Imperial College London

Department of Mathematics
Fluid Dynamics Seminar

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A tiny step towards solving the problem of turbulence: dissipation bounds in pipes of general shape

by

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A copy of this presentation with clickable links is available at https://www.imperial.ac.uk/aeronautics/fluiddynamics/2026_IC_Chernyshenko.pdf

Talk plan

I. Motivation: the great problem of turbulence

II. Known unknowns: what I will not tell you

III. Auxiliary function(al)s: the tool

IV. Bounds for energy dissipation in pipes of general shape

V. Pressure role and bounds for steady flows

VI. An unusual inviscid flow

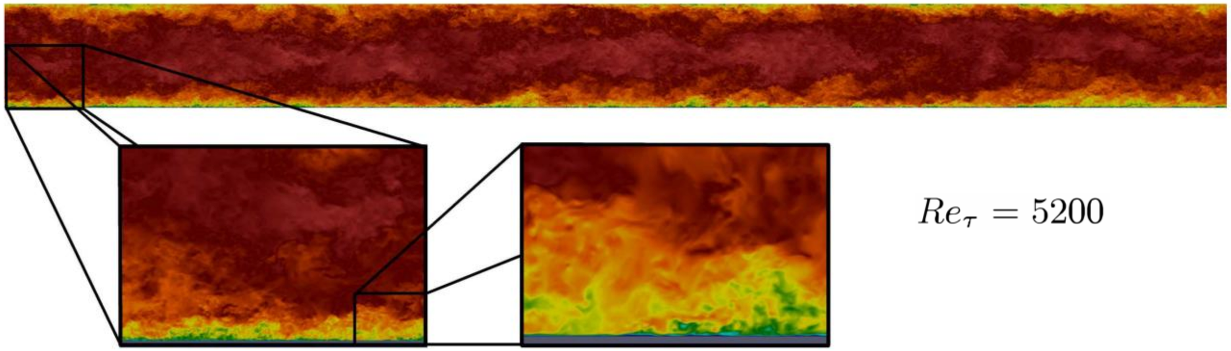
VII. Conclusion: the next step

What is the great problem of turbulence?

Timeline

1883	Osborne Reynolds: Turbulence
1895	Osborne Reynolds, RANS
1932	Horace Lamb: "… when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic."
1960s	Richard Feynman: "Turbulence is the most important unsolved problem of classical physics."
2025	Sreenivasan & Schumacher, "What is the turbulence problem, and when may we regard it as solved?", *Annual Review of Condensed Matter Physics*, *16*(1), 121-143

Do computers solve the problem of turbulence? Or can't we see the forest for the trees?



$Re_\tau = 5200$

Figure 7: Instantaneous streamwise velocity component over the entire streamwise length of the simulated channel. Zooming in on the flow highlights the multi-scale nature of the turbulence.

Channel flow calculation that required 260,000,000 CPU hours and resolved all the minute detail.

Lee, Malaya, & Moser 2013

But we wanted only the mean friction !

Acknowledgements: Lee, MK, Malaya, N & Moser, R. (2013). Petascale direct numerical simulation of turbulent channel flow on up to 786K cores. International Conference for High Performance Computing, Networking, Storage and Analysis, SC. 10.1145/2503210.2503298.
Argonne National Laboratory's Flickr page (https://commons.wikimedia.org/wiki/File:IBM_Blue_Gene_P_supercomputer.jpg) „IBM Blue Gene P supercomputer“, https://creativecommons.org/licenses/by-sa/2.0/legalcode

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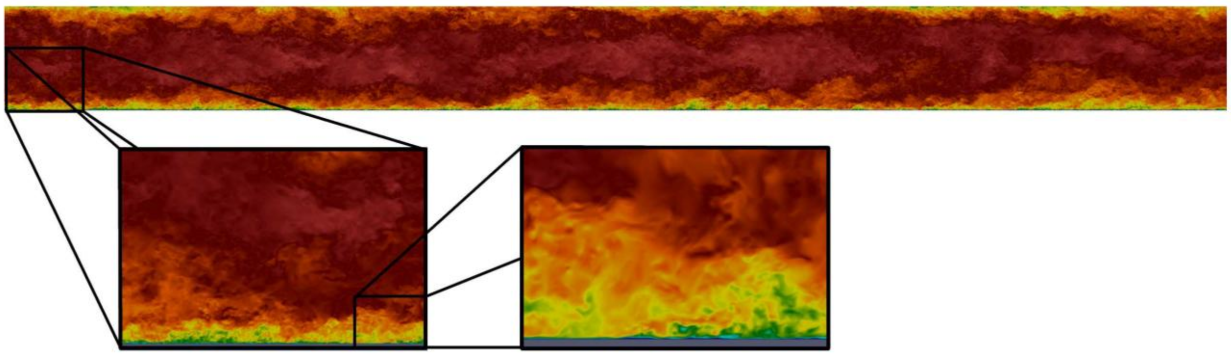


Figure 7: Instantaneous streamwise velocity component over the entire streamwise length of the simulated channel. Zooming in on the flow highlights the multi-scale nature of the turbulence.

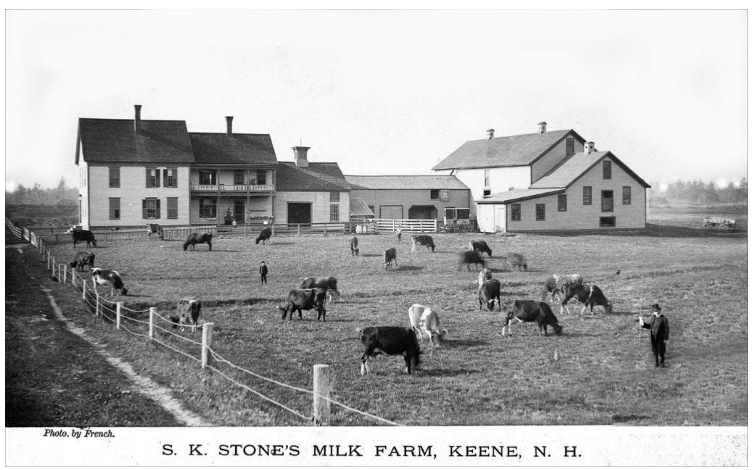
Channel flow calculation that required 260,000,000 CPU hours and resolved all the minute detail.

[Lee, Malaya, & Moser 2013]

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} + \mathbf{b},$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\overline{\Phi[\mathbf{u}(t)]} = ?$$



The problem of turbulence is how to buy a glass of milk without buying all the cows

Acknowledgements: Lee, MK, Malaya, N & Moser, R. (2013). Petascale direct numerical simulation of turbulent channel flow on up to 786K cores. International Conference for High Performance Computing, Networking, Storage and Analysis, SC. 10.1145/2503210.2503298.
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Stepping back to move forward: bounds for time averages

$$\overline{\Phi} = ? \quad \rightarrow \quad L \leq \overline{\Phi} \leq B, \quad L, B = ?$$

A bit of history

Howard (1972), Busse (1978)
…
Doering & Constantin (1994),
Constantin & Doering (1995) – Background method
…
Kerswell (1998) – Unification of the above
…
…
…
…

Background method:
the current workhorse
of bounding time averages.
Review by Fantuzzi *et al.* (2022)

Otto F, Seis C. (2011), Seis C. (2015) – "Direct method"

Chernyshenko et al. (2014) – Auxiliary functional method (of which the background method is a special case)

Relationship between these methods: Chernyshenko (2022)

…

Known unknowns: what I will not tell you

Not talking about:

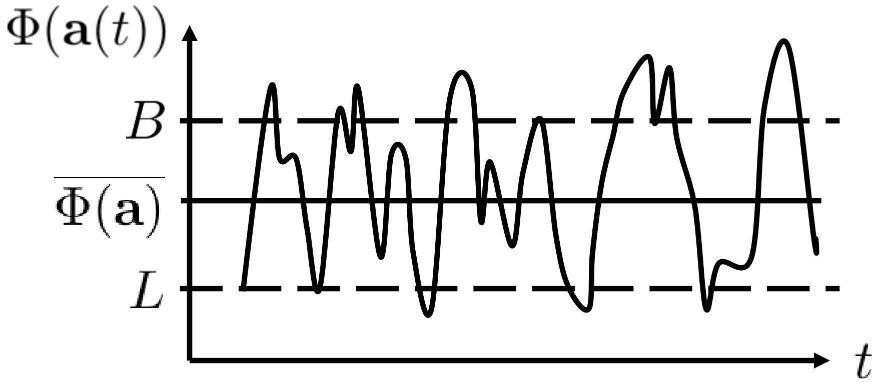
Polynomial optimisation: Goulart & Chernyshenko 2012, Chernyshenko et al. 2014, Goluskin and Fantuzzi 2019, Fuentes *et al*. 2019

Bounds for systems with noise Chernyshenko et al. 2014, Fantuzzi *et al*. 2016

Orbits: Tobasco *et al.* 2018, Lakshmi *et al*. 2020, 2021

Auxiliary function method of bounding time averages

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \overline{\Phi(\mathbf{a}(t))} = ?$$



$$\overline{D(\mathbf{a}(t))} = 0, \;\; \Phi(\mathbf{a}) + D(\mathbf{a}) \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B$$

$$V < \infty \quad \Rightarrow \quad \overline{\frac{dV}{dt}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{dV}{dt}\, dt = \lim_{T\to\infty} \frac{V|_{t=T} - V|_{t=0}}{T} = 0$$

$$\Rightarrow \quad \overline{\Phi + \frac{dV}{dt}} = \overline{\Phi}$$

$$\Phi(\mathbf{a}) + \frac{dV}{dt} \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B$$

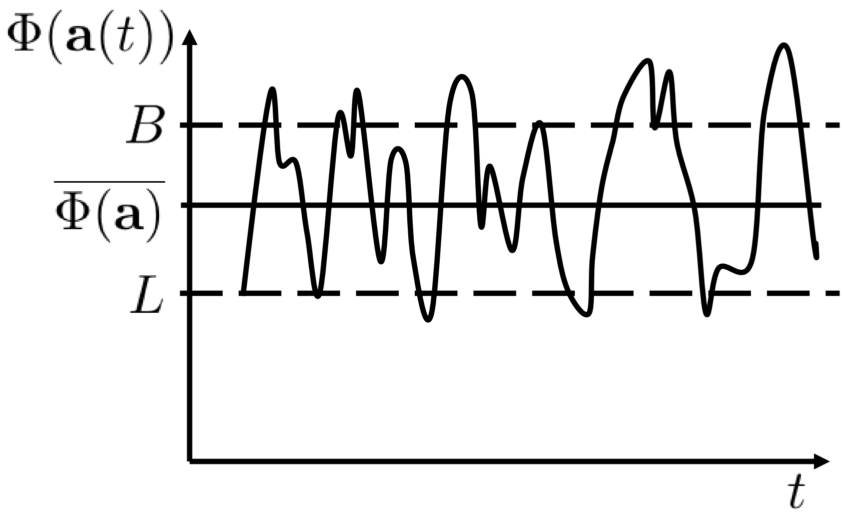
$$V(\mathbf{a}(t)) \Rightarrow \frac{dV}{dt} = \mathbf{f} \cdot \nabla V \quad \Rightarrow \quad \boxed{\Phi(\mathbf{a}) + \mathbf{f} \cdot \nabla V \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B}$$

Lie derivative

Chernyshenko et al. 2014

Bounds for time averages can be optimised

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \overline{\Phi(\mathbf{a}(t))} = ?$$



$$L = \max_{V(\mathbf{a}),C} C$$

$$\text{s.t. } \mathbf{f} \cdot \nabla V + \Phi(\mathbf{a}) - C \geq 0 \; \forall \, \mathbf{a}$$

The largest lower bound problem
is dual
to the smallest time average problem.

$$L = \min_{\mathbf{a}_0} \overline{\Phi}(\mathbf{a}(t))$$

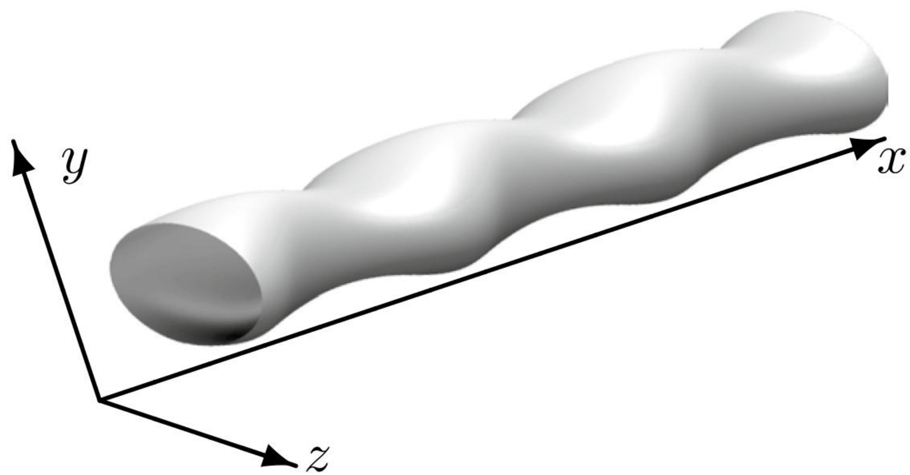
$$\text{s.t.} \quad \frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \mathbf{a}(0) = \mathbf{a}_0$$

Theorem: Auxiliary functions provide arbitrarily sharp bounds on time averages.

Tobasco *et al.* 2018, Rosa & Temam 2020

Korda et al. 2021

Is there Re-independent bound for energy dissipation?



$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \nu \nabla^2 \mathbf{u} + g \mathbf{e}_x,$$

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{\text{wall}} = 0,$$

$$\mathbf{u}(x+L) = \mathbf{u}(x), \quad p(x+L) = p(x).$$

Inner product and norm

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle_\Omega = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\Omega, \quad ||\mathbf{u}||^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

$$\langle \mathbf{u}, \mathbf{v} \cdot \nabla \mathbf{w} \rangle = - \langle \mathbf{w}, \mathbf{v} \cdot \nabla \mathbf{u} \rangle \text{ and } \langle \mathbf{u}, \nabla^2 \mathbf{v} \rangle = - \langle \nabla \mathbf{u}, \nabla \mathbf{v} \rangle$$

Energy conservation

$$\frac{1}{2} \frac{d||\mathbf{u}||^2}{dt} = -\nu ||\nabla \mathbf{u}||^2 + \langle \mathbf{u}, g \mathbf{e}_x \rangle$$

Dissipation Production

$$\nu \overline{||\nabla \mathbf{u}||^2} > D_L \ ?$$

The simplest auxiliary functional is linear:

$$\boxed{V[\mathbf{u}] = \langle \mathbf{b}, \mathbf{u} \rangle}$$

$\mathbf{b}|_{\text{wall}} = 0$, $\mathbf{b}(x+L) = \mathbf{b}(x)$, and $\nabla \cdot \mathbf{b} = 0$.

$$\mathcal{L}V[\mathbf{u}(t)] = \frac{dV[\mathbf{u}(t)]}{dt}$$

$$\left\langle \mathbf{b}, \frac{\partial \mathbf{u}}{\partial t} \right\rangle$$

$$\nu||\nabla \mathbf{u}||^2 + \mathcal{L}\langle \mathbf{b}, \mathbf{u} \rangle = \nu||\nabla \mathbf{u}||^2 + \left\langle \mathbf{b}, -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x \right\rangle \overset{?}{>} D_L \ \forall \mathbf{u}$$

$$\nu||\nabla \mathbf{u}||^2 + \left\langle \mathbf{u}, \mathbf{u}\cdot\nabla\mathbf{b} \right\rangle - \nu\langle \nabla\mathbf{b}, \nabla\mathbf{u} \rangle + \langle \mathbf{b}, g\mathbf{e}_x \rangle \overset{?}{>} D_L \ \forall \mathbf{u}$$

Good guy — Bad guy — Bad guy — Good guy

Constantin & Doering (1995)

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$$\nu||\nabla \mathbf{u}||^2 + \left\langle \mathbf{u}, \mathbf{u} \cdot \nabla \mathbf{b} \right\rangle - \nu \langle \nabla \mathbf{b}, \nabla \mathbf{u} \rangle + \langle \mathbf{b}, g\mathbf{e}_x \rangle \overset{?}{>} D_L \ \forall \mathbf{u}$$

Good guy | Bad guy | Bad guy | Good guy

$$= \nu\left(||\nabla \mathbf{b} - \nabla \mathbf{u}||^2 - ||\nabla \mathbf{b}||^2 - ||\nabla \mathbf{u}||^2\right)/2$$

Good | Bad | Bad

Constantin & Doering (1995)

The simplest auxiliary functional is linear:

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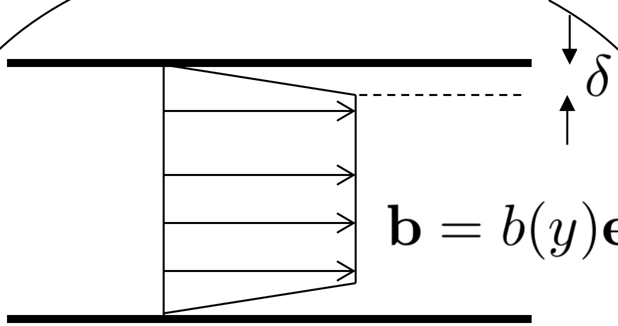
$$\nu||\nabla \mathbf{u}||^2 + \mathcal{L}\langle \mathbf{b}, \mathbf{u} \rangle = \nu||\nabla \mathbf{u}||^2 + \left\langle \mathbf{b}, -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \nu \nabla^2 \mathbf{u} + g\mathbf{e}_x \right\rangle \overset{?}{>} D_L \ \forall \mathbf{u}$$

$$\nu||\nabla \mathbf{u}||^2 + \left\langle \mathbf{u}, \mathbf{u} \cdot \nabla \mathbf{b} \right\rangle - \nu \langle \nabla \mathbf{b}, \nabla \mathbf{u} \rangle + \langle \mathbf{b}, g\mathbf{e}_x \rangle \overset{?}{>} D_L \ \forall \mathbf{u}$$

Good guy   Bad guy   Bad guy   Good guy

$$= \nu\left(||\nabla \mathbf{b} - \nabla \mathbf{u}||^2 - ||\nabla \mathbf{b}||^2 - ||\nabla \mathbf{u}||^2\right)/2$$

Good   Bad   Bad



$$\mathbf{b} = b(y)\mathbf{e}_x$$

$$\delta \sim \nu, \quad \mathbf{b} \sim 1$$

Poincaré $\Rightarrow$ $$\left|\left\langle \mathbf{u}, \mathbf{u} \cdot \nabla \mathbf{b} \right\rangle\right| \leq C\delta^2 ||\nabla \mathbf{u}||^2 \left|\left|\nabla \mathbf{b}\right|\right|_\infty$$

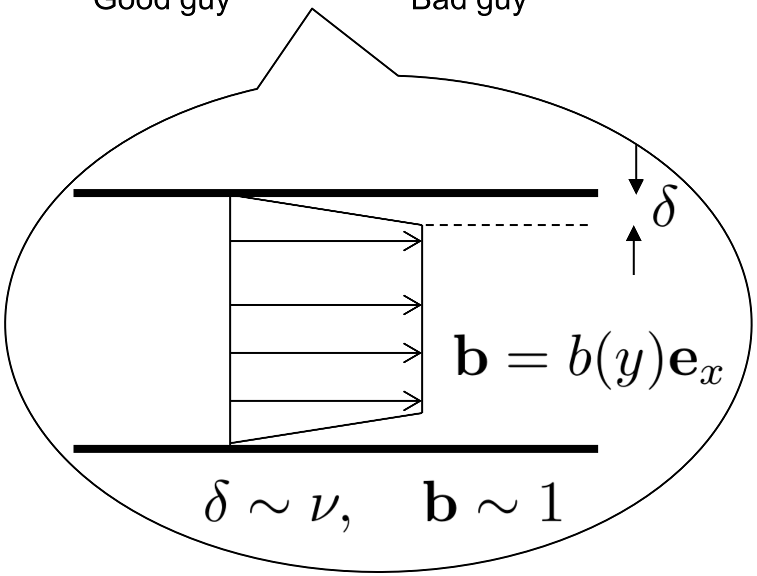
Constantin & Doering (1995)

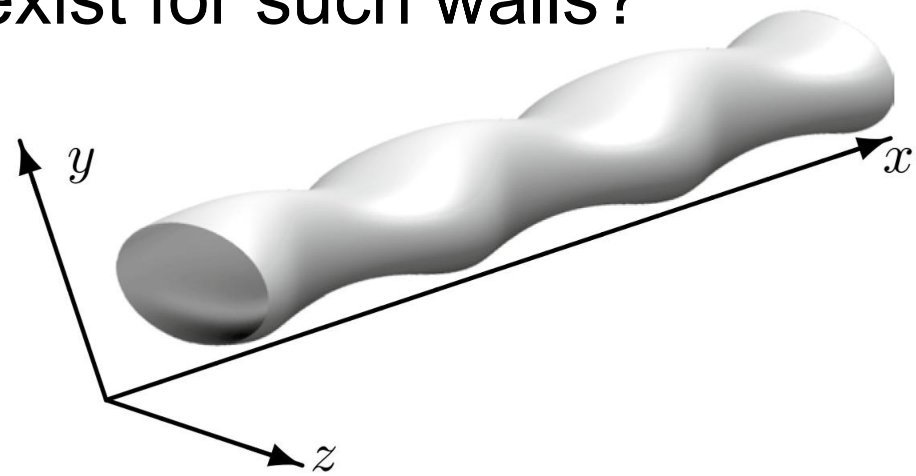
$$\boxed{\overline{\nu||\nabla \mathbf{u}||^2} \geq \langle \mathbf{b}, g\mathbf{e}_x \rangle - \frac{\nu}{2}||\nabla \mathbf{b}||^2}$$

Are curvy walls just harder to study, or do Re-independent bounds not exist for such walls?

$$\nu||\nabla \mathbf{u}||^2 + \langle \mathbf{u}, \mathbf{u}\cdot\nabla \mathbf{b}\rangle + \;\ldots$$

Good guy   Bad guy





$$\langle \mathbf{u}, \mathbf{u}\cdot\nabla\mathbf{b}\rangle = \langle \mathbf{u}, \mathbf{u}\cdot\left(\nabla\mathbf{b} + \nabla\mathbf{b}^\top\right)\rangle/2$$

Poincaré

$$\overline{\nu||\nabla \mathbf{u}||^2} \geq \langle \mathbf{b}, g\mathbf{e}_x\rangle - \frac{\nu}{2}||\nabla\mathbf{b}||^2$$

$$V[\mathbf{u}] \overset{?}{=} \langle \mathbf{b}, \mathbf{u}\rangle$$

$$L = \max_{V(\mathbf{a}),C} C$$

$$\text{s.t. } \mathbf{f}\cdot\nabla V + \Phi(\mathbf{a}) - C \geq 0$$

Theorem: Auxiliary functions provide arbitrarily sharp bounds on time averages.

Tobasco *et al.* 2018, Rosa & Temam 2020

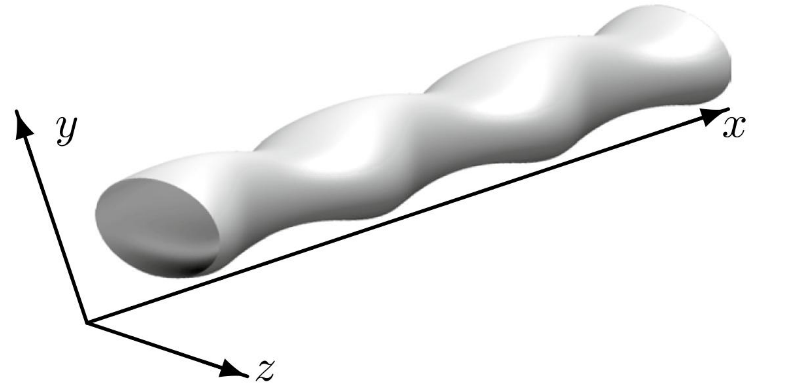
Works on Re-independent bounds for walls of general shape are rare

Wang (1997) – the first ever!

Goluskin & Doering (2016) – convection

Tilgner (2021) – assumes finite TKE

Chernyshenko (2025) – below



$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x$$

$$\frac{d\langle \mathbf{e}_x, \mathbf{u}\rangle}{dt} = F_b - F_p - F_f, \quad \overline{F_b} = \overline{F_p} + \overline{F_f}$$

$$F_b = g\langle \mathbf{e}_x, \mathbf{e}_x\rangle, \quad F_p = \int_{\text{wall}} p\mathbf{e}_x \cdot \mathbf{n}\, d\sigma$$

$$\text{and } F_f = -\nu \int_{\text{wall}} (\mathbf{e}_x \times \boldsymbol{\omega})\cdot \mathbf{n}\, d\sigma$$

$\overline{F_f}$ is given, $\overline{F_b}$ is unknown

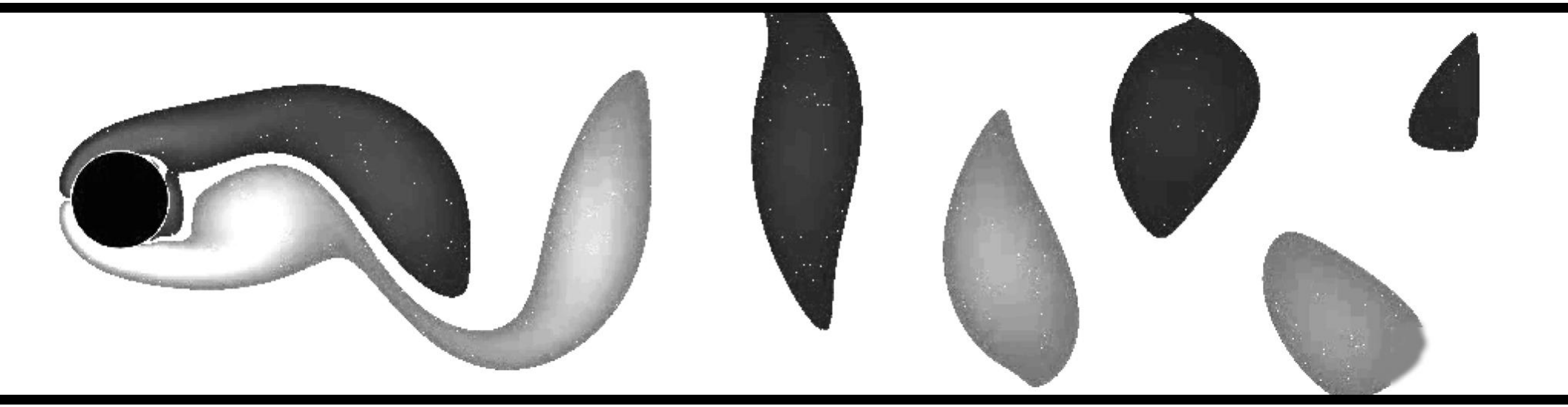
↓

Background flow zero in the bulk is used

↓

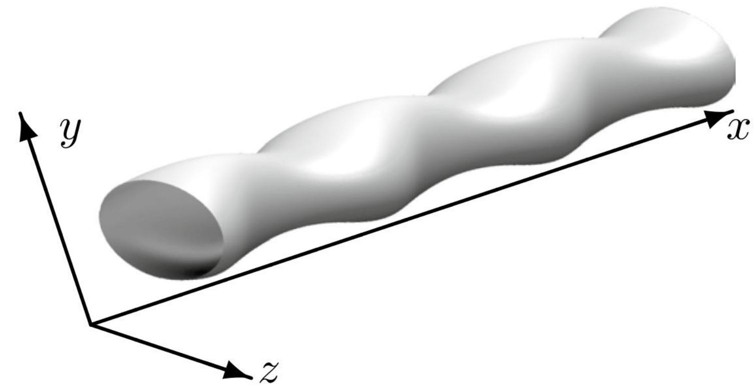
The bound obtained

Can the friction drag tend to zero but the pressure drag remain finite as Re tends to infinity?



Steady flow should be easier to study, but …

$$-\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x = 0,$$
$$\nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u}|_{\text{wall}} = 0,$$
$$\mathbf{u}(x+L) = \mathbf{u}(x),\ \ p(x+L) = p(x)$$

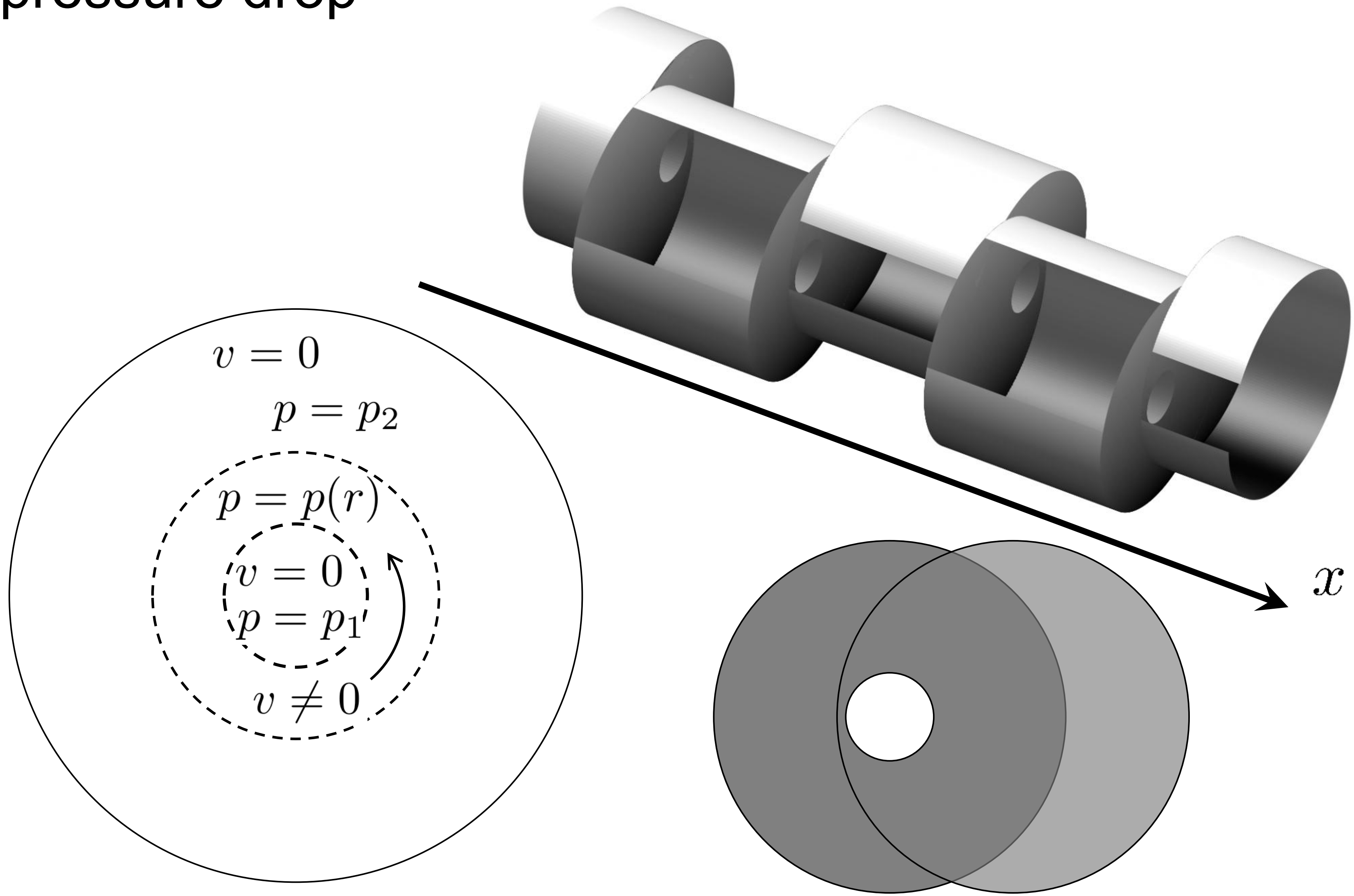


$$\nu\|\nabla\mathbf{w}\|^2 + \left\langle \mathbf{b}_{\mathbf{w}}, -\mathbf{w}\cdot\nabla\mathbf{w} - \nabla p + \nu\nabla^2\mathbf{w} + g\mathbf{e}_x \right\rangle \geq D_L \quad \forall\mathbf{w}$$
$$\Rightarrow \nu\|\nabla\mathbf{u}\|^2 \geq D_L$$

$\langle \mathbf{b}_{\mathbf{w}}, -\mathbf{w}\cdot\nabla\mathbf{w}\rangle$ is not bad anymore
$\left\langle \mathbf{b}_{\mathbf{w}}, \nu\nabla^2\mathbf{w}\right\rangle$ can be dealt with as before
$\langle \mathbf{b}_{\mathbf{w}}, g\mathbf{e}_x\rangle$ might produce Re-independent bound

Unless $\exists\, \mathbf{w}, p : -\mathbf{w}\cdot\nabla\mathbf{w} - \nabla p + g\mathbf{e}_x = 0$

Inviscid pipe flow with zero flow rate but nonzero pressure drop



Can an inviscid pipe flow with zero flow rate but nonzero pressure drop be the high-Re limit?

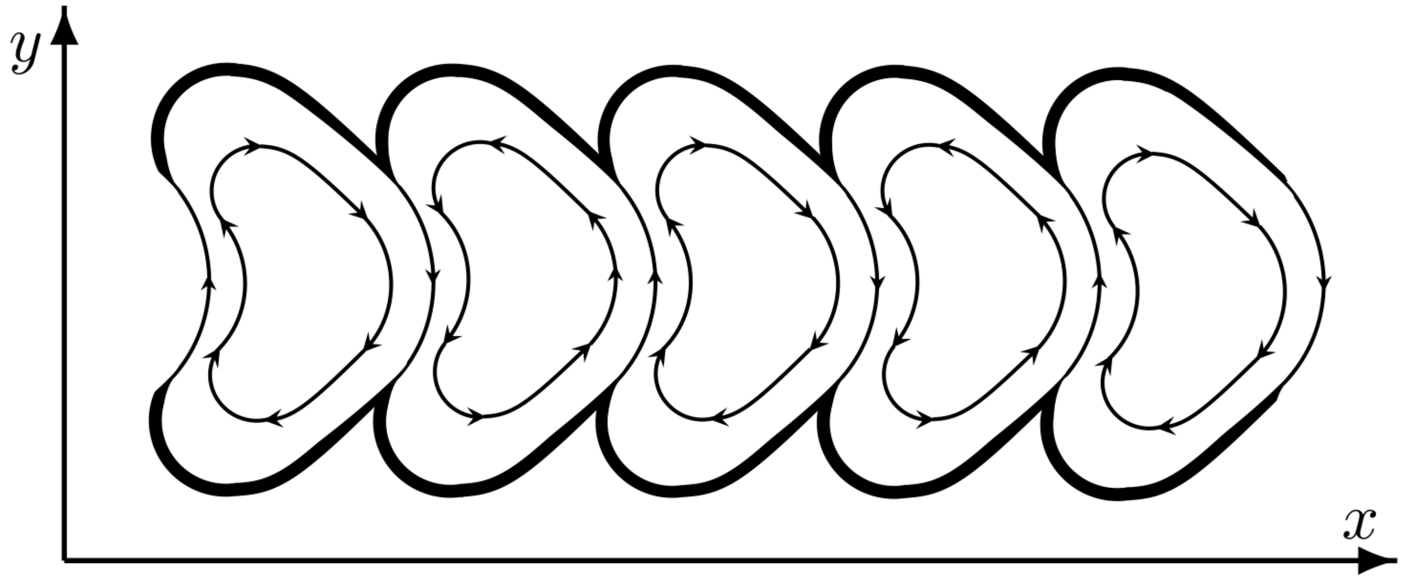
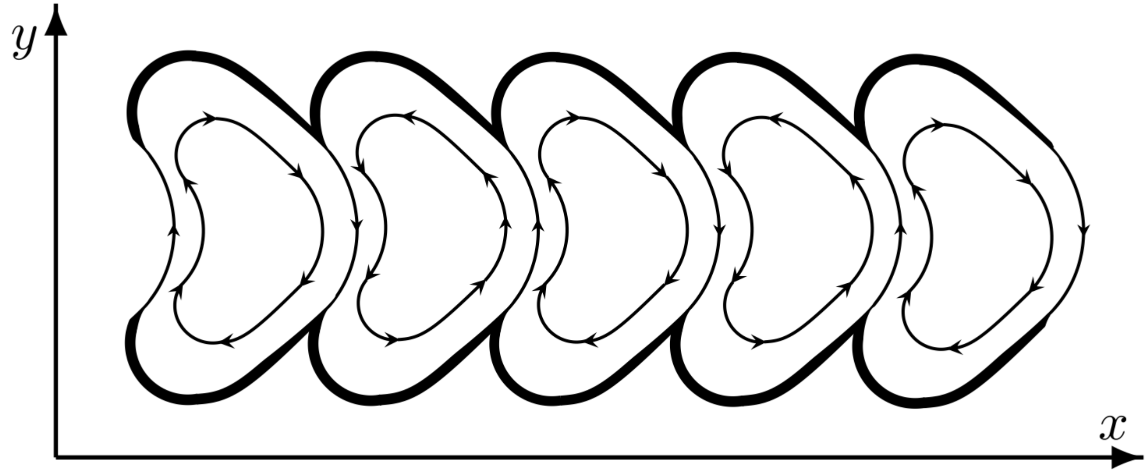


Fig. 3 Hypothetical inviscid flow. The velocity and the Bernoulli constant jump across the streamlines separating the neighbouring cells. The pressure is continuous. The body force is compensated by the curvature of the streamlines in the middle and the wall pressure. The orientation corresponds to $g < 0$, that is the body force is directed to the left.

Summary

- **A great problem of turbulence: to buy a glass of milk without buying all the cows.**
- **Bound the sum of the functional of interest and the Lie derivative of an auxiliary functional.**
- **Bounding the friction drag is doable but bounding the pressure drag is hard.**
- **A challenge for asymptoticians: exotic high-Re steady flow**



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If such asymptotic solution for $\nu \to 0$ would be constructed, it would be a counterexample to the existence of a finite ν-independent lower bound for the energy dissipation rate.

Link to the paper: [Chernyshenko (2025)]

A copy of this presentation with clickable links is available at https://www.imperial.ac.uk/aeronautics/fluiddynamics/2026_IC_Chernyshenko.pdf