following relation, valid as $\epsilon \to 0$;

$$N(0,\epsilon)\epsilon + L\int_{\epsilon}^{L} n_0(l)\, dl \approx L \tag{B1}$$

where $N(0,\epsilon)$ is the minimum number of boxes of size ϵ needed to cover the points of a set (here the points of discontinuity of $u_{||}$), and $n_0(l)$ is the probability density function for two *consecutive* points of discontinuity to be at a distance l from each other.

(B1) can be generalised as follows ($q \geq 2$): for $\epsilon \rightarrow 0$,

$$N(q,\epsilon)\epsilon + L\int_{\epsilon}^{L} n_{q-1}(l)\,dl \approx L; \tag{B2}$$

$N(q,\epsilon)$ is the minimum number of boxes of size ϵ needed to cover part of the set so that each box covers at least q points of that set; $n_{q-1}(l)$ is the probability density which determines the chance for exactly $q-1$ points of the set to be between two other points of that set at a distance l from each other.

It is shown in Vassilicos & Hunt (1991) that for (spiral) 1-d accumulation patterns of the form $x_n \sim n^{-a}$ ($a > 0$),

$$N(0,\epsilon) \sim \epsilon^{-D_{K0}} \tag{B3a}$$

and

$$D_{K0} = \frac{1}{1+a}. \tag{B3b}$$

The same proof can be used to show that for $q \geq 2$,

$$N(q,\epsilon) \sim \epsilon^{-D_{Kq}} \tag{B4}$$

and

$$D_{Kq} = \frac{1}{1+a}. \tag{B5}$$

If we now assume a set of points to have non-trivial generalised Kolmogorov capacities D_{K0} and D_{K2}, D_{K3}, *etc.* as defined by (B4) (which were first introduced under the denomination of generalised dimensions by Hentschel & Procaccia (1983) in the more restricted context of fractals), then by differentiating (B1) and (B2) with respect to ϵ we obtain ($q \geq 1$)

$$n_{q-1}(l) \sim l^{-D_{q-1}}, \tag{B6}$$

$$D_0 = D_{K0} \tag{B7a}$$

and for $q \geq 2$

$$D_{q-1} = D_{Kq}. \tag{B7b}$$

It is clear that $N(q,\epsilon) \geq N(q',\epsilon)$ if $q \leq q'$, and therefore

$$D_{K0} \geq D_{K2} \geq D_{K3} \geq D_{K4} \geq \dots \tag{B8}$$

which implies, in particular, that all generalised Kolmogorov capacities are not greater than 1 as $D_{K0} \leq 1$. From (B7) and (B8),

$$1 \geq D_0 \geq D_1 \geq D_2 \geq \dots \tag{B9}$$

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