Deconvolution of Well Test Data as a Nonlinear Total Least Squares Problem

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Abstract

Finding a good algorithm for the deconvolution of pressure and flow rate data is one of the long-standing problems in well test analysis. In this paper we give a survey of methods which have been suggested in the past 40 years, and develop a new formulation in terms of the logarithm of the response function. Contents of the paper, as presented, have not been reviewed by the Society of Petroleum Engineers and are subject to correction by the author(s). The material, as presented, does not necessarily reflect any position of the Society of Petroleum Engineers, its officers, or members. Papers presented at SPE meetings are subject to publication review by Editorial Committees of the Society of Petroleum Engineers. Electronic reproduction, distribution, or storage of any part of this paper for commercial purposes without the written consent of the Society of Petroleum Engineers is prohibited. Permission to reproduce in print is restricted to an abstract of not more than 300 words; illustrations may not be copied. The abstract should contain conspicuous acknowledgement of where and by whom the paper was presented. Write Librarian, SPE, P.O. Box 833836, Richardson, TX 75083-3836, USA, fax 01-972-952-9435.

Introduction

The purpose of well test analysis is to determine geological properties of hydrocarbon or water reservoirs from measurements of wellbore pressure and production rate over time. This involves three steps:

1. Estimating the reservoir response from the data,
2. matching the shape of the response function against a library of type curves to identify a suitable reservoir model, and
3. fitting the parameters of this model to the data.

Here the second step is mainly qualitative, while the first and third steps are entirely quantitative.

It is solely the first step with which we shall be concerned in this paper. As a function of time, the pressure drop is the convolution product of rate and reservoir response; this is the content of Duhamel's principle. Thus, estimating the reservoir response essentially amounts to inverting a convolution integral, and is therefore an instance of a widely encountered mathematical problem called deconvolution.

In the simple case of a single flow period with constant rate, the response function can be obtained (up to scale) as the derivative of the pressure drop with respect to the logarithm of time. The standard method of obtaining response estimates is therefore to perform numerical differentiation on the pressure data. This method has a number of serious limitations:

- Numerical differentiation has the effect of amplifying measurement errors with which the data are always contaminated due to limited gauge accuracy as well as external sources of noise. The effect of gauge resolution and the radius of investigation of a well test have been studied in a number of publications; see Daungkaew, Hollaender and Gringarten and references there.
- The method extends to multirate tests, where the derivative is formed with respect to the superposition time. However, this extension is justified not so much by rigorous error analysis as by experimental evidence that it faithfully preserves the “visual shape” of model features for a set of standard type curves.
- Moreover, even this extension cannot give response estimates beyond the longest flow period with constant rate, which reduces the radius of investigation even further.

In the narrower sense, the term deconvolution refers to the variable rate problem only; this includes the case with multiple constant flow periods. The variable rate problem has received scant, but recurring attention over the last 40 years by researchers both in Petroleum and Water Resources Engineering; see references there. In the next section we shall briefly sketch the main developments and summarize their merits and shortcomings. As for the latter ones, the two methods...
which were tested with simulated data seem to share a strong sensitivity to data uncertainty once the error level reaches the order of 5% in the pressure or 1% in the rates, which in practice appears to be a fairly common level of uncertainty, at least as far as flow rates are concerned. Results obtained from such noisy signals are often affected by oscillations which can render them uninterpretable in the worst case.

In the following two sections we develop a new method which contains two novel ideas.

The first of these is an error measure for deconvolution that accounts for uncertainties not only in the pressure, but also in the rate data, which are usually much less accurately measured. The resulting formulation is what is known as a Total Least Squares (TLS) problem in the Numerical Analysis literature and as an Errors-In-Variables (EIV) problem in Statistics. TLS has become a standard approach in many disciplines, but its application to well test analysis seems to be new.

The second idea concerns the way in which the solution space is chosen to reflect prior knowledge about the solution. This can be done implicitly by the way the solution space is parametrized, or explicitly in the form of constraints on the parameters. In the Petroleum Engineering literature, the explicit approach has a long history going back to the paper by Coats, Rapoport, McCord, and Drews\(^6\) who used sign constraints on the response and its first two derivatives. Later, Kuchuk, Carter and Ayestaran\(^12\) took up this approach in a least squares fashion. Another case of was considered by Bayg"un, Kuchuk and Arikan\(^2\) who used constraints on the autocorrelation sequence and the energy of the solution vector in order to keep the solution smooth and reduce oscillations.

Our approach uses the implicit alternative instead. Its main idea is to encode the response function in a more natural way such that sign constraints are not necessary. This has the unwelcome consequence of rendering the problem nonlinear; however some of this complication is offset by the reduction in the number of constraints. In fact, our algorithm uses no constraints at all, but just a single regularizing function which is chosen as such that sign constraints are not necessary. This has the undesirable consequence of rendering the solution smooth and reduce oscillations.

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We explain our method in considerable detail and report preliminary numerical experiments with both simulated and field data. These experiments suggest that the method is capable of producing smooth, interpretable reservoir response estimates from data contaminated with fairly large errors. However, so far practical experience is still limited, and more comprehensive tests are needed. Further work is in progress.

**Deconvolution and well test analysis**

The foundation of well test analysis is Duhamel’s principle, which states that the pressure drop \(\Delta p\) at the wellbore is the convolution of the flow rate \(q\) and the reservoir impulse response \(g\) as functions of time:

\[
\Delta p = p_0 - p = q * g, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

which is shorthand for

\[
\Delta p(t) = p_0 - p(t) = \int_0^T q(\tau)g(t - \tau)\,d\tau \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)
\]

Here and throughout the paper, \(p_0\) denotes the average initial reservoir pressure and \(T\) the test duration. Flow rate and response are assumed zero for times \(t < 0\). Since the introduction of pressure derivative analysis,\(^4\) reservoir model identification is usually based on plots of

\[
\log_{10} \{tg(t)\} \text{ over } \log_{10} t. \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3)
\]

Estimating this quantity from measurements of pressure and flow rate is a task which, at least in principle, amounts to inverting the convolution integral (2); hence the name “deconvolution”.

In the case of a single flow period with constant rate, the relation between the pressure drop and the reservoir response simplifies to

\[
tg(t) = \frac{d\Delta p(t)}{d\ln t}. \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (4)
\]

Thus the standard method of obtaining response estimates is to perform numerical differentiation on the pressure data with respect to the logarithm of time.\(^4\) The limitations of this method were discussed in the introduction.

Methods for the general case can be broadly classified into two groups, time domain methods and spectral methods.

**Time domain methods** discretize the convolution integral (1) using interpolation schemes for rate and response, and proceed to solve the resulting linear system, which usually involves the choice of an error measure and often also a set of constraints on the search space of solutions. They can thus be characterized in terms of the following ingredients:

- Two sets of interpolating functions \(\{\theta_j\}\) and \(\{\psi_k\}\) such that rate and response can be written as

  \[
  q(t) = \sum_j q_j \theta_j(t), \quad g(t) = \sum_k x_k \psi_k(t), \ldots \ldots (5)
  \]

  where \(q_j\) are the measured rates and \(x_k\) the response coefficients to be found by deconvolution. Substituting (5) transforms (1) to a system of linear equations which can be written in matrix form as

  \[
  \Delta p = Qx, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6)
  \]

  where the vector \(\Delta p\) has components \(\Delta p_i = \Delta p(t_i)\), the matrix \(Q\) has coefficients

  \[
  Q_{ik} = \sum_j q_j \{\theta_j \ast \psi_k\}(t_i), \quad \ldots \ldots \ldots \ldots \ldots \ldots (7)
  \]
and \( x \) denotes the vector of response coefficients \( x_k \). The system (6) can be solved directly if the matrix \( Q \) is square and invertible; this is what early attempts suggested. In practice, the second condition can hardly ever be met; the matrix \( Q \) is often near rank deficient, and data are usually affected by measurement errors. Both facts tend to conspire in such a way as to render results obtained by direct inversion uninterpretable. There are two possible strategies in this situation which are often combined: The first is to impose constraints on the search space of response coefficients \( x_k \); the second is to choose the number of coefficients much smaller than the number of equations, and to seek a solution of the resulting overdetermined system in a least squares sense. We shall explain both alternatives in turn.

- **Constraints** on the search space of vectors \( x \) express prior knowledge about the solution, for instance from physical principles. The most obvious constraint in the case of well test analysis is that the reservoir response cannot attain negative values. Thus, assuming \( \psi_k(s) \geq 0 \) for all \( k \) and \( s \), we have to insist on

\[
x_k \geq 0 . \quad \text{................................. (8)}
\]

In fact the first and second derivatives of the reservoir response satisfy similar sign constraints, as was shown by Coats et al.,\(^6\) who to our knowledge were also the first to use constraints to determine response functions in the Petroleum Engineering literature. More recently, Kuchuk and collaborators\(^2,12\) have taken up this approach and experimented with various sets of constraints.

- **An error measure** (or objective function) \( E(x) \), is primarily a measure of the failure of a vector \( x \) to solve the underlying linear system; it can also be used to enforce other “desirable” features of a solution, like smoothness. The solution is found by minimizing \( E \) over all vectors \( x \) in the search space.

In our case, the most obvious quantity to minimize is of course the difference between the two sides of (6), i.e. the pressure match vector

\[
\varepsilon = \Delta p - Qx . \quad \text{................................. (9)}
\]

Mathematically the most convenient measure of this difference is its 2-norm

\[
E_{2,2}(x) = \| \varepsilon \|_2^2 := \sum \varepsilon_i^2 \quad \text{................................. (10)}
\]

The deconvolution problem can thus be restated as

\[
\min_{x} \| \Delta p - Qx \|_2^2 . \quad \text{................................. (11)}
\]

This is a linear least squares problem for which standard solution procedures are available. A modern implementation would be based on the singular value decomposition of the matrix \( Q \); see, for instance, Golub and van Loan,\(^10\) sec. 3.5.

A choice which can be used to enforce a degree of smoothness in the solution is the weighted sum

\[
E_{RLS}(\lambda, x) = \| \varepsilon \|_2^2 + \lambda \| Dx \|_2^2 \quad \text{................................. (12)}
\]

where \( D \) is a discretized derivative operator such that \( Dx \) approximates the vector of derivatives of the solution \( g \) between nodes, and \( \lambda \geq 0 \) is an adjustable weight. In this formulation, deconvolution becomes a regularized least squares problem. The solution \( x \) of this problem will of course depend on \( \lambda \). Two main criteria for the selection of this parameter have been suggested in the mathematical literature: the L curve criterion\(^9\) which can also be used in the more general Total Least Squares setting to be discussed in the following section, and cross validation,\(^9\) which in practice is limited to the regularized least squares problem. We shall come back to this issue later.

Table 1 summarizes what we see as the three main publications to date on time domain methods. We have not implemented any of these methods ourselves, and our comments on their performance are therefore based on the results shown in these papers. Only the last two methods seem to have been tested with simulated signals contaminated with varying levels of noise. From these tests it appears that the two constrained least squares methods give reasonable results for noise levels of up to 2% in the pressure, but that the deconvoluted response can contain large oscillations if the error level increases to 5%. Moreover, results with as little as 1% error in continuously measured rates seem to render results obtained with the linear set of constraints completely uninterpretable.\(^{12}\)

In practice, even higher error levels of up to 10% are known to occur in measured rates. None of the methods seems to have been tested with errors at this level.

**Spectral methods** have received comparatively less attention so far. They are based on the convolution theorem of spectral analysis which states that the convolution product commutes with spectral transforms like the Laplace and Fourier transforms. Applied to our deconvolution problem (1) this means

\[
\Delta p = g \ast q \iff \Delta \hat{p} = \hat{g} \cdot \hat{q} , \quad \text{................................. (13)}
\]

where quantities with bar denote the transforms of the quantities without bar. Thus an estimate for the transform of \( g \) can be obtained as

\[
\hat{g} = \frac{\Delta \hat{p}}{\hat{q}} \quad \text{................................. (14)}
\]

provided that \( \Delta \hat{p} \) and \( \hat{q} \) can be estimated from the data. Finally \( g \) could be estimated by applying the inverse transform to \( \hat{g} \). Alternatively, Bourgeois and Horne\(^3\) who investigated the case of the Laplace transform suggested that the model identification step could be done in Laplace space.

Duhamel’s principle in the form of eq. (2) highlights a problem with this method in general: The convolution theorem (13) holds strictly only if \( T = \infty \), i.e. if we know that both pressure
drop and flow rate are zero after the end of the test. However, in practice we can only know this for the rate; to wait until the pressure has reached equilibrium level would not be economically sensible for most tests.

Numerical experiments we carried out ourselves show that the estimates $\hat{g}$ obtained with the incomplete pressure signal up to the finite test duration $T$ tend to follow the Laplace transform of the correct response function truncated at time $T$. This observation could be practically relevant if there were reliable numerical methods available to invert Laplace transforms taken over finite intervals. However, this appears to be a notoriously difficult problem itself, and the only algorithm we found in the literature did not seem accurate enough for our purpose.

For the rest of the paper we shall be concerned with time domain methods only.

**Deconvolution as a bilinear TLS problem**

It was already mentioned that rate uncertainties are an important source of errors in the deconvolved response, and that none of the deconvolution methods suggested to date accounts for this type of uncertainty. However, only a minor modification of the error model is required in order to allow for rate errors, as we will presently show.

Notations are as follows: $p$ and $q$ continue to denote measured pressure and rate signals. The true, but unobserved signals are $p + \varepsilon$ and $q + \delta$, where $\varepsilon$ and $\delta$ are signals representing the measurement errors. These unobserved signals satisfy Duhamel’s principle (cf. (1)):

$$ p + \varepsilon = p_0 - y * g, \quad y = q + \delta $$

Interpolating the functions on the right hand side according to

$$ y(t) = \sum_j y_j \theta_j(t), \quad g(t) = \sum_k x_k \psi_k(t) $$

and evaluating the resulting relation at times $t_i, i = 1 \ldots m$ at which the pressure measurements are taken, one obtains

$$ p_i + \varepsilon_i = p_0 - \sum_{j,k} A_{ijk} (q_j + \delta_j) x_k $$

where $A_{ijk}$ denotes the rank 3 convolution tensor

$$ A_{ijk} = \{ \theta_j * \psi_k \}(t_i) $$

We combine this set of equations with an error measure which is a weighted sum of the squared norms of the two error sequences:

$$ E_{TLS}(\nu, \varepsilon, \delta) = \| \varepsilon \|_2^2 + \nu \| \delta \|_2^2 $$

where $\nu \geq 0$ is an adjustable weight. For instance, the choice

$$ \nu_{rel} = \frac{\| \Delta p \|_2^2}{\| p \|_2^2} $$

minimizes the sum of relative squared errors,

$$ \frac{\| \varepsilon \|_2^2}{\| \Delta p \|_2^2} + \frac{\| \delta \|_2^2}{\| p \|_2^2} = E_{TLS}(\nu_{rel}, \varepsilon, \delta) $$

In this setting, the deconvolution problem can be stated as

$$ \min_{p_0, \varepsilon, \delta} \left[ \frac{\| \varepsilon \|_2^2}{\| \Delta p \|_2^2} + \frac{\| \delta \|_2^2}{\| p \|_2^2} \right] \text{ subject to (17).} $$

This formulation is an example of what is known as a Total Least Squares (TLS) problem in Numerical Analysis and as an Errors-In-Variables (EIV) problem in Statistics. A different, but equivalent formulation involving the true rates $y$ is

$$ E_{TLS} = \| r(p_0, x, y) \|_2^2 $$

where the vector

$$ r(p_0, x, y) = \left[ \frac{\varepsilon}{\sqrt{\nu} \delta} \right] = \left[ \frac{p_0 - p - x.A.y}{\sqrt{\nu} (y - q)} \right] $$

 whose 2-norm is minimized is called the TLS residue. Here $x.A.y$ denotes the vector with components

$$ \sum_{j,k} A_{ijk} y_j x_k, \quad i = 1 \ldots m. $$

Note that the residue is linear in each of the three unknowns $p_0, x$ and $y$ alone, but quadratic in the vector $(p_0, x, y)$ of all unknowns together!

This is an instance of a separable nonlinear least squares problem. One of the standard algorithms for this class of problems is the Variable Projection algorithm (see, for instance, Björck,3 [§9.4.1]). The algorithm solves the more general prob-

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Table 1. Selection of time domain methods published in the Petroleum Engineering literature.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Year</th>
<th>Error measure</th>
<th>Constraints</th>
<th>Solution method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coats &amp; al.6</td>
<td>1964</td>
<td>None (linear system)</td>
<td>Signs of $g$ and its first two derivatives</td>
<td>Linear programming</td>
</tr>
<tr>
<td>Kuchuk &amp; al.12</td>
<td>1990</td>
<td>$E_{LS}$</td>
<td>Signs of $g$ and its first two derivatives</td>
<td>Least squares with linear inequality constraints (active set algorithm)</td>
</tr>
<tr>
<td>Baygün &amp; al.2</td>
<td>1997</td>
<td>$E_{LS}$</td>
<td>$| Dg |_2$ (D a discretized derivative), auto-correlation coefficients of $t_j(t)$</td>
<td>Low-dimensional nonlinear least squares with nonlinear constraints</td>
</tr>
</tbody>
</table>

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*Other norms may be more appropriate if the number and time spacing of pressure and rate data is very different. We shall come back to this issue in the section on “Practical details”.*
The matrix $F(z)$ and the vector $v(z)$ are arbitrary differentiable functions of $z$. Unlike most standard algorithms for the linear least squares problem, this algorithm is iterative and thus requires an initial guess $(y_0, z_0)$.

We have implemented a version of this method and also a regularized version with error measure

$$E_{RTLS} = \| \varepsilon \|_2^2 + \nu \| \delta \|_2^2 + \lambda \| Dz \|_2^2$$

and regularization parameter $\lambda \geq 0$, for which the residue is given by

$$r(p_0, x, y) = \left[ \frac{\varepsilon}{\sqrt{\nu}} \delta \right] \sqrt{\lambda Dx}$$

Preliminary experiments suggest that deconvolved results still show large oscillations when rates are contaminated by errors of the order of 10%. Responses computed from field data containing a number of coefficients with negative sign and were not interpretable. Therefore we shall not include any numerical results from this method.

Our experience suggests that sign constraints are still necessary if the Total Least Squares approach is used to deconvolve linear response estimates, i.e., estimates of the function $f_g(t)$ itself. An alternative strategy to treat the oscillation problem is to encode the solution in a more natural way that makes sign constraints unnecessary. The obvious candidate for such an encoding is the logarithm of the function $f_g(t)$, which is the quantity usually plotted over log$_{10} t$ in the diagnostic plot. This is the approach we shall develop in the following section.

**Deconvolution as a nonlinear TLS problem**

To avoid a profusion of numerical factors, we shall replace the decadic logarithms by natural ones; thus we aim to estimate

$$z(\sigma) = \ln \{ f_g(t) \}, \quad \sigma = \ln t, \quad t \in [0, T].$$

(The diagnostic plot is then simply a plot of $z(\sigma)$/ln 10 against $\sigma$/ln 10.) Substituting (29) into the superposition principle (1), one obtains

$$\Delta p(t) = \int_{-\infty}^{\ln T} e^{z(\sigma)} q(t - e^\sigma) d\sigma.$$  (30)

We seek to match a piecewise linear function $z$ to the data. Thus we choose nodes $\sigma_k$ such that

$$-\infty = \sigma_0 < \sigma_1 < \sigma_2 < \ldots < \sigma_n = \ln T$$

and interpolate $z$ linearly between values $z_k = z(\sigma_k)$ which are to be computed:

$$z(\sigma) = \alpha_k + \sigma \beta_k$$

where we assume $\beta_1 = 1$ to model wellbore storage, and

$$\beta_k = \frac{z_k - z_{k-1}}{\sigma_k - \sigma_{k-1}}, \quad \alpha_k = z_k - \beta_k \sigma_k.$$  (33)

As before, we interpolate the actual rate function $y$ in the form (16). Substituting these expressions in (30), we obtain the following model for the pressure signal:

$$p(t) = p_0 - \sum_{j=1}^{N} y_j G_j(z, t),$$

$$G_j(z, t) = \sum_{k=-1}^{n} \theta_j(t - e^\sigma) e^{\alpha_k + \sigma \beta_k} d\sigma.$$  (35)

This model is still linear in the flow rates $y_j$, $j = 1..N$ and the initial pressure $p_0$, but nonlinear in the response parameters $z_k$. The parameters can be fitted using the Variable Projection algorithm in its general form. Combining the initial pressure and the actual flow rates to a vector

$$y = (p_0, y_1, y_2, \ldots, y_N),$$

forming the error sequence $\varepsilon$, and comparing equations (26) and (28), one finds the following expressions for the coefficients of the matrix $F(z)$ and the vector $v(z)$:

$$F_{ij}(z) = \begin{cases} -1 & i = 1..m, j = 0, \\ \sqrt{\nu} & i = m + j, j = 1..N, \\ 0 & \text{otherwise}, \end{cases}$$

$$v_i(z) = \begin{cases} -p_i & 1 \leq i \leq m, \\ \sqrt{\nu} q_{i-m} & m < i \leq m + N, \\ -\sqrt{\lambda} (Dz)_{i-m-N} & i > m + N, \end{cases}$$

where the index ranges are $i = 1..m + N + d$ and $j = 0..N$ and $d$ is the number of rows in the matrix $D$ (usually $n - 1$).

For the numerical experiments reported below we have implemented the case of stepwise rate functions, in which $q_j$ and $y_j$ refer to constant flow rates in some time interval $I_j$. Expressions for the quantities required by the Variable Projection algorithm are given in the appendix.

**Practical details**

There are still a number of choices to be made before the method is completely specified. They are listed in the following table together with the default settings we used in the experiments reported below.
Some of these choices and the way they interact require a little more explanation:

- The first node \( \sigma_1 \) should be chosen early enough to ensure that the region \( \sigma < \sigma_1 \) corresponds to wellbore storage, which is what our model assumes. If \( \sigma_1 \) is chosen too late, some of the early time features may not be visible.

- If \( Dz \) is chosen as the vector of derivatives of \( z \), then for an arbitrary spacing of nodes we want

\[
Dz = (\beta_2, \beta_3, \ldots, \beta_n) \quad \text{...} \quad (39)
\]

(not including the unit derivative for \( \sigma < \sigma_1 \), and thus we need to choose

\[
D_{kl} = \begin{cases} 
-(\sigma_k - \sigma_{k-1})^{-1} & \text{if } l = k, \\
(\sigma_k - \sigma_{k-1})^{-1} & \text{if } l = k + 1, \\
0 & \text{otherwise}.
\end{cases} \quad \text{...} \quad (40)
\]

(cf. (33)). If the nodes are uniformly spaced with step size \( h \),

\[
\sigma_k = \sigma_1 + (k - 1)h, \quad k = 1 \ldots n, \quad \text{...} \quad (41)
\]

then

\[
D = \begin{pmatrix} 
-\frac{1}{h} & \frac{1}{h} & \frac{1}{h} & \cdots & \frac{1}{h} \\
\frac{1}{h} & -\frac{2}{h} & \frac{1}{h} & \cdots & \frac{1}{h} \\
\frac{1}{h} & \frac{1}{h} & -\frac{2}{h} & \cdots & \frac{1}{h} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{h} & \frac{1}{h} & \frac{1}{h} & \cdots & -\frac{2}{h} \\
\end{pmatrix} \quad \text{...} \quad (42)
\]

Penalizing derivatives in this fashion is not the only way to impose smoothness on the deconvolved results; another conceivable choice is to penalize \textit{curvature} instead. However so far we have not investigated this alternative.

- Just as \( \nu_{rel} \) (20) is a natural unit for the relative weight in the error measure, there is such a unit for the regularization parameter \( \lambda \) once a regularization matrix \( D \) is chosen, as we will now show. To ensure that the energy measure is invariant under subdivision of node intervals, we normalize \( D \) to unit Frobenius norm:

\[
\| D \|_F^2 := \sum_{l,k} D_{kl}^2 = 1. \quad \text{...} \quad (43)
\]

Then a natural, scale-invariant measure of the “relative smoothness” of \( z \) is

\[
\frac{\| Dz \|_2}{\| z \|_2} \quad \text{...} \quad (44)
\]

A dimensionless formulation of the deconvolution problem is to minimize a weighted sum of the two relative errors and this smoothness measure,

\[
\frac{\| \varepsilon \|_2^2}{\| \Delta p \|_2^2} + \frac{\| \delta \|_2^2}{\| q \|_2^2} + \lambda \frac{\| Dz \|_2^2}{\| z \|_2^2} \quad \text{...} \quad (45)
\]

where \( \varepsilon \) and \( \lambda \) are the dimensionless forms of their counterparts \( \nu \) and \( \lambda \). Comparing this expression with \( E_{RTLS} / \| \Delta p \|_2^2 \) (see (27)), one obtains

\[
\nu = \nu_{rel} \quad \text{and} \quad \lambda = \lambda_{rel} \text{, ...} \quad (46)
\]

with \( \nu_{rel} \) as in (20) and

\[
\lambda_{rel} = \frac{\| \Delta p \|_2^2}{\| \Delta p \|_2^2} \quad \text{...} \quad (47)
\]

Here in practice we would suggest replacing \( z \) by the initial solution \( z_0 \) in order to avoid updating \( \lambda_{rel} \).

As the number of nodes is usually much smaller than the number of pressure data points, (47) is likely to overestimate the optimal regularization parameter by a factor of roughly \( m/n \). It may therefore be more appropriate to replace the discrete 2-norms by their continuous counterparts defined by

\[
\| f \|_2^2 := \int_0^T |f(t)|^2 dt \quad \text{...} \quad (48)
\]

or at least to divide the 2-norms by the number of vector components. Similar remarks apply to \( \nu_{rel} \). If the number and time sampling of pressure and rate data is very different, as is likely when only an average rate is given for each flow period.

- Once the units \( \nu_{rel} \) and \( \lambda_{rel} \) have been determined, we choose a range of scalings around them, for instance,

\[
\nu = 10^{-3} \ldots 10^3, \quad \lambda = 10^{-3} \ldots 10^3, \quad \text{...} \quad (49)
\]

and compute one iteration with each resulting pair \((\nu, \lambda)\). For each value of \( \nu \) we then plot the curves

\[
(\rho, \zeta) := (\| \varepsilon \|_2, \| Dz \|_2) \quad \text{...} \quad (50)
\]
parametrized by $\lambda$. These are the “L curves” mentioned to above; they are simply a visualization of the compromise between the smoothness of $z$ and the degree of its compatibility with Duhamel’s principle, given the data. The name “L curve” refers to the shape these curve have in regularized linear least squares problems, where the theory predicts a distinct point of maximal inflection which is chosen as the optimal value of $\lambda$. The observation remains essentially true in our nonlinear case, even though the curves look a little different (see Fig. 6 for an example); thus we adopt the selection principle and pick the pair $(\nu, \lambda)$ corresponding to the point of maximal inflection which is closest to the origin.

- It remains to choose the starting point for the iteration, i.e. the pair $(y_0, z_0)$.

For $y_0$ there is an obvious choice, namely an estimate for the initial pressure $p_0$ obtained from the data followed by the measured rates. As an estimate for $p_0$ we simply used the maximum of the pressure data sequence; alternatively the first measurement could be used.

As starting point $z_0$ for the response we use wellbore storage (unit slope) before the first node and a constant value afterwards,

$$z_0(\sigma) = \begin{cases} \ln c_0 + \sigma & \text{if } \sigma \leq \sigma_1, \\ \ln c_1 & \text{if } \sigma \geq \sigma_1. \end{cases} \quad (51)$$

Here continuity at the first node implies

$$\ln c_0 = \ln c_1 - \sigma_1; \quad (52)$$

the value of $c_1$ is determined such that the best pressure match is obtained. In detail, evaluating (30) with the model (51) leads to

$$\Delta p(t) = c_0 \Delta p_1(t) + c_1 \sum_j q_j \mu_j(t), \quad (53)$$

where

$$\mu_j(t) = \int_{\sigma_1}^{\ln T} \theta_j(t - e^\sigma) \, d\sigma \quad (54)$$

and

$$|\Delta p_1(t)| \leq \exp(\sigma_1) \cdot \max \{ |\eta_j|, j = 1..N \}. \quad (55)$$

By choosing the first node early enough, the contribution $\Delta p_1$ from the early part can be made arbitrarily small. Thus we only evaluate the remaining part at the times $t_i$ of the pressure measurement, obtaining an error sequence which in vector notation is given by

$$\varepsilon = c_1 M q - \Delta p, \quad (56)$$

where the components $\Delta p_i = p_0 - p_i$ are computed with the starting value of $p_0$ and $M$ denotes the matrix

$$M = \begin{pmatrix} \mu_1(t_1) & \mu_2(t_1) & \ldots & \mu_N(t_1) \\ \mu_1(t_2) & \mu_2(t_2) & \ldots & \mu_N(t_2) \\ \vdots & \vdots & & \vdots \\ \mu_1(t_m) & \mu_2(t_m) & \ldots & \mu_N(t_m) \end{pmatrix}. \quad (57)$$

The desired minimum of $\|\varepsilon\|_2$ is attained for

$$c_1 = \frac{\Delta p^T M q}{\|M q\|^2}. \quad (58)$$

Our practical experience so far suggests that the method is stable in the sense that results do not depend on the starting point of the iteration. The choice we just described has the merit of being relatively easy to compute and not too distant from interpretable response curves, which ought to reduce the number of iterations required for convergence.

- Finally, a stopping criterion needs to be given. We stopped the iteration when the drop in the two-norm $||r||_2$ of the residue fell below 1% of its initial value in the simulated examples, and below 0.1% in the field examples. For both sets of examples, this took at most 5 iterations.

### Numerical results

#### Tests with simulated data.

Figures 1–4 show results for a simulated example; all quantities are dimensionless. The simulated reservoir behaviour is radial flow with wellbore storage coefficient $C_D = 100$, skin coefficient $S = 5$ and a sealing fault at a distance of $d = 300$ wellbore radii according to the model given by Agarwal, Al-Hussainy and Ramey. The observation data are shown in black. Three levels of measurement rate error are indicated as a function of time. The duration of the longest flow period is $2 \times 10^4$, which corresponds to the dashed line at $\tau = 4.3$ in Fig. 1. The duration of the entire test is $2 \times 10^3$, which corresponds to the dashed line at $\tau = 5.3$. Thus by construction the difference between the two reservoir models emerges only after the longest rate period, and is therefore invisible for a conventional test.

Clean data are shown in black. Three levels of measurement errors are simulated and colour-coded as follows:

<table>
<thead>
<tr>
<th>Colour</th>
<th>Pressure error</th>
<th>Rate error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Green</td>
<td>0.5 %</td>
<td>—</td>
</tr>
<tr>
<td>Yellow</td>
<td>0.5 %</td>
<td>1 %</td>
</tr>
<tr>
<td>Red</td>
<td>0.5 %</td>
<td>10 %</td>
</tr>
</tbody>
</table>
Here the error levels are defined as the relative variances of the two error signals, $\|\epsilon\|^2 / \|\Delta p\|^2$ and $\|\delta\|^2 / \|q\|^2$. The error level in the pressure signal would correspond to the following set of assumptions: Permeability 100 mD, reservoir thickness 50 ft, flow rate 1000 barrels per day, viscosity 0.6 cP, pressure uncertainty 2.5 psi.

The remaining process parameters were as follows: $\nu = 44.87$, $\lambda = 424.8$, and $c_1 = 0.5$. Fig. 4 shows deconvolved responses obtained with and without rate optimization. The starting values are shown as black dots. Responses converge rapidly towards the true derivative type curve; the maximum number of iterations taken to reach the stopping criterion was 5. However, for 10% rate error the result with rate adaptation is visibly smoother towards the end than the one without, even though regularization parameters are the same. Increasing the regularization parameter beyond the level chosen does not improve the deconvolved results, but tends to flatten features and to slow down convergence. This shows that unless rate data are very accurate, regularization alone is not always capable of producing interpretable results and should therefore be complemented by a more suitable error model that accounts for rate errors.

The apparent stabilization for late times in the case with flow rate optimization is of course a fluke and happens at the wrong level. The remaining discrepancy at early times is probably due to their lesser statistical weight on the linear time scale used in the pressure match, and could be reduced by lowering the stopping threshold.

**Tests with field data.** Figures 5–8 show results obtained from the same set of field data with Bourdet’s method and with the method suggested in this paper. The data are from the now abandoned Maureen field and comprise 7 constant rate periods and 858 pressure data points. The entire test duration is 65 hours with no flow for the first 34 hours, which makes the usable test duration about 31 hours. These consist of two short and one long flow period interspersed by short buildups and followed by a final buildup of 17 hours, which is the longest constant rate period and was therefore chosen for the conventional analysis shown in Fig. 5. The fitted model (shown in colours) is characterized by wellbore storage, skin, partial penetration, radial flow and a sealing fault.

Figure 6 shows the L curves for our method obtained with the entire data set. The optimal parameter combination (marked by a grey dot) was found to be

$$\nu = 4.48 \times 10^{-4} \left( \frac{\text{psi}}{\text{bopd}} \right)^2, \quad \lambda = 3.16 \times 10^5 \text{ psi}^2 \quad (60)$$

Figures 7 and 8 show the first 4 iterations of deconvolved response and pressure match for these values using 20 and 41 nodes, respectively. The nodes are spaced uniformly on the logarithmic scale between $10^{-3}$ and 31 hours, whereas the conventional method can only interpret 17 hours. Thus our method extends the interpretable time by a factor of almost 2, corresponding to one third of a log cycle.

We added a number of flow periods during the longest drawdown to match obvious discontinuities in the pressure signal; this is visible in the detailed pressure match in Figure 7. The initial values for the response and the pressure data are shown as black dots. Responses in Figures 7 and 8 are rate-normalized whereas the conventional response shown in Figure 5 is not; this explains the scale factor between them which is of the order of the last rate (4100 bopd).

In both deconvolved responses, the *early time* features are much less pronounced than in the conventional plot; this problem was already noted in the context of the simulated example. Here it is probably compounded by the lack of data at early times. Besides the hump due to skin seems to appear later in the deconvolved plot.

Both deconvolved and conventional type curves show the -1/2 slope characteristic of partial penetration, which is known to be true. Their *middle and late time* behaviour is quite different though: The deconvolved plots show a late time stabilization at a much lower level compared with the rest of the plot. Moreover, the plot with 41 nodes shows an intermediate level of stabilization which does not appear in the two other plots and could be interpreted as composite or multi-layered behaviour.

The end of the conventional response curve is badly affected by measurement noise which is evidently amplified by numerical differentiation. By construction, this source of error is not present in our method, and thus the deconvolved responses obtained with our method generally look smooth and interpretable provided the regularization parameter is carefully chosen.

The method is currently implemented in *Mathematica*, a list-based programming environment. Computation times for 41 nodes, 3 flow periods and 858 data points were about 250 seconds per iteration, and thus about 1000 seconds for the four iterations shown and for a single weight and regularization parameter. Additionally a much larger share of the total computing time is spent on finding the optimal parameter pair, even though this part of the computation was done with only two iterations. Future implementations in a compiled language are expected to run considerably faster.

**Conclusions**

1. The standard way in which the well test deconvolution problem has been treated in the Petroleum Engineering literature is as a linear Least Squares problem with explicit sign and energy constraints.

2. This strategy fails to produce interpretable results at error levels of the order of up to 5% in the pressure signal and 1% in the rate signal, as reported by the authors.
Fig. 1 – Pressure and derivative type curve for radial flow (black) and a sealing fault at 300 wellbore radii (blue) with wellbore storage and skin coefficients CD=100 and S=5, and interpolated derivative type curve with nodes given by eq. (59). The two vertical lines mark the end of the longest flow period and the end of the test.

Fig. 2 – Pressure matches for simulated data with 0.5% error in the pressure and 10% in the rates. Data in black, pressure curves with the deconvolved response in red. Dashed curve without rate optimization, solid curve with rate optimization.

Fig. 3 – Rates (left) and pressure signals (right) obtained from the model response shown in Fig. 1. Clean data in black. Simulated errors: 0.5 % in the pressure only (green); dto. with 1 % error in rates (yellow); dto. with 10 % error in the rates (red).

Fig. 4 – True type curve and deconvolved responses for the simulated tests shown in Fig. 3. Left: results obtained by optimizing initial pressure and responses only. Right: results obtained by optimizing initial pressure, responses, and rates. Black dots mark the initial response $z_0$. The two vertical lines indicate the end of the longest flow period and the end of the test.
Fig. 5 – Field example: Response estimate obtained from final buildup only with Bourdet’s algorithm (black) together with fitted reservoir model (pressure drop in red, derivative in purple). (Output from Interpret 2000.)

Fig. 6 – L curves for the field example with 41 nodes. Weight $\nu$ ranging from $10^{-6}$ (orange, hidden) to $10^{-1}$ (dark blue) and regularization parameter $\lambda$ ranging from $10^{4.5}$ (top) to $10^{7.5}$ (bottom of plot). The grey dot marks the optimal pair.

Fig. 7 – Deconvolved response (in psi/bopd) and detail of the pressure match (in psi) over time (in hours) for the field example with 20 nodes and additional flow periods. Initial response and pressure data in black. Iterations: 1st (light green), 2nd (green), 3rd (blue), 4th (purple).

Fig. 8 – Deconvolved response (in psi/bopd) and pressure match (in psi) over time (in hours) for the field example with 41 nodes and additional flow periods. Initial response and pressure data in black. Iterations: 1st (light green), 2nd (green), 3rd (dark blue), 4th (red).
3. In this paper we have introduced a new method which reformulates deconvolution as a nonlinear Total Least Squares problem. The two main improvements are

- a nonlinear encoding of the reservoir response which makes explicit sign constraints obsolete, and
- a modified error model which accounts for errors in both pressure and rate data.

4. Preliminary experiments with simulated and field data suggest that the new method is capable of deconvolving smooth, interpretable response functions from data contaminated with errors of up to 10% in rates provided that error weight and regularization parameter are suitably chosen.

Nomenclature

\( A_{ijk} \) = rank 3 convolution tensor, eqn. (18)
\( c_0, c_1 \) = coefficients in initial response \( z_0 \), eqn. (51)
\( d \) = number of rows in matrix \( D \)
\( D \) = discretized derivative matrix
\( E \) = general error measure
\( E_{LS} \) = error measure for Least Squares, eqn. (10)
\( E_{TLS} \) = error measure for Total Least Squares, eqn. (23)
\( E_{RTLS} \) = error measure for regularized Total Least Squares, eqn. (23)
\( F(z) \) = matrix in separable nonlinear LS residue, eqn. (26)
\( g(t) \) = reservoir impulse response (dimensionless or in psi/bopd), eqn. (2)
\( G_j(z,t) \) = functions in nonlinear convolution model, eqn. (35)
\( G_{j\ell}(z,t) \) = functions in nonlinear convolution model, eqn. (A-3)
\( h \) = uniform step size between nodes \( \sigma_k \)
\( I_j, I_{j\ell} \) = intervals, eqns. (A-1), (A-2)
\( m \) = number of pressure data points
\( M \) = matrix, eqn. (57)
\( n \) = number of nodes for the deconvolved response
\( N \) = number of rate data points
\( p(t) \) = continuous pressure (dimensionless or in psi)
\( p, p_i \) = (vector of) pressure data (dimensionless or in psi)
\( p_0 \) = average initial pressure (dimensionless or in psi)
\( q(t) \) = continuous flow rate (dimensionless or in bopd)
\( q, q_j \) = (vector of) discrete flow rates
\( Q \) = matrix, eqn. (7)
\( r \) = residue vector, eqns. (24), (26), (28)
\( t \) = time (dimensionless or in hours)
\( T \) = test duration (dimensionless or in hours); as superscript: transpose of a vector
\( v(z) \) = vector in separable nonlinear LS residue, eqn. (26)
\( x, x_k \) = (vector of) coefficients of linear response function, eqn. (5) (dimensionless or in psi/bopd)
\( y, y_j \) = (vector of) true rates (dimensionless or in bopd); from eqn. (36): preceded by true initial pressure
\( y_0 \) = starting values for vector \( y \)
\( z(\sigma) \) = continuous logarithmic response function (diagnostic plot), eqn. (29)
\( z, z_k \) = (vector with components) \( z(\sigma_k) \)
\( z_0 \) = starting values for vector \( z \)
\( \alpha_k, \beta_k \) = coefficients of logarithmic response interpolation, eqn. (33)
\( \delta, \delta_j \) = (vector of) absolute rate errors (dimensionless or in bopd)
\( \delta_{\ell} \) = radius of interval \( I_{\ell} \), eqn. (A-2)
\( \Delta t \) = step size between nodes \( \sigma_{l-1} \) and \( \sigma_l \) in the case of arbitrary node spacing
\( \Delta p(t) \) = continuous pressure drop signal (dimensionless or in psi)
\( \Delta p, \Delta p_i \) = (vector of) pressure drop data (dimensionless or in psi)
\( \varepsilon, \varepsilon_i \) = (vector of) absolute pressure data errors (dimensionless or in psi)
\( \zeta \) = \( \| Dz \|_2 \), eqn. (50)
\( \theta_j(t) \) = interpolating functions for rate (dimensionless), eqns. (5), (16)
\( \lambda \) = regularization parameter, eqns. (12), (23)
\( \lambda = \lambda / \lambda_{rel} \)
\( \lambda_{rel} \) = natural unit for regularization parameter, eqn. (47)
\( \mu, \mu_{\ell} \) = midpoint of interval \( I_{\ell} \), eqn. (A-2)
\( \nu \) = relative error weight, eqns. (23), (23)
\( \nu = \nu / \nu_{rel} \)
\( \nu_{rel} \) = natural unit for relative error weight, eqn. (20)
\( \rho \) = 2-norm of least squares residue, eqn. (50)
\( \sigma \) = natural logarithm of time (dimensionless or log(hours))
\( \sigma_k \) = nodes for logarithmic response interpolation (dimensionless or log(hours))
\( \tau \) = integration variable
\( \psi_k \) = interpolating functions for linear response, eqns. (5), (16)
\( \| \cdot \| \) = 2-norm of a continuous function, eqn. (48)
\( \| \cdot \|_2 \) = 2-norm of a vector, eqn. (10)
\( \| \cdot \|_F \) = Frobenius norm of a matrix, eqn. (43)
\( f \) = Laplace transform of a function \( f \)

Acknowledgments

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We are also grateful to Professor Gene Golub (SCCM Stanford and Imperial College London) for his advice on numerical methods and particularly on Total Least Squares.
Appendix: Formulae for step-wise rate functions

So far we have left the interpolation scheme for the flow rates completely general. For the numerical experiments reported below we have implemented the case of stepwise rate functions, in which \( q_j \) and \( y_j \) refer to constant flow rates in some time interval \( I_j \) and

\[
\theta_j(t) = \begin{cases} 
1 & \text{if } t \in I_j, \\
0 & \text{if } t \notin I_j,
\end{cases} \quad (A-1)
\]

In this case, let \( I_\beta = [a_\beta, b_\beta] = [\mu_\beta - \delta_\beta, \mu_\beta + \delta_\beta] \) denote the interval

\[
I_\beta = \{ \sigma \in [\alpha_{-1}, \alpha_1] : t - e^\sigma \notin I_j \}. \qquad (A-2)
\]

For the sake of brevity we shall omit the index pair \( j \ell \) in \( \alpha, b, \mu \)

and \( \delta \). Then we obtain \( G_j(z, t) = \sum_{j=1}^n G_\beta(z, t) \) with

\[
G_\beta(z, t) = \begin{cases} 
0 & \text{if } I_\beta = \emptyset, \\
\exp(\alpha(z) + b_\beta z)/b_\beta & \text{if } \alpha = -\infty, \\
2\delta e^{\alpha_1 + \mu_\beta z} \sinh(\delta_\beta z) & \text{if } \beta = 0, \\
2\beta e^{\alpha_1 + \mu_\beta z} \sinh(\delta_\beta z) & \text{if } \beta \neq 0.
\end{cases} \quad (A-3)
\]

One further quantity needs to be specified for the Variable Projection algorithm: the Jacobian of the residual vector \( r(y, z) \). In the general case its coefficients are

\[
\frac{\partial r_i}{\partial z_k} = \begin{cases} 
\sum_{j=1}^n b_j \frac{\partial G_j}{\partial z_k}(z, t_i) & 1 \leq i \leq m, \\
0 & m < i \leq m + N, \\
\sqrt{\lambda} D_i - m - N, k & i > m + N.
\end{cases} \quad (A-4)
\]

Thus it suffices to give the derivatives of the quantities \( G_{\beta_l}(z, t) \):

\[
\frac{\partial G_{\beta_l}}{\partial z_{-\ell}} = \begin{cases} 
2(\mu - \sigma_{-\ell})e^{\alpha \ell}/\Delta_l & l > 1, \beta_l = 0, \\
(\mu - \sigma_{-\ell} - 1)/\Delta_l & \sinh(\delta_\beta)/1 + \delta \cos(\delta_\beta) & \text{otherwise},
\end{cases} \quad (A-5)
\]

\[
\frac{\partial G_{\beta_l}}{\partial z_{\ell}} = \begin{cases} 
2(\mu - \sigma_{-\ell} + 1)/\Delta_l & l = 1, \alpha = +\infty, \\
2e^{\alpha_1 + \mu_\beta z} \sinh(\delta_\beta) & l = 1, \alpha = -\infty, \\
2\delta(\mu - \sigma_{-\ell})e^{\alpha \ell}/\Delta_l & l > 1, \beta_l = 0, \\
2e^{\alpha_1 + \mu_\beta z} \sinh(\delta_\beta) & l > 1, \beta_l = 0, \\
(\sigma_{-\ell} - 1)/\Delta_l & \sinh(\delta_\beta)/1 + \delta \cos(\delta_\beta) & \text{otherwise},
\end{cases}
\]

where \( \Delta_l = \sigma_{-\ell} - \sigma_{-\ell-1} \), and

\[
\frac{\partial G_{\beta_l}}{\partial z_{k}}(z, t) = 0 \quad \text{if } k \neq \ell, l. \quad \text{……………….. \( (A-5) \)}
\]