Kelvin Waves Cascade in Superfluid Turbulence

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We study numerically the interaction of four initial superfluid vortex rings in the absence of any dissipation or friction. We find evidence for a cascade of Kelvin waves generated by individual vortex reconnection events which transfers energy to higher and higher wave numbers k. After the vortex reconnections occur, the energy spectrum scales as k^{-1} and the curvature spectrum becomes flat. These effects highlight the importance of Kelvin waves and reconnections in the transfer of energy within a turbulent vortex tangle.

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When studying a physical system which is dynamically complex, an important issue to consider is the effect of nonlinearity on the distribution of energy over the degrees of freedom of the system. For example, in the case of a classical viscous flow, the nonlinear terms of the Navier-Stokes equation redistribute the energy over various scales of motion without affecting the total energy budget, and the celebrated Richardson cascade of eddies leads to Kolmogorov's $k^{-5/3}$ dependence of energy on wave number k. The aim of our Letter is to describe a form of energy cascade of helical waves on vortex filaments (Kelvin waves). Our argument is that vortex reconnections leave behind regions of high curvature which generate Kelvin waves. Nonlinear interactions between these waves transfer energy to higher Kelvin wave numbers k'. We use the term "Kelvin wave number" to distinguish between this wave number k' (the wave number along the vortex filament) and the magnitude $k = |\mathbf{k}|$ of the wave vector $\mathbf{k}$ of the Fourier spectrum of three-dimensional space x, y, z. We investigate the Kelvin wave cascade process through direct numerical simulations of vortex filament dynamics and show that this wave cascade is clearly visible in the spectra of vortex line curvature, torsion, and line velocity.

Our results apply directly to the study of turbulence in helium II, which manifests itself as a tangle of superfluid vortex filaments [1–11]. A superfluid vortex filament is well described by the classical [12] theory of vortex filaments because the vortex core radius $a \approx 10^{-8}$ cm is microscopic and the circulation $\Gamma = 9.97 \times 10^{-4}$ cm^2/sec is fixed by quantum constraint. We represent a vortex filament as a curve $\mathbf{s} = \mathbf{s}(\xi, t)$ in three-dimensional space (where ξ is arclength and t is time) [13]. At each position $\mathbf{s}$ along a filament we define the unit vectors $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{b}}$ along the tangent, normal, and binormal directions, respectively, the local curvature $c(\xi) = |\mathbf{s}''|$ and the torsion $\tau(\xi) = |\hat{\mathbf{b}}'|$, where a prime denotes derivative with respect to arclength. The curve moves with velocity $\mathbf{v}_L$ at the point $\mathbf{s}$ given by

$$\mathbf{v}_L = \frac{d\mathbf{s}}{dt} = \frac{\Gamma}{4\pi} \int \frac{(\mathbf{r} - \mathbf{s}) \times d\mathbf{r}}{|\mathbf{r} - \mathbf{s}|^3}. \tag{1}$$

In writing (1) we neglect the friction against the normal fluid component; that is to say we concentrate our attention to low temperature superfluid turbulence [2,7]. Equation (1) is used with the extra assumption that vortex filaments reconnect when they approach each other at a sufficiently short distance; this assumption was proved [14] by using the Gross-Pitaevskii (GP) model of a Bose-Einstein condensate. The GP model reduces to our Euler vortex filament model when variations of the wave function over scales of the order of the superfluid healing length ξ are neglected ($\xi \approx a$). Our numerical method is described with more detail elsewhere [15,16]. Our reconnection technique is standard [7,13,17] and is based on the great separation of scale between the vortex core $a \approx 10^{-8}$ cm and the minimum scale $\delta = 10^{-3}$ cm which is resolved numerically, and the large separation between the reconnection time scale $a/c \approx 10^{-12}$ sec, where c is the sound speed, and the time step $\Delta t = 10^{-4}$ sec. When vortex filaments (discretized into points) approach each other by less than a distance δ, reconnection is performed instantaneously by switching the labels which tell which point is linked to which. The time step Δt is chosen so that $\Delta t < V_p \delta = 0.5 \times 10^{-3}$ sec where V_p is the velocity of the Kelvin wave with the highest wave number $k'_{\max} = \delta/2 = 500$ cm^{-1} which we define using the minimum scale δ and $\omega = \Gamma k'^2/(4\pi)\{\ln[2/(k'a)] - 0.5772\}$ is the angular velocity of the wave.

Our calculation begins with four superfluid vortex rings placed on opposite sides of a cube (Fig. 1) and oriented so that they all move toward the center [18]. Unlike other more complex configurations [7,10], this configuration is simple enough to investigate the effects of an individual reconnection. The four rings approach each other and undergo four symmetric reconnections at time $t_c = 0.059$ sec. Each reconnection introduces a cusp which then relaxes, generating large amplitude Kelvin waves. As time

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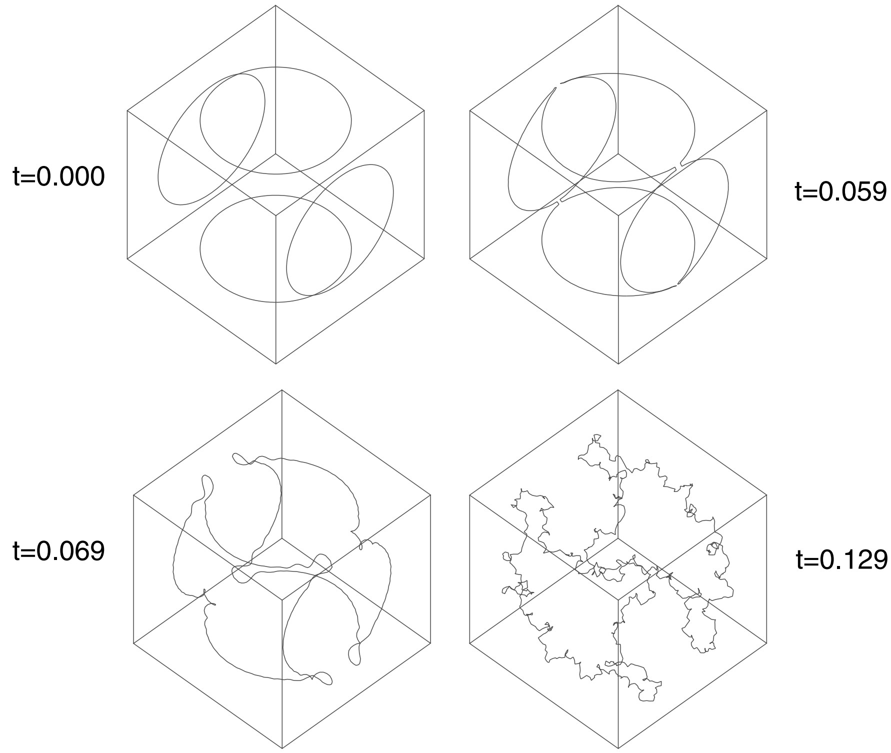

FIG. 1. Vortex configurations at $t = 0.0$ sec (initial state, 576 points used; the vortex rings have a radius of 0.023 cm and the cube has size of 0.0639 cm), $t = 0.059$ sec (first reconnection), $t = 0.69$ sec (note the Kelvin waves), and $t = 0.129$ sec (927 points used; note the crinkled shape of the filaments).

proceeds, the vortex filaments assume a crinkled shape because there is no friction with the normal fluid to smooth small scale waves [7]. During the approach of the vortex rings to reconnection very little vortex wave is visible on the rings. Since vortex lines approaching reconnection tend to twist to an antiparallel orientation, our initial conditions in which the vortices are approaching each other in an antiparallel manner may be preventing the generation of strong vortex waves before reconnection, hence isolating the effects of reconnections.

Since our model is incompressible there is no loss of energy by the generation of sound either during the reconnection event itself or as an effect of the motion of the filaments [19]. In a real system these processes would represent dissipation processes even in a pure inviscid superfluid at absolute zero temperature. The loss of vortex energy to sound emission will be most effective at some high Kelvin wave numbers k_{sound} [6]. We need therefore a mechanism to transfer energy to high Kelvin wave numbers. We show in this Letter that isolated (and rare) reconnection events along with the nonlinear transport of energy between wave modes provide this mechanism.

To study this energy transfer we first consider the spectrum $E_V(k)$ of the superfluid velocity field $\mathbf{v}_s$, hereafter referred to as the energy spectrum (or the spectrum of the Eulerian velocity) which is such that

$$\frac{\rho_s}{2}\iiint v_s^2(x,y,z)\,dx\,dy\,dz = \int_0^\infty E_V(k)\,dk\,, \qquad (2)$$

where ρ_s is the superfluid density. The results of this calculation are shown in Fig. 2. Isolated and straight vortex filaments have a $1/r$ velocity field and thus have a k^{-1} energy spectrum. For our initial conditions of four large rings the energy spectrum shows approximately an exponential behavior (see the two bottom curves of Fig. 2). After the reconnections occur and the reconnected vortex lines begin to move farther apart, the energy spectrum develops approximately a k^{-1} form (see the two top curves of Fig. 2). The energy spectrum in the high k region rises after the

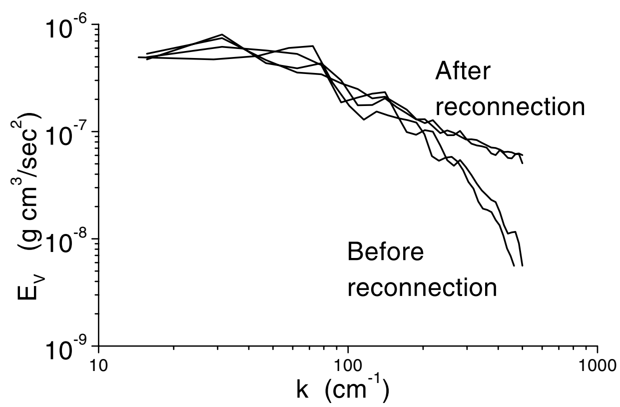

FIG. 2. Velocity spectra $E_V(k)$ before the reconnections (lower two curves at $t = 0.0$ sec and $t = 0.030$ sec) and after the reconnections (upper two curves at $t = 0.069$ sec and $t = 0.089$ sec). The spectra are obtained by discretizing the computational box into 64^3 mesh points.

reconnections, directly illustrating the transfer of energy to high k values by the reconnection events (an estimate of k_{sound} from Vinen's theory [6] gives an order of magnitude $k_{\text{sound}} > 10^2\ \text{cm}^{-1}$ consistent with the high end of our spectrum). The energy spectrum after reconnection has the same k^{-1} form of the spectrum of an isolated vortex filament. This is a bit unfortunate since it is difficult to distinguish the exponential form and the k^{-1} form at intermediate values of k, so the shape of $E_V(k)$ is relatively unaffected by the cascade.

To study this energy transfer in more detail and confirm the Kelvin wave cascade, it is more instructive to consider the curvature spectrum

$$\frac{1}{2} \int c^2(\xi)\, d\xi = \int_0^\infty E_C(k')\, dk', \tag{3}$$

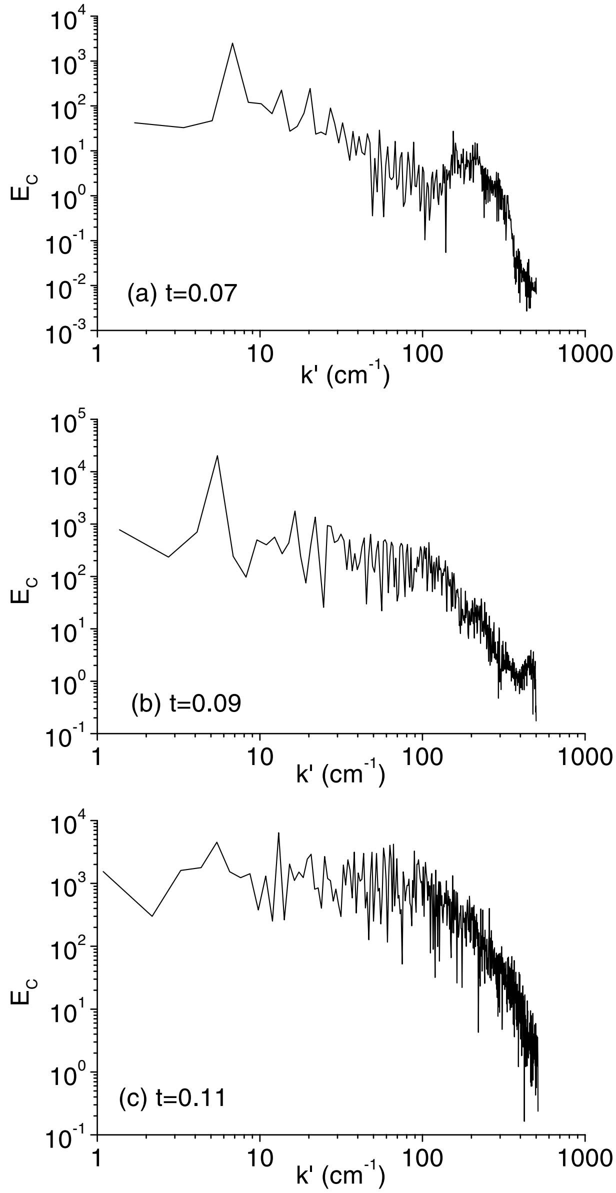

FIG. 3. Curvature spectra $E_C(k')$. (a) $t = 0.069$ sec; (b) $t = 0.089$ sec; (c) $t = 0.109$ sec.

where the right-hand-side integral is taken over the Kelvin wave number k'. The spectrum $E_C(k')$ is initially a delta function (representing the curvature of the four rings) and it remains a sharp function until the time t_c of the reconnection event. Curvature spectra at different times $t > t_c$ are shown in Fig. 3. As soon as the reconnection takes place $E_C(k')$ becomes nonzero at all Kelvin wave numbers, so two energy transfer mechanisms are operating: the instantaneous transfer of energy to a wide range of curvatures by the reconnection event and the following redistribution of that energy by nonlinear interactions. The effect of the reconnection is nonlocal in curvature space while the nonlinear wave interactions are primarily local interactions, exchanging energy between neighboring wavelength scales [20].

Given enough time for the nonlinear interaction between the Kelvin waves to equilibrate in some statistical sense, the spectrum E_C becomes constant (Fig. 3c). It takes some time to reach this equilibration, and during this process the plateau region of the spectrum followed by a roll-off spreads from low k' to high k' due to the local nature of the Kelvin wave cascade. Figure 3 shows that it takes only a short time (≈ 0.03 sec after the reconnection) to reach this equilibrium value over a significant range of the spectrum. This indicates that the nonlinear interaction of the Kelvin waves is not weak and cannot be ignored in theories of superfluid turbulence. Note that the roll-off is well resolved and occurs at wave numbers much lower than the limit k'_{max} which cause the spectrum to terminate due to the finite resolution.

Why should $E_C(k')$ be constant? The parameters of the system are Γ, ρ_s, the characteristic vortex separation distance ℓ, and the rate of vortex energy dissipation ϵ_{sound} to sound emission at high Kelvin wave numbers k_{sound} and above. If an equilibrium cascade of Kelvin waves is achieved [6] in the range $\ell^{-1} \ll k' \ll k_{\text{sound}}$ then we argue that the large and small scale parameters ℓ and ϵ_{sound} do not affect $E_C(k')$ in that range; hence $E_C(k') = E_C(\rho_s, \Gamma, k')$. The curvature spectrum $E_C(k')$ is dimensionless—see (3)—and dimensional constraints imply that $E_C(k')$ must be independent of ρ_s, Γ, and k' and

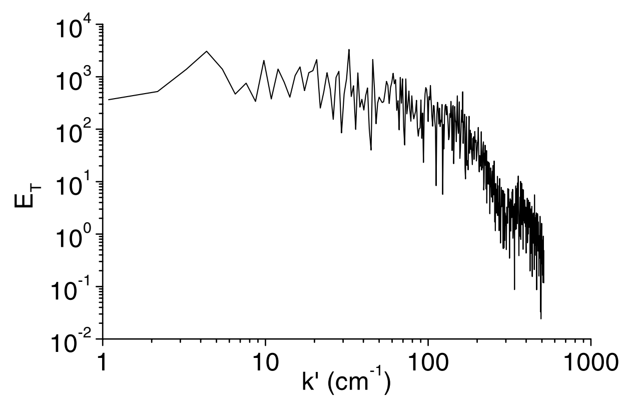

FIG. 4. Torsion spectrum $E_T(k')$ at $t = 0.109$ sec.

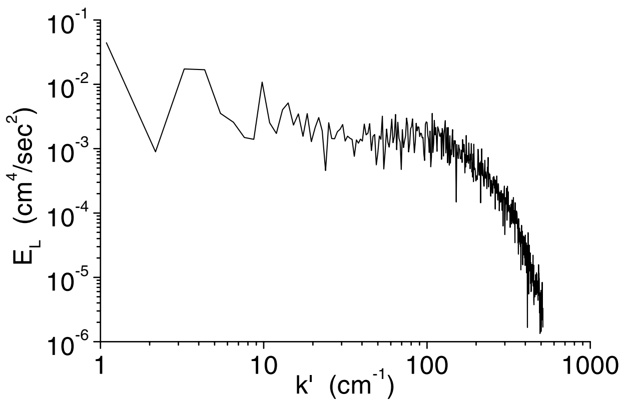

FIG. 5. Lagrangian velocity spectrum $E_L(k')$ at $t = 0.109$ sec.

be simply equal to a dimensionless constant. Figure 3 confirms this. An identical dimensional argument can be made that the spectrum $E_T(k')$ of the torsion is also equal to a dimensionless constant in the Kelvin wave cascade range of wave numbers, and evidence for this is given in Fig. 4. The spectrum $E_L(k')$ of the (Lagrangian) velocity $\mathbf{v}_L$ is such that

$$\frac{1}{2}\int v_L^2(\xi)\,d\xi = \int_0^\infty E_L(k')\,dk'. \tag{4}$$

Because $v_L \approx \Gamma c$ to the leading approximation [13], and because E_C does not depend on k', neither does E_L, and in fact $E_L \sim \Gamma^2$. Figure 5 confirms that E_L does not depend on k'.

We now consider the energy spectrum $E_V(k)$. Vinen [6] introduced a "smoothed" length of vortex line per unit volume, obtained after all the Kelvin waves have been removed, and considered $E_K(k')dk'$, the energy per unit length of the smoothed vortex line associated with Kelvin waves in the range k' to $k' + dk'$. By dimensional analysis he found that $E_K(k') \sim \rho_s \Gamma^2 k'^{-1}$. We notice that, because the fluctuations of the velocity field are induced by the Kelvin wave fluctuations on the filaments, it is reasonable to expect that $E_V(k) \sim E_K(k')$ with $k \sim k'$. Because the length of smoothed vortex lines scales with ℓ, we have $E_V(k) \sim \ell E_K(k')$. Using Vinen's result, we obtain $E_V \sim \ell \rho_s \Gamma^2 k^{-1}$, in agreement with the k^{-1} dependence observed after reconnections in Fig. 2.

In conclusion we have found direct numerical evidence of the cascade process in the interaction of Kelvin waves directly after individual reconnection events on vortex filaments. The effect of this cascade on the spectra for curvature, vortex line velocity, and torsion is strong. The computed spectra confirm our scaling arguments. These results highlight the importance of reconnections and Kelvin waves on the transfer of energy within a turbulent superfluid vortex tangle. This work should also stimulate more efforts in the development of micro-instrumentation: existing measurements of velocity spectra [5] cannot yet resolve the small scales under discussion here, due to the relative large size of the probes.

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