Partial Differential Equations 2019
Coursework 2

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1. *(Analyticity of harmonic functions.) In lectures we saw an outline on how to show that harmonic functions are analytic by extending the formula involving the fundamental solution to a* complex *analytic function. Here we give an alternative argument.*

   Let $u \in C^2(\Omega)$ be harmonic in $\Omega \subset \mathbb{R}^d$ (bounded, connected, open, as in lectures). In lectures, we proved that u is smooth in Ω and also that for any closed ball $\overline{B}(\xi, a) \subset \Omega$ we have the estimate

   $$|\partial_{\xi_i} u(\xi)| \leq \frac{d}{a} \max_{\overline{B}(\xi,a)} |u(x)| \,. \tag{1}$$

   (a) Let $\overline{B}(\xi, a) \subset \Omega$ and $|u(x)| \leq M$ hold in $\overline{B}(\xi, a)$. Use (1) to prove that
   $$|D^\alpha u(\xi)| \leq \left(\frac{m}{a} d\right)^m M \quad \text{for } |\alpha| = m.$$
   Hint: Apply (1) successively to the k^{th} derivatives in the balls $|x - \xi| \leq a\frac{m-k}{m}$ for $k = 0, 1, ..., m-1$.

   (b) Prove that u is analytic in Ω. Hint: Note $m^m \leq m! e^m$.

2. (Fritz John, 4.2) Prove the weak maximum principle for solutions of the two-dimensional elliptic equation

   $$Lu = au_{xx} + 2bu_{xy} + cu_{yy} + 2du_x + 2eu_y = 0$$

   where a, b, c, d, e are continuous functions of x and y in $\bar{\Omega}$ and $ac - b^2 > 0$ (ellipticity) as well as $a > 0$ hold. HINT: Prove it first under the strict condition $Lu > 0$, then use $u + \epsilon v$ for $v = \exp\left[M(x - x_0)^2 + M(y - y_0)^2\right]$ with appropriate M, x_0, y_0.

3. Prove the following *basic version of the Banach Alaoglu theorem* (which we used in connection with the difference quotients): Let (u_k) be a bounded sequence in a Hilbert space H, i.e. $\|u_k\|_H \leq C$. Then there exists a subsequence which converges weakly in H. Hint: Use the following outline

(a) Pick an ONB (e_k) and use a diagonal argument to show that for a subsequence of the (u_k), denoted $(u_n^{(n)})$ (arising from a Cantor diagonal argument) we have that

$$\langle u_n^{(n)}, e_k \rangle \to v_k \in \mathbb{R} \quad \text{holds for all } e_k.$$

(b) Show that $\sum_{k=1}^{\infty} |v_k|^2 < \infty$ and hence $v = \sum_k v_k e_k \in H$.

(c) Show that $u_n^{(n)} \rightharpoonup v$.

4. (Best constant in Poincare's inequality; F. John Chapter 5) Show that if there exists a function $u \in C^2\left(\overline{\Omega}\right)$ vanishing for $\partial\Omega$ for which the quotient

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

reaches its smallest value λ, then u is an eigenfunction to the eigenvalue λ, i.e. $\Delta u + \lambda u = 0$ in Ω. In fact λ must be the smallest eigenvalue belonging to an eigenfunction in $C^2\left(\overline{\Omega}\right)$.