The Shifted Rayleigh Filter: A New Algorithm for Bearings Only Tracking

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Abstract—A new algorithm, the ‘shifted Rayleigh filter’, is introduced for two or three-dimensional bearings only tracking problems. In common with other ‘moment matching’ tracking algorithms such as the extended Kalman filter and its modern refinements, it approximates the prior conditional density of the target state by a normal density; the novel feature is that an exact calculation is then performed to update the conditional density in the light of the new measurement. The paper provides the theoretical justification of the algorithm. It also reports on simulations involving variants on two scenarios, which have been the basis of earlier comparative studies. The first is a ‘benign’ scenario where the measurements are comparatively rich in range related information; here the shifted Rayleigh filter is competitive with standard algorithms. The second is a more ‘extreme’ scenario, involving multiple sensor platforms, high dimensional models and noisy measurements; here the performance of the shifted Rayleigh filter matches the performance of a high order bootstrap particle filter, while reducing the computational overhead by an order of magnitude.

Index Terms—bearings only tracking, nonlinear filtering, moment matching filters.

I. INTRODUCTION

The purpose of bearings only tracking is to determine the current position of an object or ‘target’ from a series of noise corrupted measurements of its direction, taken from some observer platform. Bearings only tracking is a special case of nonlinear filtering, that is, of recursive Bayesian estimation of a signal process, describing the evolution of the state of an observed system, given values of a measurement process. The difficulties of estimating target range from sensors only providing bearings information, for certain target and sensor platform configurations, are well known [1], [2].

The challenge of bearings only filtering is dealing with the presence of a nonlinearity in the equations that describe the relationship between target position and the bearings only measurements. In nonlinear filtering problems of this nature, no suitable description of the conditional distribution of the state, given present and past measurements, is available for computations. All practical algorithms then are based on approximations.

We propose in this paper a new technique for bearings only tracking, which we refer to as the shifted Rayleigh filter. This is, inevitably, based on the approximation of conditional expectations. But the algorithm incorporates calculations that exploit, in an apparently new way, the essential structure of the nonlinearities present in bearings only tracking. Furthermore, the nature of the approximations involved in the construction of the filter is particularly transparent. We provide a theoretical justification of the algorithm and report on the results of simulations assessing its performance.

A distinctive feature of the shifted Rayleigh filter is the mathematical model of the noisy measurement process on which it is based. A special case of this model relates the state bearings measurement \( b_t \) (interpreted as a vector of ‘direction cosines’) to the state \( x_t \) at time \( t \) through the equation

\[
b_t = \Pi (H x_t + w_t) .
\]

Here, \( \Pi \) denotes the projection of 3-dimensional space onto the unit sphere (in the case of bearings only measurements in 3-dimensional space), or of the plane onto the unit circle (for measurements in the plane). \( H \) is a matrix chosen so that \( H x_t \) is the vector of Cartesian coordinates of the displacement of the target from the observer, associated with the state vector \( x_t \) at time \( t \). The term \( w_t \) represents white noise (‘measurement’ noise). What is unusual here is that the noise \( w_t \) is modelled as additive noise present in an ‘augmented’ measurement, \( y_t = H x_t + w_t \), of the Cartesian coordinates of target relative to the platform, which is projected onto the sphere (or circle) to generate the ‘actual’ bearings measurement. This contrasts with the traditional ‘angle plus white noise’ model. The covariance of the noise process \( w_t \) can be chosen in different ways to take account of a variety of mechanisms, by which the bearings measurements are corrupted by noise.

One iteration of the shifted Rayleigh filter is summarized as follows. It takes, as starting point, a normal approximation to the conditional distribution of the state at time \( t - 1 \). The exact conditional distribution of the state at time \( t \), taking account also of the new measurement, is calculated. This conditional distribution is then approximated by a normal distribution, by matching the first and second moments of the true distribution, in preparation for the next iteration.

We see that the only approximation introduced by the algorithm is to replace a conditional distribution by a ‘matched’ normal distribution at a single point in each iteration. (This is what we meant by ‘transparency’ of the approximations involved; the isolation of the approximation also allows for further analysis and potential improvement).

It is useful to divide tracking methods into two categories. The ‘density’ category, which contains particle filters [3], IMM filters [4], and the range-parameterized extended Kalman filter [5], comprises algorithms that aim at direct approximation of the conditional densities of the state. Algorithms in the ‘moment matching’ category are based on local linearization or discrete approximation [6] and the description of distributions...
in terms of their first and second moments. This category includes the extended Kalman filter and its refinements [7], [8], [9], ‘pseudomeasurement’ approaches to bearings only tracking [10], [7], [11], [12] and the unscented Kalman filter [6]. Density trackers are more versatile and have the potential for greater accuracy, but are generally much more computationally demanding than moment matching trackers.

The shifted Rayleigh filter is a moment matching filter and offers the computational savings of such algorithms. At the same time, it aims to improve on performance of other moment matching algorithms applied to bearings only tracking problems, because it is based on an exact calculation of the updated conditional density of the state at each iteration, when the prior density is taken to be normal.

Simulations have been carried out to compare the shifted Rayleigh filter (and refinements to allow for multiple sensor platforms, presence of clutter and manoeuvring targets) with standard algorithms applied to bearings only tracking problems. These include extended Kalman filters, unscented Kalman filters, pseudo-measurements filters and particle filters. The results of the simulations are reported in this paper and elsewhere [13], [14], [15].

Broadly speaking the performance of all of the filters examined in these simulations is comparable for ‘benign’ bearings only tracking scenarios where the measurements are comparatively rich in range related information. But in more challenging scenarios involving, for example, high dimensional models and high noise and clutter levels, the shifted Rayleigh filter is often superior to alternative moment matching filters (when they work at all), in terms of accuracy of estimates and robustness (insensitivity of the estimates to the choice of modelling parameters). The simulations also demonstrate that, for the scenarios considered, the computational requirements of the shifted Rayleigh filter are an order of magnitude less than those of particle filters, for which the number of particles have been adjusted to give comparable accuracy of estimation.

Finally we remark on the computational demands of the shifted Rayleigh filter as compared with other moment matching filters. The shifted Rayleigh filter makes use of nonlinear functions that are not in closed form and calls on non-standard library functions (for example, the ‘complementary error function’ in MATLAB) for their evaluation. It might be thought that, for this reason, the algorithm would be computationally demanding. This though is not the case; calling these functions require only a little more computation time than the evaluation of, say, a cosine. In fact we have found that for high dimensional problems, such as those involving multiple sensor platforms, the shifted Rayleigh filter is more computationally efficient that the extended Kalman filter (because the shifted Rayleigh algorithm avoids the computation of Jacobians) or the unscented Kalman filter (because it avoids costly function evaluations associated with this algorithm).

Some points on notation. Given a vector process \( \{r_i\} \) and integers \( i, j \) with \( j \geq i \), then \( r_{i:j} \) denotes the finite sequence \( \{r_i, r_{i+1}, \ldots, r_j\} \). To simplify notation, we use the same symbol for a random variable and also for its values. Vectors will be denoted by bold lower case letters (or underscoring in the case of Greek symbols) and matrices by capital letters. The Euclidean length of a vector \( d \) is written \( ||d|| \). We denote by \( I_{r \times r} \) the \( r \times r \) identity matrix and by \( O_{m \times n} \) the \( m \times n \) matrix of zeros. If \( z_1, z_2 \) are scalars, the function \( \arctan(z_2/z_1) \) is to be interpreted in the ‘four quadrants’ sense; that is, as the angle of rotation of the vector \( (z_1, z_2)^T \) from the first axis in the direction of the second axis.

II. FORMULATION OF THE BEARINGS ONLY TRACKING PROBLEM

Let \( r \) be the dimension of the space in which bearings measurements are taken \((r = 2 \text{ or } 3)\). A \( d \)-vector process \( x_0, x_1, \ldots \), which describes the motion of the target and platform in \( R^d \), an \( r \)-vector process \( d_1, d_2, \ldots \), which describes the position or ‘displacement’ of a nominal target ‘centre’ relative to the sensor platform and an \( r \)-vector process \( b_1, b_2, \ldots \), which describes the sequence of noisy bearings measurements, are governed by the following equations:

\[
\begin{align*}
x_t &= Fx_{t-1} + u_t^{m-1} + v_{t-1} \quad \text{(the system equation)} \\
d_t &= Hx_t + u_t^{m} \quad \text{(the displacement equation)} \\
b_t &= \Pi(d_t + w_t) \quad \text{(the measurement equation)}
\end{align*}
\]

for \( t = 1, 2, \ldots \).

Here \( \Pi \) denotes the projection of \( r \)-vectors onto the unit sphere in 3-dimensional space (if \( r = 3 \)) or onto the unit circle (if \( r = 2 \)). It is convenient in formulating the shifted Rayleigh filter to regard the bearing as an \( r \)-vector \( b \) of unit length in the direction of the displacement vector perturbed by a noise term. For targets in 3-dimensional space, a more conventional description of bearing is given in terms of azimuth \( \eta \) and elevation \( \epsilon \) of target position relative to the sensor platform, perturbed by noise. In the noise free case, the two are related as follows:

\[
b = (\cos(\eta) \cos(\epsilon), \sin(\eta) \cos(\epsilon), \sin(\epsilon))^T.
\]

In the preceding equations the processes \( v_0, v_1, \ldots \) and \( w_1, w_2, \ldots \) are sequences of independent, Gaussian random variables, with covariances \( Q_0, Q_1, \ldots \) and \( Q_1^m, Q_2^m, \ldots \), respectively. They are the system and measurement noise processes, respectively. \( u_0^m, u_1^m, \ldots \) and \( u_0^m, u_1^m, \ldots \) are deterministic sequences, included in the model to increase its versatility, for instance to compensate for the effects of a moving coordinate frame.

A precise formulation of the model requires specification of:

- \( r \) dimension of augmented measurement space \((r = 2 \text{ or } 3)\)
- \( d \) dimension of state space
- \( F \) \( d \times d \) system matrix
- \( H \) \( r \times d \) augmented measurement matrix
- \( Q^s \) \( r \times d \) system noise covariance matrix
- \( Q^m \) \( r \times r \) measurement noise covariance matrix
- \( u_t^s \) \( d \) vector system input signal
- \( u_t^m \) \( r \) vector measurement input signal

The last four terms \((Q^s, Q^m, u_t^s \text{ and } u_t^m)\) may depend on measurements taken prior to time \( t \).
Of particular interest will be cases when the measurement noise covariance takes the form:

\[ Q_i^{m} = \sigma^2 E[|d_i|^2 | \theta_{1:i-1}]I_{2 \times 2} + Q^{fr}. \]  

(1)

Here, \( \sigma^2 \) is a variance parameter for the noise on the sensor and \( Q^{fr} \) represents the covariance of ‘translational’ noise, so called because this term can be used to take account of noisy perturbations of either the target or the observer platform. We elaborate on this particular form of the measurement noise covariance in the next section.

The tracking problem is to recursively estimate the conditional expectation and conditional covariance of the state \( x_t \), given present and past bearings measurements \( b_{1:t} \), for \( t = 1, 2, \ldots \).

III. THE BEARING MODEL

In the derivation of the shifted Rayleigh filter, the vector bearing \( b \) is modelled by adding noise to the ‘displacement’ vector \( d \) to generate the ‘augmented’ measurement \( y \) and by projecting this onto the unit sphere (if \( r = 3 \) or the circle (if \( r = 2 \)):

\[
\begin{align*}
    y &= d + w \\
    b &= \Pi(d + w).
\end{align*}
\]

(2)

Here \( w \) is a Gaussian random variable with covariance \( E[|d|^2 | \theta_1]I_{2 \times 2} + Q^{fr} \), independent of \( d \).

This is clearly a natural model in those cases where the dominant noise on the bearing arises from random translations of the target (or random fluctuations of the ‘contact’ point of the line of sight over an extended target) that can be characterized by a covariance matrix \( Q^{fr} \). If, however, it is the sensor noise that dominates (i.e. \( \sigma^2 > 0, Q^{fr} \approx O_{2 \times 2} \)), the relevance of the model is less clear.

In this section we justify the use of our model (2) and relate it to the standard model for the bearing sensor noise. For simplicity we shall suppose that \( r = 2 \) and \( Q^{fr} = O_{2 \times 2} \) (planar measurements and no translational noise).

The standard models for bearing noise differ from that in (2) in that the measured bearing is expressed in a traditional ‘signal plus white noise’ form:

\[
\theta = \arctan(d_1/d_2) + n.
\]

(3)

Here, \( n \) is an \( N(0, \sigma^2) \) variable, independent of the displacement vector \( d \), that represents the sensor noise. Note that, in this particular version, the bearing angle is being interpreted in the cartographic sense; that is, as clockwise from ‘north’.

A. The Displacement Vector

The displacement \( d \) describes the position of the target relative to the sensor platform. It is related to the state vector \( x \) according to the equation:

\[
d = Hx + u^m.
\]

(4)

(The affine nature of this equation is crucial to the derivation of the shifted Rayleigh filter.) In many applications it is natural to take the first two pairs of state elements \((x_{11}, x_{21}), (x_{31}, x_{41})\) as the Cartesian coordinates of the nominal ‘centres’ of the

This conforms to the above model when we choose

\[
H = \begin{bmatrix}
1 & 0 & -1 & 0 & \ldots & 0 \\
0 & 1 & 0 & -1 & \ldots & 0
\end{bmatrix}
\]

and \( u^m = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

In other applications, it is computationally more efficient to represent the position \( x^0 \) of a moving sensor platform as a function of time, such as \( x^0 = x^0 + tc \). In these circumstances, we might take the displacement vector to be (4) in which

\[
H = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0
\end{bmatrix}
\]

and \( u^m = -(x^0 + tc) \).

B. The Measurement Noise Model

We now relate the shifted Rayleigh bearing model (2) to the traditional model (3). As a first step it is helpful to consider the following variant on the shifted Rayleigh noise model

\[
\begin{align*}
    y' &= d + |d|e' \\
    b' &= \Pi(y').
\end{align*}
\]

(5)

in which \( e' \) is an \( N(0, \sigma^2 I_{2 \times 2}) \) distributed random variable, independent of \( d \). The only difference here is in the noise terms \( w \) and \( |d|e' \) used to construct the augmented measurements \( y \) and \( y' \). Notice however that these noise terms are matched to second order: they have identical first and second moments and they are both uncorrelated with \( d \). Indeed

\[
E[|d|e'] = 0
\]

\((w) \) and \(|d|e' \) have zero mean

\[
E[|d|e'd^T] = E[|d|e'e'd^T] | d | = 0 \times I_{2 \times 2}
\]

\((d) \) and \(|d|e' \) are uncorrelated

\[
\text{cov}(|d|e') = E[|d|^2e'e'^T] = E[|d|^2] \sigma^2 I_{2 \times 2}
\]

\((w) \) and \(|d|e' \) have covariance \( E[|d|^2] \sigma^2 I_{2 \times 2} \).

Now let us examine the angular representation \( \theta' \) of the modified vector bearing \( b' \). It is shown in [14] that it is related to the true bearing \( \arctan(d_1/d_2) \) as follows:

\[
\theta' = \arctan(d_1/d_2) + \nu',
\]

(6)

where \( \nu' \) is a zero mean random variable, restricted to \([-\pi, +\pi]\), independent of \( d \), with density \( \alpha_{\nu}(\cdot) \):

\[
\alpha_{\nu}(\theta) = \frac{e^{-\nu^2}}{2\pi \sqrt{2\pi} \sigma} F_{\text{normal}}(\cos \theta) e^{\frac{1}{2}(\cos^2 \theta - \cos^2 \nu')}. \]

Here, \( F_{\text{normal}} \) is the cumulative distribution function of the normal \( N(0,1) \) variable. Notice that \( \theta' \), given by (6), is very close to the bearing \( \theta \) in the standard model (3); indeed the only difference between \( \theta' \) and \( \theta \) is that the densities of the noise terms used in their construction are \( \alpha_{\nu}(\cdot) \) and the normal \( N(0, \sigma^2) \) density respectively. Fig. 1 plots these two density functions for a range of \( \sigma \) values. We see that, for \( \sigma \) less than a radian, the uniform variation between the densities is
It also introduces a scalar variable $d$ of the target state used to construct the augmented measurement Rayleigh filter noise model (2), in which the noise term $w$ used to calculate $\gamma_t$ differs from $w$, but has identical first and second moments, and is uncorrelated with $d$. A consequence of this is that, in applications of the shifted Rayleigh filter to problems initially formulated in terms of the standard model (5), it is generally perfectly reasonable to identify the two versions of the noise variance parameter $\sigma^2$ appearing in the two models (2) and (5). In the simulations in the final sections, we shall do this without further comment.

IV. THE SHIFTED RAYLEIGH FILTER

The algorithm recursively generates a sequence

$$\hat{x}_{0|0}, \hat{x}_{1|1}, \ldots$$

of $d$-vectors and a sequence

$$P_{0|0}, P_{1|1}, \ldots$$

of $d \times d$ covariance matrices. For each $t$, $\hat{x}_{t|t}$ and $P_{t|t}$ are interpreted as estimates of the conditional mean and covariance of the target state $x_t$, given measurements up to time $t$, $b_{1:t}$. In addition, the algorithm introduces, for times $t = 1, 2, \ldots$, the following variables:

$$\hat{x}_{t|t-1}$$ $d$-vector predicted mean of $x_t$

$$P_{t|t-1}$$ $d \times d$-matrix predicted covariance of $x_t$

$$V_t$$ $r \times r$-matrix predicted covariance of $y_t$

$$K_t$$ $d \times r$-matrix Kalman gain

$$\gamma_t$$ estimated conditional mean of $|y_t|$ given $b_t$

$$\delta_t$$ estimated conditional variance of $|y_t|$ given $b_t$.

It also introduces a scalar variable $z_t$ used to calculate $\gamma_t$ and $\delta_t$.

Algorithm:

Initial Data:

$$\mathbf{m}_{x_0}$$ $d$-vector state mean at time $t = 0$

$$P_0$$ $d \times d$ covariance matrix of state at time $t = 0$.

Set

$$(\hat{x}_{0|0}, P_{0|0}) = (\mathbf{m}_{x_0}, P_0).$$

For $t = 1, 2, \ldots$ calculate $(\hat{x}_{t|t}, P_{t|t})$ from $(\hat{x}_{t-1|t-1}, P_{t-1|t-1})$ using the value of $(\hat{x}_{t-1|t-1}, P_{t-1|t-1})$ obtained from the previous recursive step and the measurement $b_t$ at time $t$, by means of the following relationships:

Prediction Step:

$$\hat{x}_{t|t-1} = F\hat{x}_{t-1|t-1} + u_{t-1}$$

$$P_{t|t-1} = FP_{t-1|t-1}F^T + Q_t^m$$

Now let

$$V_t = HP_{t|t-1}H^T + Q_t^m$$

Correction Step:

$$K_t = P_{t|t-1}H^T V_t^{-1}$$

$$z_t = (b_t^TV_t^{-1}b_t)^{-1/2} b_t^TV_t^{-1}(H\hat{x}_{t|t-1} + u_{t-1})$$

$$\gamma_t = (b_t^TV_t^{-1}b_t)^{-1} - \frac{1}{2} \rho_2(z_t)$$

$$\delta_t = (b_t^TV_t^{-1}b_t)^{-1} \left( 3 + 2z_t\rho_3(z_t) - 2\rho^2_2(z_t) \right)$$

$$\hat{x}_{t|t} = (I - K_tH)\hat{x}_{t|t-1} - K_t u_{t|m} + \gamma_t K_t b_t$$

$$P_{t|t} = (I - K_tH)P_{t|t-1} + \delta_t K_t b_t b_t^T K_t^T$$

In these formulæ, $\rho_1(z)$ is the mean of what we choose to call a ‘shifted Rayleigh’ variable:

$$\rho_1(z) = \frac{\int_0^\infty s e^{-(z+s)^2/2} ds}{\int_0^\infty s^{-1} e^{-(z+s)^2/2} ds}.$$

In the case when $r = 2$ and $z = 0$, this will be recognized as the mean of the standard Rayleigh variable.

$\rho_2(z)$ is a positive increasing function that is asymptotic to $-\frac{2}{z}$. 

$\rho_3(z)$ behaves similarly. A straightforward analysis reveals that $\rho_2(z)$ and $\rho_3(z)$ can be expressed as

$$\rho_2(z) = z e^{-z^2/2} + \sqrt{2\pi} (z^2 + 1) F_{\text{normal}}(z)$$

$$\rho_3(z) = z + 2 \rho_2(z).$$

Here, as before, $F_{\text{normal}}(z)$ is the cumulative distribution function of a normal $N(0,1)$ variable. In spite of the appearance of the first equation, these functions are easy to compute; $F_{\text{normal}}(z)$ is just the rescaling $\frac{1}{2}\text{erfc}(\frac{z}{\sqrt{2}})$ of the complementary error function $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$, which is a standard library function in numerical computing packages.
To appreciate the structure of the shifted Rayleigh filter, it is helpful to consider the form the filter would take if access were provided not just to the bearing \( b_t = y_t / |y_t| \), but to the augmented measurement \( y_t \) itself:

\[
y_t = H x_t + u_t^m + w_t.
\]

In this case, the evolving conditional expectation of the state would be governed by the Kalman filter equations, key ingredients of which are:

\[
\begin{align*}
\hat{x}_{t|t} &= (I - K_t H) \hat{x}_{t|t-1} - K_t u_t^m + K_t y_t \\
\hat{P}_{t|t} &= (I - K_t H) \hat{P}_{t|t-1}
\end{align*}
\]

We see that the shifted Rayleigh filter reproduces the equations of the Kalman filter, except in two respects. First, in the updating equation for \( \hat{x}_{t|t} \), the augmented measurement \( y_t \) (information about which is available only through its bearing), is replaced by \( E[y_t | b_t] \). To see this we note, from the analysis of Section V, that \( \gamma_t = E[|y_t| | b_t] \). (Strictly speaking, this relationship is valid only when the conditional density of the state at time \( t \), given \( b_{t-1} \), is normal.) Then

\[
E[y_t | b_t] = E[|y(t)| | b_t | b_t] = \gamma_t b_t,
\]

consistent with this interpretation. Second, the updating equation for the conditional covariance \( \hat{P}_{t|t} \) in the shifted Rayleigh filter includes an extra term \( \delta_t K_t b_t b_t^T K_t^T \), to take account of the fact, once again, that only partial information about \( b_t \), namely its bearing, is available and therefore the error covariance is ‘increased’. The extra term is precisely the conditional covariance of \( K_t y_t \) given \( b_{t-1} \). Indeed \( \delta_t \) is the conditional variance of \( |y_t| \) given \( b_t \). (This also follows from the analysis in Section V.) So

\[
\text{cov}[K_t y_t | b_t] = \text{cov}[|y(t)| | K_t b_t | b_t] = \text{var}[|y(t)| | b_t | K_t b_t b_t^T K_t^T] = \delta_t K_t b_t b_t^T K_t^T
\]

as claimed.

We emphasize that these salient features of the shifted Rayleigh filter are not merely empirical justifications to the Kalman filter to take account of the fact that the measurements provide the bearing \( b_t \) of \( y_t \), not \( y_t \) itself. Rather they are the outcome of an exact analysis of the updated conditional density of the state, under the assumption that the prior distribution is normal.

Some Extensions:

For simplicity, we have developed the ideas behind the shifted Rayleigh filter just for time invariant tracking problems involving a single bearings only measurement sensor. These ideas trivially extend to the time varying case. They may be further extended to allow for several sensors and also for measurements that include affine functions of the state as well as the bearing. Consider, for instance, situations involving two bearings only sensors, mounted on spatially separated platforms. Here, the measurement process can be modelled by the projections of two augmented measurements, thus

\[
b_t^i = \Pi(H^i x_t + u_t^i + w_t^i),
\]

the index \( i \) identifying the relevant sensor. In this situation, at each time \( t \), the prediction and correction steps are applied in turn to the bearings vector of each sensor. For more general formulations in which some observations are bearings and others are affine functions of the state, the affine observations can be incorporated by a standard Kalman filtering algorithm before the application of the steps for the bearings observations.

V. Analysis

This section provides a theoretical justification for the shifted Rayleigh filter algorithm. The system and measurement process equations of Section II relate random variables \( x_t \) and \( b_t \) to a random variable \( x_{t-1} \) (via random variables \( v_{t-1} \) and \( w_t \) ). At each iteration, the input to the algorithm is a normal distribution \( N(\hat{x}_{t-1|t-1}, P_{t-1|t-1}) \), interpreted as the distribution of \( x_{t-1} \) and a vector, interpreted as a value of \( b_t \). The output of the algorithm is a vector \( \hat{x}_{t|t} \) and a covariance matrix \( \hat{P}_{t|t} \). The following proposition tells us that, according to this interpretation, the algorithm calculates the conditional mean and covariance of \( x_t \) given \( b_t \). For the set-up of Section II this suggests that, if the conditional distribution of \( x_{t-1} \), given measurements up to time \( t-1 \), closely approximates a normal distribution, then one iteration of the algorithm will yield a close approximation to the true conditional mean and covariance of \( x_t \), given measurements up to time \( t \).

**Proposition 5.1:** Let \( Q^s \) and \( Q^m \) be given covariance matrices of dimension \( d \times d \) and \( r \times r \) respectively. Let \( x_{t-1}, v_{t-1} \) and \( w_t \) be independent random variables distributed as

\[
x_{t-1} \sim N(\hat{x}_{t-1|t-1}, P_{t-1|t-1}), \quad v_{t-1} \sim N(0, Q^s), \quad w_t \sim N(0, Q^m)
\]

in which \( \hat{x}_{t-1|t-1} \) is a given \( d \)-vector and \( P_{t-1|t-1} \) is a given \( d \times d \) covariance matrix. Define \( x_t \) and \( b_t \) by

\[
\begin{align*}
x_t &= F x_{t-1} + u_t^s + v_{t-1} \\
b_t &= \Pi((H x_t + u_t^m + w_t)
\end{align*}
\]

(\( F, H, \) etc. are as in Section II). Then

\[
E[x_t | b_t] = \hat{x}_{t|t} \quad \text{and} \quad \text{cov}[x_t | b_t] = \hat{P}_{t|t},
\]

where \( \hat{x}_{t|t} \) and \( \hat{P}_{t|t} \) are calculated from \( \hat{x}_{t-1|t-1} \) and \( P_{t-1|t-1} \) and \( b_t \) via the shifted Rayleigh filter equations (7)–(15).

We precede the proof with some key moment calculations, the results of which are summarized as a lemma:

**Lemma 5.1:** Let \( y \) be an \( r \)-dimensional (\( r = 2 \) or \( 3 \)) random variable with distribution \( N(m, R) \) (\( R > 0 \)). Let \( s \) and \( \hat{\theta} \) be

\[
s = |R^{-\frac{1}{2}} y| \quad \text{and} \quad \hat{\theta} = |R^{-\frac{1}{2}} y|^{-1} R^{-\frac{1}{2}} \hat{y}.
\]

(\( s \) and \( \hat{\theta} \) are the length and projection (the ‘direction cosines’) of a normalized version \( R^{-\frac{1}{2}} y \) of \( y \). Then the conditional density of \( s \) given \( \hat{\theta} \) is the ‘shifted Rayleigh’ density given by

\[
p(s | \hat{\theta}) = \frac{s^{r-1} e^{-\frac{1}{2}(s-\hat{\theta})^2}}{\int_0^{\infty} s^{r-1} e^{-\frac{1}{2}(s-\hat{\theta})^2} ds}.
\]
where
\[ z = m^T R^{-\frac{1}{2}} \hat{\rho} \]

Furthermore, the moments of this conditional distribution satisfy the relation, for \( n \geq 0 \),
\[ E[s^n] = (n + r - 2)E[s^{n-2}] + zE[s^{n-1}] \]

In particular, the mean \( \rho_r(z) \) and variance \( \sigma_r^2(z) \) are related by the following formula:
\[ \sigma_r^2(z) = r + z \rho_r(z) - (\rho_r(z))^2 \]

**Proof:** The random variable \( y' = R^{-\frac{1}{2}} y \) has mean \( R^{-\frac{1}{2}} m \) and density
\[ p(y') = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}(y' - R^{-\frac{1}{2}} m)^T (y' - R^{-\frac{1}{2}} m)} \]

In the case \( r = 2 \), consider the polar-coordinates representation of \( y' \), in terms of the random variables \( s \) and \( \alpha \):
\[ s = ||y'|| \]
and
\[ \alpha = \arctan(y'_2/y'_1) \]
that is, \( \cos(\alpha) = y'_1/||y'|| \) and \( \sin(\alpha) = y'_2/||y'|| \). On calculating the density of \( y' \) following a change of coordinates from Cartesian to polar, we find that
\[ p(s, \alpha) = \frac{s^{-1}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}((s-z)^2 + \frac{1}{2}(s^2 - m^T R^{-1} m - z^2))} \]
where
\[ z = m^T R^{-\frac{1}{2}} (\cos(\alpha), \sin(\alpha))^T = m^T R^{-\frac{1}{2}} \hat{\rho} \]
It follows that
\[ p(s|\hat{\rho}) = \frac{1}{K(s)} s^{-1} e^{-\frac{1}{2}(s-z)^2} \]
where \( K(s) \) is the normalizing factor \( \int_{0}^{\infty} \sigma^{r-1} e^{-\frac{1}{2}(\sigma-z)^2} d\sigma \), as claimed. Note that the term \( s^{-1} \) in the numerator of this expression is just the determinant of the Jacobian of the transformation between Cartesian and polar coordinates. The \( r = 3 \) case is treated similarly, but by consideration of the spherical-coordinates representation of \( y' \).

Let \( s \) be a shifted Rayleigh variable, governed by this distribution. Then
\[ E[s^n] = \frac{1}{K(s)} \int_{0}^{\infty} \sigma^{n+r-1} e^{-\frac{1}{2}(\sigma-z)^2} d\sigma \]
\[ = \frac{1}{K(s)} \left( \int_{0}^{\infty} (\sigma^{n+r-2} (\sigma-z) e^{-\frac{1}{2}(\sigma-z)^2}) d\sigma \right) + \frac{1}{K(s)} \left( \int_{0}^{\infty} (z \sigma^{n+r-2} e^{-\frac{1}{2}(\sigma-z)^2}) d\sigma \right) \]
\[ = \frac{1}{K(s)} \left( (n+r-2) \int_{0}^{\infty} \sigma^{n+r-3} e^{-\frac{1}{2}(\sigma-z)^2} d\sigma \right) \]
\[ + zE[s^{n-1}] \]
\[ = (n+r-2)E[s^{n-2}] + zE[s^{n-1}] \]
as asserted. (An integration by parts has been used to derive the third line above.) This is the desired relationship between moments. The concluding formulae follows from this equation, in the special case \( n = 2 \), since \( \sigma_r^2(z) = E[s^2] - (\rho_r(z))^2 \).

**Proof:** (Proposition 5.1) Throughout this section, we take \( \dot{x}_{t|t-1}, \dot{I}_{t|t-1} \), etc. to be as defined in (7)–(15). Define \( y_t \), the ‘augmented’ measurement, to be
\[ y_t = Hx_t + u_t^\alpha + w_t \]
We shall exploit the following decomposition formula for \( x_t \):
\[ x_t = \xi_t + (I - K_t H) \dot{x}_{t|t-1} - K_t u_t^m + K_t y_t \]
(16)
where \( \xi_t \) is a random variable with the following property: \( \xi_t \) is independent of \( y_t \), and \( \dot{x}_{t|t-1} \), and
\[ \xi_t \sim N(0, (I - K_t H) P_{t|t-1}) \]
(17)
This is essentially a ‘stochastic’ phrasing of the Kalman filter ‘correction’ equations. Here, \( K_t \) is the gain matrix given by (10). Furthermore, conditioned on \( \dot{x}_{t-1|t-1} \),
\[ y_t \sim N(H \dot{x}_{t|t-1} + u_t^m, V_t) \]
Define the random variables
\[ s_t = ||V_t^{-\frac{1}{2}} y_t|| \]
and
\[ \theta_t = ||V_t^{-\frac{1}{2}} y_t||^{-1} V_t^{-\frac{1}{2}} y_t \]
that is, \( s_t \) and \( \theta_t \) are the range and bearing vector of the normalized augmented measurement \( V_t^{-\frac{1}{2}} y_t \). Notice that
\[ \theta_t = (b_t^T V_t^{-\frac{1}{2}}) V_t^{-\frac{1}{2}} b_t \]
(18)
and
\[ b_t = (\theta_t^T V_t\theta_t)^{-\frac{1}{2}} V_t^\frac{1}{2} \theta_t \]
(19)
It follows from these relationships that conditioning on the random variable \( b_t \) is equivalent to conditioning on \( \theta_t \).
Since \( V_t^{-\frac{1}{2}} y_t = s_t \theta_t \), we can decompose \( y_t \) as:
\[ y_t = s_t V_t^{-\frac{1}{2}} \theta_t \]
It follows then from (16) that
\[ x_t = \xi_t + (I - K_t H) \dot{x}_{t|t-1} - K_t u_t^m + s_t K_t V_t^\frac{1}{2} \theta_t \]
(20)
The conditional expectation of \( x_t \), given \( b_t \) can be expressed
\[ E[x_t|b_t] = E[\xi_t] + (I - K_t H) \dot{x}_{t|t-1} - K_t u_t^m + E[s_t|b_t] K_t V_t^\frac{1}{2} \theta_t \]
To arrive at this relationship, we have used the facts that \( \xi_t \) and \( b_t \) are independent (since \( \xi_t \) and \( y_t \) are independent), and that conditioning on \( b_t \) is equivalent to conditioning on \( \theta_t \).
But \( E[\xi_t] = 0 \). From Lemma 5.1 we know that
\[ E[s_t|\theta_t] = \rho_r(z_t) \]
where
\[ z_t = \theta_t^T V_t^{-\frac{1}{2}} \]
\[ (H \dot{x}_{t|t-1} + u_t^m) \]
\[ = (b_t^T V_t^{-\frac{1}{2}} b_t)^{-\frac{1}{2}} b_t^T V_t^{-1} (H \dot{x}_{t|t-1} + u_t^m) \].
It follows from (18) that

$$E[\mathbf{x}_t|\mathbf{b}_t] = (I - K_t\mathbf{H})\hat{x}_{t-1} - K_t\mathbf{u}_t + \gamma_t\mathbf{K}_t\mathbf{b}_t.$$ 

Since $\xi_t$ and $\mathbf{b}_t$ are independent, we can also deduce from (17) and (20) that

$$\text{cov}\{x_t|\mathbf{b}_t\} = \text{cov}\{s_t|\xi_t\} + \text{cov}\{s_t|\mathbf{b}_t\}K_tV_t^2\mathbf{b}_t\mathbf{b}_t^T V_t^2 K_T^T.$$ 

But, by Lemma 5.1,

$$\text{cov}\{x_t|\mathbf{b}_t\} = \begin{cases} 2 + z_t \rho_2(z_t) - (\rho_2(z_t))^2 & \text{if } r = 2 \\ 3 + z_t \rho_3(z_t) - (\rho_3(z_t))^2 & \text{if } r = 3 \end{cases}.$$ 

We conclude from (17) and (18) that

$$\text{cov}\{x_t|\mathbf{b}_t\} = (I - K_t\mathbf{H})P_{t-1} + \delta_t \mathbf{K}_t \mathbf{b}_t \mathbf{b}_t^T K_T^T.$$ 

All the assertated relationships have now been confirmed; the proof is complete.

VI. SCENARIO I: COMPARISONS WITH A PSEUDO-MEASUREMENT FILTER

In the following two sections we report on simulation studies to assess the performance of the shifted Rayleigh filter in relation to other tracking algorithms.

We consider first a scenario featuring in an earlier comparative study of an extended Kalman filter, a pseudo-measurement filter and a particle filter [12]. A target moves along a horizontal track, with zero vertical displacement, according to a white-noise acceleration model. Let $x_t^1$ be the horizontal distance travelled in meters and let $x_t^2$ be the velocity, after $t$ seconds. The state of the target $x_t = [x_t^1, x_t^2]^T$ evolves according to the equations:

$$x_t = \begin{bmatrix} 1 & h \hline 0 & 1 \end{bmatrix} x_{t-1} + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} v_{t-1},$$

(21)

in which the sample period $h = 1$ second. Here, $v_{t-1}$ is a zero mean Gaussian white noise process with variance 0.01 $(m/s)^2$. The observer platform follows an approximately parallel track at a constant average speed. The horizontal and vertical displacements of the platform $x_t^p = [x_t^{p,1}, x_t^{p,2}]^T$ are governed by the equations:

$$x_t^{p,1} = 4t + \tilde{x}_t^{p,1}, \quad t = 0, 1, 2, \ldots$$

$$x_t^{p,2} = 20 + \tilde{x}_t^{p,2}, \quad t = 0, 1, 2, \ldots$$

in which $\tilde{x}_t^{p,1}$ and $\tilde{x}_t^{p,2}$ are zero-mean Gaussian white noise processes, both with variance $q = 1$.

Measurements are made of

$$\eta_t^m = \eta_t + u_t$$

where $\eta_t$ is the angle (in radians) of the line-of-sight of this target from the horizontal; that is

$$\eta_t = \arctan \frac{x_t^{p,2}}{x_t^{p,1} - x_t^{p,1}^T}.$$ 

The sensor noise is Gaussian white noise $u_t$, of variance $\sigma^2 = (0.05)^2$ rad$^2$ (that is, (2.86)$^2$ degrees$^2$). It is assumed that all white noise processes in this model are independent of each other.

The prior distribution of $x_0|)$ is assumed to be Gaussian with mean $\hat{x}_0|0)$ and covariance $P_{0|0}$:

$$\hat{x}_0|0) = \begin{bmatrix} 0 \\ \tan(\eta_0^m) \end{bmatrix}$$

and

$$P_{0|0} = \begin{bmatrix} 20 \tan(\eta_0^m) + \frac{400\sigma^2}{\sin^4(\eta_0^m)} & 0 \\ 0 & 1 \end{bmatrix}.$$ 

(For a justification of this initialization data, see [4], p. 387.)

The tracking problem is to provide a recursive estimate of the current target state $x_t$, given present and past measurements $\eta_t^m$, for $t = 1, 2, \ldots$. The configuration of the observer platform and target is illustrated in Fig. 2.

![Fig. 2. The target-observer geometry in Scenario I.](image)

A. Implementation of the shifted Rayleigh filter

For this scenario, the shifted Rayleigh filter is as described in Section IV, with the following parameter choices:

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$Q^s = 10^{-2} \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad Q^{tr} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$u_t^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_t^m = \begin{bmatrix} -4t \\ 20 \end{bmatrix}.$$ 

Here, $Q^s$ is the covariance of the effective perturbations to the state produced by range acceleration. $Q^{tr}$ is the contribution to the augmented measurement covariance $Q_t^m$ of the platform perturbations. For this scenario, we choose $Q_t^m$ to have the structure described at the end of Section II:

$$Q_t^m = Q^{tr} + \sigma^2(||Hx_t|0|1 + u_t^m||^2 + \text{trace}(HP_t|0|1H^T)I_{2\times2},$$ 

where $\sigma^2 = (0.05)^2$. 

B. Implementation of the pseudomeasurement filter

A brief description of the version of the pseudo-measurements filter used in [12] is as follows:

Take zero to be the output of a measurement equation

\[ 0 = H_t \hat{x}_t + \nu_t \]

in which \( \nu_t \) is a white noise process with covariance \( Q^m_t \). Take

\[ \zeta_t = -\left( \dot{x}_{t|t-1}^1 - 4t \right) \sin(\eta^m_t) - 20 \cos(\eta^m_t) \]

to be the measurement residual. Here

\[ H_t = [\sin(\eta^m_t), 0] \]

and

\[ Q^m_t = q + \sigma^2(\dot{x}_{t|t-1}^1 - 4t) \cos(\eta^m_t) + 20 \sin(\eta^m_t)^2 + \sigma^2[q + \nu_{t|t-1}^1 \cos^2(\eta^m_t)] . \]

The pseudomeasurement filter is the Kalman filter for a system that results from appending this measurement equation to the target dynamics equation (21). Accordingly, the estimate of the conditional mean and covariance are updated by means of the following equations

\[
\begin{align*}
\dot{x}_{t|t-1} &= F \dot{x}_{t-1|t-1} \\
P_{t|t-1} &= FP_{t-1|t-1}F^T + Q^s \\
V_t &= H_tP_{t|t-1}H_t^T + Q^m_t \\
K_t &= P_{t|t-1}H_t^TV_t^{-1} \\
\dot{\hat{x}}_t &= \dot{x}_{t|t-1} + K_t \zeta_t \\
P_t &= (I - K_tH_t)P_{t|t-1}
\end{align*}
\]

in which

\[
F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q^s = 10^{-2} \begin{bmatrix} 0.25 & 0.5 & 0.5 \\ 0.5 & 1 \end{bmatrix} .
\]

C. Comparative Performance

Simulations were carried out to compare the shifted Rayleigh filter and the pseudomeasurements filter for this scenario, Monte Carlo averages being taken over 500 runs. In each run, though the ‘true’ initial state \( \hat{x}_0 \) was held at \((80,1)^T\), its prior mean and covariance were randomly sampled according to the prescribed formulae, along with the system and measurement noise processes. Fig. 3 shows the RMS error for both filters. The performance of the two filters is very similar.

The conditions of the simulation were those adopted in [12], where the pseudomeasurement filter is shown to have similar performance characteristics to those of the particle filter and a version of the extended Kalman filter. It can be inferred that the shifted Rayleigh filter has very similar performance to these filters for the scenario under consideration.

VII. SCENARIO II: COMPARISONS WITH A PARTICLE FILTER

For the scenario considered next, we compare the performance of the shifted Rayleigh filter against that of a well-tested ‘bootstrap’ filter. The scenario considered next is a variant on one used by by Marrs [16] and Karan [17] et al. to assess the performance of a particle filter. The aim is to track a single target from a number of drifting sonobuoys. A monitoring aircraft estimates the positions of the drifting sonobuoys, by observing the direction of arrival of sensor transmissions. The sonobuoys track the position of the target by means of noisy bearings measurements. We assume, in this preliminary assessment of the shifted Rayleigh filter, no clutter and no missed measurements. (Marrs and Karan et al. allow both these features.) Modifications to the shifted Rayleigh filter to allow for clutter are assessed in [14].

The simulations reported in [16] and [17] cannot be used to compare directly the performance of the shifted Rayleigh filter and particle filters, since only a qualitative discussion of the simulations is provided.

In our model of this scenario the target follows an integrated Brownian motion of low intensity (that is, a ‘nearly constant’ velocity model) and each of three sonobuoy sensors moves as the sum of a low intensity Brownian motion and an integrated Brownian motion, which is common to all three, that represents the effect of a drift current.

The state is 12-dimensional:

\[
x_t = (x_{0,t}, \dot{x}_{0,t}, y_{0,t}, \dot{y}_{0,t}, x_{1,t}, y_{1,t}, \ldots, x_{2,t}, y_{2,t}, x_{3,t}, y_{3,t}, u_{1,t}, u_{2,t})^T ;
\]

the first four components represent the \((x,y)\) coordinates of the position and velocity of the target, the next six, the coordinates of the positions of the three sonobuoys, and the last two, those of the drift current effecting all three sonobuoys. The state satisfies the discrete stochastic equation

\[ x_{t+1} = Fx_t + \nu_{t+1} \]
where
\[ F = \text{diag} \left( \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, I_{2\times2}, I_{2\times2}, I_{2\times2}, I_{2\times2} \right) \]
\[ + h \begin{bmatrix} O_{1\times10} \\ D \end{bmatrix}. \]

Here, \( h \) is the sample period and \( D \) is the matrix \([O_{2\times4}, I_{2\times2}, I_{2\times2}, O_{2\times2}]^T\). The covariance matrix of the system noise \( \nu_i \) has the form
\[ Q^i = \text{diag} (q_1 R, q_1 R, q_2 I_{2\times2}, q_2 I_{2\times2}, q_2 I_{2\times2}, q_3 I_{3\times2}) \]
where \( R \) is the \( 2\times2 \) symmetric matrix with rows \([h^3/3, h^2/2]\) and \([h^2/2, h]\).

Six simultaneous measurements are made at each time step. Three of these are measurements of the bearing angles of the sonobuoys from the monitoring platform, and three the bearing angles of the target from the sonobuoys. The first three are modelled (in the traditional form) as
\[ \theta_{i,t} = \arctan \left( \frac{x_{ms,t} - x_{i,t}}{y_{ms,t} - y_{i,t}} \right) + \epsilon_{i,t} \]
for \( i = 1, 2, 3 \), where \((x_{ms,t}, y_{ms,t})\) is the position of the monitoring sensor platform and whose monitoring sensor noise \( \epsilon_{i,t} \) has variance \( \sigma^2 \) for each \( i \). The second three measurements of the target are similarly modelled as
\[ \eta_{i,t} = \arctan \left( \frac{x_{o,t} - x_{i,t}}{y_{o,t} - y_{i,t}} \right) + \epsilon'_{i,t} \]
with sonobuoy sensor noise \( \epsilon'_{i,t} \), of variance \( (\sigma')^2 \). The shifted Rayleigh filter and a version of the particle filter used by Marrs [16] were applied to this model. For both algorithms the parameters were set to the following values:
\[ h = 1 \text{sec}, \quad \text{the sample period between measurements} \]
\[ T = 100 \text{sec}, \quad \text{the tracking period} \]
\[ q_1 = 0.16, \quad \text{target velocity perturbation intensity} \]
\[ q_2 = 1, \quad \text{sonobuoy perturbation intensity} \]
\[ q_3 = 0.02, \quad \text{current perturbation intensity} \]
\[ \sigma = 0.014 \text{ rad} \quad (\sim 0.8^\circ), \quad \text{the standard deviation of monitoring sensor bearing noise} \]
\[ \sigma' = 0.283 \text{ rad} \quad (\sim 16^\circ), \quad \text{the standard deviation of sonobuoy sensor bearing noise} \]

The monitoring sensor platform starts at the location \((0, -300)\) and moves uniformly in a straight line to the location \((0, 300)\) (all distances in metres) over a period of 100 seconds. The changing coordinates \((x_{ms,t}, y_{ms,t})\) of the monitoring sensor platform appear as terms in \( u_0^m \). The target moves from right to left and the sonobuoys from left to right.

The prior state mean is
\[ \hat{x}_{0\mid 0} = (300, -4, -300, 4, 100, -400, \ldots, -200, -400, -200, 0, 2, 2)^T \]
and the prior state covariance is
\[ P_{0\mid 0} = \text{diag}(10000, 1, 10000, 1, 0, \ldots, 0, 0.02, 0.02). \]

Note that the initial sensor positions are assumed to be known. In the application of the shifted Rayleigh filter, a time-dependent modification of the algorithm was employed that introduced an artificial ‘clock’ time. This allowed the six simultaneous measurements at each step to be incorporated into the algorithm one at a time. In this modified algorithm, a single ‘prediction’ step was followed by six ‘correction’ steps.

Fig. 4 shows the behaviour of the estimates of target and sonobuoy positions provided by both the shifted Rayleigh and particle filters, for a typical simulation.

Fig. 4. Typical tracks of target and drifting sonobuoy sensors, together with the estimated tracks. The final estimated 90% confidence ellipses are shown for the target track, for the two methods.

Fig. 5 shows the root mean square (RMS) errors of the estimates of target positions resulting from the application of both the filters, based on Monte Carlo simulations with 100 runs. The errors can also be estimated as (the square root of) averages of the filter’s own estimates of the conditional variances; these are also shown in the figure.

Fig. 5. RMS target position error (Scenario II)

A. Performance

In these simulations, the averaged estimation error of the particle filter reduced as the number of particles employed increased. The number of particles was increased to 15,000, beyond which the improvement in estimation accuracy became
gradual. Furthermore the RMS errors are within 30% (for most of the time period) of the standard deviations derived from the conditional variances generated by the filter. It is reasonable to assume, then, that the conditional mean estimates provided by the particle filter (with 15,000 particles) are close to the (nonlinear) minimum least squares estimates of target positions.

For the shifted Rayleigh filter, the RMS errors of the estimates were initially greater than those associated with the particle filter estimates, but of comparable magnitude. It was observed however that, if the particle filter was implemented with fewer than 10,000 particles, the shifted Rayleigh filter actually did better than the particle filter. Even for this reduced number of particles, the computational demands of the shifted Rayleigh filter are very significantly less than those of the particle filter (by a factor of over 300). It should be stated however the well tested, generic particle filter employed in our simulations was chosen for its reliability. No particular attempt was made to fine tune the filter to fit the scenario.

In [16], Marrs reports on the ‘catastrophic failure’ of the extended Kalman filter to provide adequate tracks, in the scenario considered here (no clutter or missed measurements). Our successful implementation of the shifted Rayleigh filter illustrates that moment matching algorithms do have an important role, even in tracking problems involving high dimensional models and strong nonlinearities. However the moment matching needs to be carried out in a manner which takes account of the nonlinearities at hand, as is the case with the shifted Rayleigh filter.

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