Partial Differential Equations
Blowup for semi-linear wave equations

Gustav Holzegel

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1 Preliminaries

1.1 Duhamel's principle

Consider the inhomogeneous wave equation with trivial data

$$\begin{aligned} \psi_{tt} - \Delta\psi = f(t,x) \quad &\text{in } \mathbb{R}^d \times (0,\infty), \\ \psi = 0 \;\;,\;\; \psi_t = 0 \quad &\text{for } \mathbb{R}^d \times \{0\}. \end{aligned} \tag{1}$$

We define $\tilde{\psi} = \tilde{\psi}(x,t;s)$ for $s \geq 0$ to be the solution of

$$\tilde{\psi}_{tt}(\cdot, s) - \Delta\tilde{\psi}(\cdot; s) = 0 \qquad \text{in } \mathbb{R}^d \times (s,\infty), \tag{2}$$

$$\tilde{\psi}(\cdot, s) = 0 \;\;,\;\; \tilde{\psi}_t(\cdot, s) = f(\cdot, s) \qquad \text{for } \mathbb{R}^d \times \{t = s\}. \tag{3}$$

Now set

$$\psi(x,t) := \int_0^t \tilde{\psi}(x,t;s)\, ds \quad \textit{for} \quad x \in \mathbb{R}^d \;\; \textit{and} \;\; t \geq 0. \tag{4}$$

We claim this is a solution of the problem (1) and prove it explicitly for $d = 3$:

Proposition 1.1. *Let $d = 3$ and $f \in C^2\left(\mathbb{R}^d \times [0,\infty)\right)$. Then ψ defined by (4) solves (1).*

Proof. Note first that by our well-posedness result for the homogeneous wave equation, the $\tilde{\psi}(x,t;s)$ are well-defined and C^2 in all its arguments for $0 \leq s \leq t$ (why?). Hence $\psi(x,t)$ is $C^2\left(\mathbb{R}^d \times [0,\infty)\right)$. Clearly, the trivial initial conditions of (1) are also satisfied. To see that it also solves the inhomogenous wave equation we compute

$$\psi_t(x,t) = \tilde{\psi}(x,t;t) + \int_0^t \tilde{\psi}_t(x,t;s)\, ds = \int_0^t \tilde{\psi}_t(x,t;s)\, ds, \tag{5}$$

$$\psi_{tt}(x,t) = \tilde{\psi}_t(x,t;t) + \int_0^t \tilde{\psi}_{tt}(x,t;s)\, ds = f(x,t) + \int_0^t \tilde{\psi}_{tt}(x,t;s)\, ds. \tag{6}$$

Combining this with

$$\Delta\psi\,(x,t) = \int_0^t \Delta\tilde{\psi}\,(x,t;s)\,ds \tag{7}$$

yields the result. $\square$

The identical argument works in any dimension, however recall we have only explicitly proved an existence result for $d=1$ and $d=3$. In the $d=1$ case we note that the solution to (2) is given explicitly by

$$\tilde{\psi}\,(x,t,s) = \frac{1}{2}\int_{x-(t-s)}^{(x+(t-s)} f(s,\bar{x})d\bar{x} \tag{8}$$

and hence the solution to the inhomogeneous problem (1) by

$$\psi(x,t) = \frac{1}{2}\int_0^t ds \int_{x-(t-s)}^{(x+(t-s)} f(s,\bar{x})d\bar{x}\,. \tag{9}$$

This is precisely an integral over the past light cone of the inhomogeneity f. If the data in (1) is non-trivial, we just add to (9) the unique solution of the homogeneous problem arising from this data. By linearity the sum provides a solution to the inhomogeneous problem assuming the prescribed data and by energy estimates (or otherwise) there can only be one solution to this problem.

2 Local Existence for semi-linear wave equations

We now look at two Cauchy problems for a class of *semi-linear* equations. They are of the form

$$\begin{aligned} \psi_{tt} - c^2\Delta\psi &= Q\left(\partial\psi\right) \\ \psi|_{t=0} &= f \\ \psi_t|_{t=0} &= g \end{aligned} \tag{10}$$

with f and g smooth and of compact support, say, and where $Q\left(\partial\psi\right)$ is quadratic in the derivatives of ψ. Below we will look at the (global problem for the) two specific cases

$$Q = \left(\partial_t\psi\right)^2 \quad \text{and} \quad Q = \left(\partial_t\psi\right)^2 - |\nabla\psi|^2\,,$$

both in the physical dimension $d=3$. The local arguments presented in this section are easily seen to apply to *any* quadratic non-linearity and to higher dimension but let us, for concreteness focus on $Q = \left(\partial_t\psi\right)^2$. The discussion in this section will be rather informal as we want to go to the global analysis as quickly as possible. I only want to convey the basic ideas in establishing that the problem (10) is locally (for small times) wellposed:

Theorem 2.1. *There exists a $T>0$ such that there exists a unique smooth solution of (10) in $(-T,T)\times\mathbb{R}^3$.*

2.1 Solution map for the inhomogeneous wave equation

We start by considering the inhomogeneous wave equation

$$\begin{aligned} \psi_{tt} - c^2 \Delta \psi &= F(t,x) \\ \psi|_{t=0} &= f \\ \psi_t|_{t=0} &= g \end{aligned} \tag{11}$$

By the results of Section 1 given $F(t,x)$ smooth (and of compact support in space, say) we can construct a global smooth solution ψ through the Duhamel formula. Therefore, we have for any $T > 0$ a map $\Phi : C^\infty\left([0,T] \times \mathbb{R}^3\right) \to C^\infty\left([0,T] \times \mathbb{R}^3\right)$ associating to F the smooth solution ψ of the initial value problem (11).

For any T by redoing the energy estimate for the inhomogenous equation we can derive

$$E\left(\tau\right) \leq E\left(0\right) + \int_0^\tau d\tilde{\tau} E\left(\tilde{\tau}\right) + G\left(\tau\right) \ , \tag{12}$$

where

$$E\left(\tau\right) = \int_{\{t=\tau\}} d^3x \left[\frac{1}{2}\left(\partial_t \psi\right)^2 + \frac{1}{2}|\nabla_x \psi|^2\right] \tag{13}$$

and

$$G\left(\tau\right) = \int_0^\tau d\tau \int_{\{t=\tau\}} d^3x |F|^2 \,. \tag{14}$$

Gronwall's inequality yields the estimate

$$E\left(\tau\right) \leq \left[E\left(0\right) + G\left(\tau\right)\right] e^\tau \tag{15}$$

for any $\tau \in [0,T]$.

Exercise 2.2. *Show that the estimate (12) remains remain valid (perhaps with a constant multiplying the second (integral) term on the right) if the L^2-term (i.e. u^2 in the integrand) is added to the definition of $E(\tau)$.*

We now want to extend the solution map to L^2-based spaces. We define

$$\mathcal{X}^k := C\left([0,T], H^k\left(\mathbb{R}^3\right)\right)$$

as the space of continuous functions on $[0,T]$ with values in the Banach space $H^k\left(\mathbb{R}^3\right)$ and equipped with the norm $\|\psi\|_{\mathcal{X}^k} := \sup_{t\in[0,T]} \|\psi\left(t,\cdot\right)\|_{H^k(\mathbb{R}^3)}$.

Exercise 2.3. *Show that this space is complete and that smooth functions form a dense set.*

We also define the space

$$\mathcal{Y}^k := C^0\left([0,T], H^k\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,T], H^{k-1}\left(\mathbb{R}^3\right)\right)$$

as the space of continuous functions $\psi : [0,T] \to H^k\left(\mathbb{R}^3\right)$ whose derivative $H^k-\lim_{h\to 0}\frac{\psi(t+h)-\psi(t)}{h}$ exists and is a continuous function with values in $H^{k-1}\left(\mathbb{R}^3\right)$ equipped with the norm

$$\|\psi\|_{\mathcal{Y}^k} := \sup_{t\in[0,T]} \|\psi\left(t,\cdot\right)\|_{H^k(\mathbb{R}^3)} + \sup_{t\in[0,T]} \|\partial_t\psi\left(t,\cdot\right)\|_{H^{k-1}(\mathbb{R}^3)} .$$

This space is also complete and smooth functions form a dense set.

Now assume the inhomogeneity F in (11) is only in $\mathcal{X}^{k-1}$ for some $k \geq 1$. By density, we can approximate F in $\mathcal{X}^{k-1}$ by a sequence of C^∞ functions F_n. Looking at $\Box(\psi_n - \psi_m) = F_n - F_m$ (which has trivial data) we obtain from (15) that

$$\sup_{t\in[0,T]} \|\psi_n - \psi_m\|_{H^1(\mathbb{R}^3)} + \sup_{t\in[0,T]} \|\partial_t\psi_n - \partial_t\psi_m\|_{L^2(\mathbb{R}^3)} \leq Te^T \sup_{t\in[0,T]} \|F_n - F_m\|_{L^2(\mathbb{R}^3)} . \tag{16}$$

Moreover, it is clear that by commutations with $\partial_x, \partial_y, \partial_z$ one can derive

$$\sup_{t\in[0,T]} \|\psi_n - \psi_m\|_{H^k(\mathbb{R}^3)} + \sup_{t\in[0,T]} \|\partial_t\psi_n - \partial_t\psi_m\|_{H^{k-1}(\mathbb{R}^3)} \leq Te^T \sup_{t\in[0,T]} \|F_n - F_m\|_{H^{k-1}(\mathbb{R}^3)} . \tag{17}$$

It follows in particular that the map Φ extends for $k \geq 1$ to a map (denoted by the same letter)

$$\Phi : \mathcal{X}^{k-1} \to \mathcal{Y}^k . \tag{18}$$

associating an inhomogeneity $F \in \mathcal{X}^{k-1}$ with its (generalised) solution $\psi \in \mathcal{Y}^k$. Note that one still has to check that the limiting ψ satisfies the inhomogeneous wave equation. Exercise: Do this for $k \geq 2$.

We finally note from the above that the map Φ satisfies the estimates

$$\|\Phi(F)\|_{\mathcal{Y}^k} \leq \|f\|_{H^k(\mathbb{R}^3)} + \|g\|_{H^{k-1}(\mathbb{R}^3)} + Te^T\|F\|_{\mathcal{X}^{k-1}} , \tag{19}$$

and

$$\|\Phi(F_1) - \Phi(F_2)\|_{\mathcal{Y}^k} \leq Te^T\|F_1 - F_2\|_{\mathcal{X}^{k-1}} . \tag{20}$$

2.2 Non-linear solution via contraction map

We finally solve the problem (10) by an iteration scheme. More precisely we solve the inhomogeneous problem

$$\begin{aligned} \psi_{tt} - c^2\Delta\psi = F &:= (\partial_t\tilde{\psi})^2 \\ \psi|_{t=0} &= f \\ \psi_t|_{t=0} &= g \end{aligned} \tag{21}$$

and try to infer the existence of a fixed point.

Fix $k = 3$ and let

$$\|f\|_{H^k(\mathbb{R}^3)} + \|g\|_{H^{k-1}(\mathbb{R}^3)} = \frac{C}{2}$$

for some $C \geq 1$. Consider the closed ball of radius C in $\mathcal{Y}^k$, denoted B. Given $\tilde{\psi} \in B \subset \mathcal{Y}^k$ we have in particular from Sobolev embedding in $\mathbb{R}^3$

$$\sup_{t\in[0,T]} \|\tilde{\psi}\|_{L^\infty(\mathbb{R}^d)} + \sum_{\mu=0}^{3} \sup_{t\in[0,T]} \|\partial_\mu \tilde{\psi}\|_{L^\infty(\mathbb{R}^d)} \lesssim C\,,$$

where $\lesssim$ allows for some Sobolev constants. We claim that for $\tilde{\psi} \in B$ we have $(\partial_t \tilde{\psi})^2 \in \mathcal{X}^2$ and in fact $\|(\partial_t \tilde{\psi})^2\|_{\mathcal{X}^2} \lesssim C\|\tilde{\psi}\|_{\mathcal{Y}^3} \lesssim C^2$. Similarly, for $\tilde{\psi}_1$ and $\tilde{\psi}_2$ in B, we have

$$\|(\partial_t \tilde{\psi}_1)^2 - (\partial_t \tilde{\psi}_1)^2\|_{\mathcal{X}^2} \leq C\|\tilde{\psi}_1 - \tilde{\psi}_2\|_{\mathcal{Y}^3}\,.$$

Using the estimates (19) and (20) it is now easy to see that the map

$$\Phi : B \to \mathcal{X}^3$$

actually maps B to B and is a contraction for sufficiently small T depending only on C and Sobolev constants.

3 Global behaviour: Blow-up vs small data global existence

We now prove the following Theorem:

Theorem 3.1 (F. John)**.** *Let $\psi : \mathbb{R}^{3+1} \to \mathbb{R}$ be a $C^2\left(\mathbb{R}^3 \times [0,\infty)\right)$ solution of*

$$\Box\psi = (\partial_t \psi)^2 \tag{22}$$

arising from smooth data of compact support. Then $\psi \equiv 0$.

The theorem can be paraphrased by saying that any C^2 solution arising from non-trivial data of compact support must blow up in finite time. Unfortunately, the theorem does not tell us anything about what goes wrong, i.e. about the nature of the singularity. We'll prove the theorem in Section 4 below. Before doing this let us make two comments.

The first is that it is easy to construct *some* data that blow up using ODE techniques:

Exercise 3.2. *Find initial data for (22) such that the solution blows up in finite time. Hint: Choose initial data that do not depend on the spatial variables. Can you make the data of compact support? Hint: Domain of dependence.*

As a second comment we contrast Theorem 3.1 with the following theorem:

Theorem 3.3. *Consider the PDE for* $\psi : \mathbb{R}^{3+1} \to \mathbb{R}$,

$$\Box\psi = (\partial_t\psi)^2 - |\nabla\psi|^2 \, , \tag{23}$$

with data $\psi(t=0,x) = \epsilon f(x)$ *and* $\partial_t\psi(0,x) = \epsilon g(x)$ *where* $f, g \in C_0^\infty\left(\mathbb{R}^3\right)$. *Then there exists an* ϵ_0 *sufficiently small*[1] *such that for* $\epsilon \leq \epsilon_0$ *all solutions are global, i.e. in particular* u *is a smooth solution on all of* $\mathbb{R}^3 \times [0,\infty)$.

While the general theory for these small data global existence results with non-linearities satisfying the so-called null condition is due to S. Klainerman, the particular theorem above can be proven with a clever algebraic trick:

Proof. Let $\phi = e^\psi - 1$. Then $\partial_t\phi = e^\psi\partial_t\psi$ and $\partial_t^2\phi = e^\psi\left(\partial_t^2\psi + (\partial_t\psi)^2\right)$ and similarly $\Delta\phi = e^\psi\left(\Delta\psi + |\nabla\psi|^2\right)$. Consequently,

$$-\partial_t^2\phi + \Delta\phi = e^\psi\left[-\partial_t^2\psi - (\partial_t\psi)^2 + \Delta\psi + |\nabla\psi|^2\right] = 0 \, ,$$

so ϕ satisfies the homogeneous linear wave equation! The initial data for v,

$$\phi_0 := \phi(x,0) = e^{\epsilon f(x)} - 1 \quad , \quad \phi_1 := \partial_t\phi\,(x,0) = \epsilon e^{\epsilon f(x)} g(x)$$

is of size ϵ and compactly supported. Now either directly from the representation formula or from the fact that in the lecture notes we proved the estimate

$$|\phi|_{L^\infty(\mathbb{R}^3\times[0,\infty))} \leq C\left(\|\phi_1\|_{H^1(\mathbb{R}^3)} + \|\phi_0\|_{\mathring{H}^1(\mathbb{R}^3)} + \|\phi_0\|_{\mathring{H}^2(\mathbb{R}^3)}\right)$$

we deduce that

$$|\phi|_{L^\infty(\mathbb{R}^3\times[0,\infty))} \leq C_{f,g}\epsilon \, .$$

So with f and g given, we can choose ϵ so small that $|\phi|_{L^\infty(\mathbb{R}^3\times[0,\infty))} \leq \frac{1}{2}$ holds on $\mathbb{R}^3 \times [0,\infty)$. But then

$$\psi = \log(\phi + 1)$$

is well defined and smooth on $\mathbb{R}^3 \times [0,\infty)$ and solves (23). $\Box$

4 The proof of Theorem 3.1

We choose R such that $\psi(x,0) = \psi_t(x,0) = 0$ for $|x| > R$.

Step 1. We derive an equation for the spherical means. Recall that for any function $\Psi(x,t)$ we can define its spherical means

$$\overline{\Psi}\,(r,t) = \frac{1}{4\pi}\int_{|\xi|=1} \Psi(r\xi,t)dS(\xi) \, .$$

[1] depending on f and g, more precisely the $\mathring{H}^1$ and $\mathring{H}^2$ norms of f and the $\mathring{H}^1$ norm of g. For general right hand sides satisfying the null condition ϵ will in fact depend on higher H^k norms and the size of the support as well because the algebraic trick used in the proof will not be available.

Here a priori $r \geq 0$ but the formula on the right makes sense for $r < 0$ as well and clearly $\overline{U}\,(-r,t) = \overline{U}\,(r,t)$ so we can consider U as an even function of r. Turning to (22) we have By Darboux's (cf. last lecture!)

$$\overline{w} := \overline{\psi}_{tt} - \overline{\psi}_{rr} - \frac{2}{r}\overline{\psi}_r = \overline{(\psi_t)^2}\ .$$

Note that $\overline{\psi}$ and $\overline{\psi}_{rr}$ as well as $\frac{2}{r}\overline{\psi}_r$ and $\overline{\psi}_t$ are even functions in r that vanish for $r > R + t$ by domain of dependence. Note also that we can write

$$(r\overline{\psi})_{tt} - (r\overline{\psi})_{rr} = r\overline{w} = r\overline{(\psi_t)^2}$$

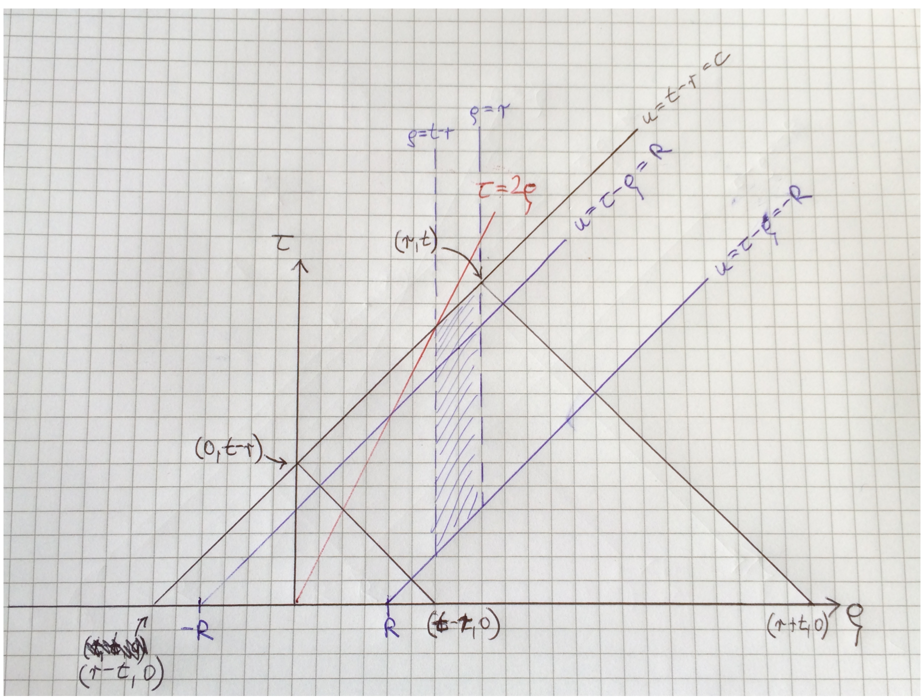
Step 2. We use the Duhamel formula for the inhomogeneous wave equation derived in the first section to obtain

$$\overline{\psi}\,(r,t) = \overline{\psi}_0(r,t) + \frac{1}{2r}\int_{T(r,t)} \rho\overline{(\psi_t)^2} d\rho d\tau \tag{24}$$

with (the homogeneous solution realising the data being)

$$\overline{\psi}_0(r,t) = \frac{(r+t)\overline{\psi}(r+t,0) + (r-t)\overline{\psi}(r-t,0)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \rho\overline{\psi}_t(\rho,0)d\rho\ . \tag{25}$$

and $T(r,t)$ the characteristic triangle with vertex at (t,r) and basis on the ρ-axis.



Step 3. We now restrict attention to the region

$$\Sigma := \{(\rho,\tau) \mid \rho + R < \tau < 2\rho\} \,.$$

In the picture it is the region to the east of the red line intersected with the region to the north of $u = R$. If $(r,t) \in \Sigma$ we easily see that $\psi_0(r,t) = 0$ because $r + t \geq R$ and $r - t < -R$ and because the integral in (25) can be written as

$$\int_{-R}^{R} d\rho \rho \overline{\psi}_t(\rho, 0) = 0$$

since one integrates an odd function over an interval symmetric about the origin. We conclude

$$\overline{\psi}\,(r,t) = \frac{1}{2r} \int_{T(r,t)} \rho \overline{(\psi_t)^2} d\rho d\tau \,. \tag{26}$$

We now define the trapezoid

$$T^{\star}\,(r,t) = \text{trapezoid with vertices at } (r,t),\ (0,t-r),\ (t-r,0),\ (t+r,0)$$

Writing the integral in (26) as a sum of the integral over the trapezoid and an integral over the characteristic triangle with vertex at $(0, t-r)$ we conclude that the second part vanishes because we are integrating an odd function over an interval that is symmetric around the origin. We conclude

$$\overline{\psi}(r,t) = \frac{1}{2r} \int_{T^{\star}(r,t)} \rho \overline{(\psi_t)^2} d\rho d\tau \geq \frac{1}{2r} \int_{T^{\star}(r,t)} \rho (\overline{\psi}_t)^2 d\rho d\tau \,, \tag{27}$$

where (for the inequality) we have used that $\overline{(\psi_t)^2} \geq (\overline{\psi}_t)^2$ by Cauchy-Schwarz and that $\rho \geq 0$ holds in $T^{\star}(r,t)$. Since the integrand is non-negative we can restrict the integration region further to

$$S\,(r,t) = \{(\rho,t) \mid t - r < \rho < r \quad \text{and} \quad -R < u = \tau - \rho < t - r\} \,,$$

(the shaded region in the above picture) to obtain the estimate

$$\overline{\psi}(r,t) \geq \frac{1}{2r} \int_{S(r,t)} \rho (\overline{\psi}_t)^2 d\rho d\tau = \frac{1}{2r} \int_{t-r}^{r} \rho d\rho \int_{\rho - R}^{\rho + t - r} d\tau (\overline{\psi}_t)^2 \tag{28}$$

valid for any $(r,t) \in \Sigma$.

Step 4. For $(r,t) \in \Sigma$ we fix the characteristic $u = t - r =: c$ and consider the points along that characteristic, i.e. points parametrised by $(\rho, \rho + c)$ with $\rho \geq c$. Our goal will be to derive an ordinary differential inequality along that characteristic.

Since $\bar{\psi} = 0$ on $u = -R$ we can write by the fundamental theorem of calculus

$$\overline{\psi}(\rho, \rho + c) = \int_{\rho - R}^{\rho + c} \partial_t \overline{\psi}\,(\rho,\tau)\ d\tau \,.$$

Cauchy-Schwarz tells us that

$$|\overline{\psi}(\rho, \rho+c)|^2 \leq (R+c) \int_{\rho-R}^{\rho+c} |\partial_t \overline{\psi}\,(\rho, \tau)\,|^2 \, d\tau \,.$$

With this (28) becomes

$$\overline{\psi}(r, r+c) \geq \frac{1}{2(R+c)r} \int_c^r \rho |\overline{\psi}(\rho, \rho+c)|^2 d\rho \tag{29}$$

for points $(r,t) \in \Sigma$ lying on the fixed characteristic $t - r = c$. Defining

$$\beta\,(r) = \int_c^r \rho |\overline{\psi}(\rho, \rho+c)|^2 d\rho$$

we can write (29) as

$$\beta' \geq \frac{1}{4(R+c)^2 r} \beta^2 \tag{30}$$

which is the desired ordinary differential inequality along the characteristic $t - r = c$. Assume $\beta \neq 0$ for some $r = r_0$ along the characteristic. Then $\beta(r) \geq \beta(r_0)$ for all $r \geq r_0$. Integrating (30) from $r = r_0$ to $r > r_0$ we find

$$\frac{1}{\beta(r_0)} \geq \frac{1}{\beta(r_0)} - \frac{1}{\beta(r)} \geq \frac{1}{4} \frac{1}{(R+c)^2} \log \frac{r}{r_0}$$

which leads to a contradiction for sufficiently large r. We conclude $\beta = 0$ and (by definition of β) $\overline{\psi} = 0$ along the entire characteristic $t - r = c$. Since this works for any $(t,r) \in \Sigma$ we conclude $\overline{\psi} = 0$ in Σ. Going back to (27), which holds for any $(r,t) \in \Sigma$, we conclude $\overline{(\psi_t)^2} = 0$ in $T^\star\,(r,t)$ for any $(r,t) \in \Sigma$. Choosing (t,r) with $t - r$ arbitrarily close to R and letting $r \to \infty$ we deduce that $\overline{(\psi_t)^2} = 0$ is identically zero for $t > R$. Since the spherical average of a non negative function is zero, the function itself has to be zero hence $\psi_t = 0$ identically for $t > |x|$. In particular also $\psi_{tt} = 0$ for $t > |x|$ and this means on a constant t slice with $t > |x|$, the function ψ satisfies $\Delta\psi = 0$ and is of compact support. This immediately implies $\psi = 0$ for $t > |x|$. Now by our uniqueness proof (which works backwards and forwards) $\psi = 0$ globally.

5 Final Remarks

It is easy to see the proof of Theorem 3.1 will go through verbatim for $\Box\psi = F$ with $F \geq a\,(\partial_t\psi)^2$ for $a > 0$ a constant. In fact, one can prove the result for more complicated non-linearities including quasi-linear ones. More details in the original paper of F. John (CPAM 34 (1981), 29-51) or the PDE notes of P. Constantin (available from his Princeton webpage) which I have been following closely here.

Similarly, Theorem 3.3 can be proven for much more general non-linearities *having a null-structure*. The proof, however is much more complicated and exploits the fact that certain derivatives of the linear wave equation decay faster than others along the light cone. This would be the topic of a proper course on non-linear wave equations!