Workshop on LMI Methods for the Analysis and Control of PDEs

Department of Engineering Sciences, University of Oxford

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Progress on bounds for dissipation in flows of general geometry reinforces the need for methods applicable to stochastic PDE

by

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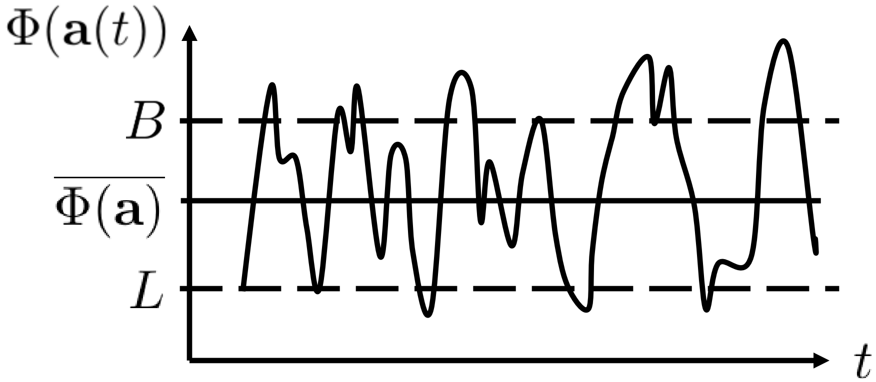
https://www.imperial.ac.uk/aeronautics/fluiddynamics/ChernyshenkoResearch/index.php

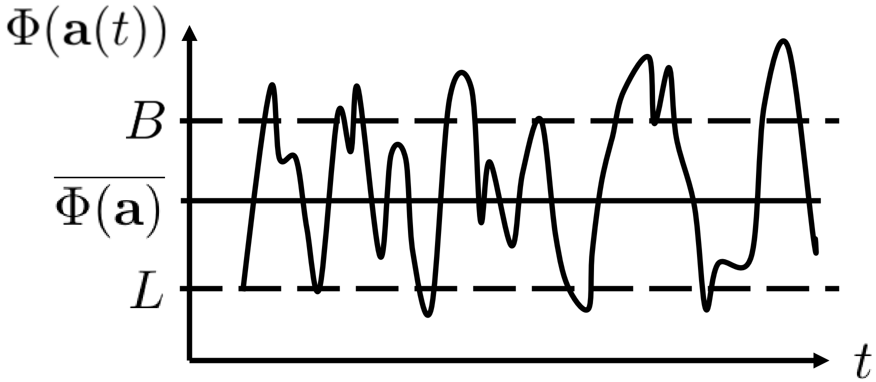
Reminder. Auxiliary function method of bounding time averages



$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \overline{\Phi(\mathbf{a}(t))} = ?$$

Chernyshenko et al. 2014

Reminder. Auxiliary function method of bounding time averages



$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \overline{\Phi(\mathbf{a}(t))} = ?$$

$$\Phi(\mathbf{a}) \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B$$

Chernyshenko et al. 2014

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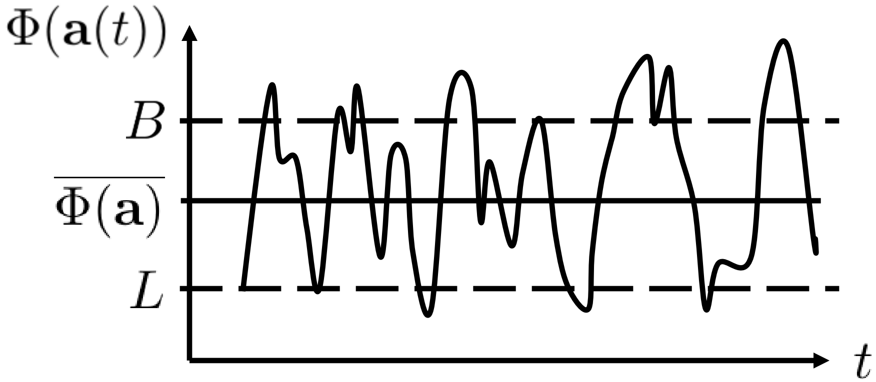

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \overline{\Phi(\mathbf{a}(t))} = ?$$

$$\overline{D(\mathbf{a}(t))} = 0, \;\; \Phi(\mathbf{a}) + D(\mathbf{a}) \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B$$

Chernyshenko et al. 2014

Reminder. Auxiliary function method of bounding time averages

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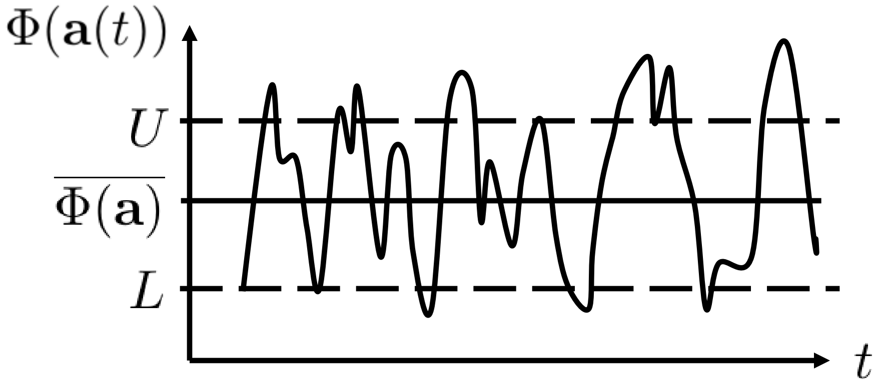

$$\overline{D(\mathbf{a}(t))} = 0, \ \Phi(\mathbf{a}) + D(\mathbf{a}) \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B$$

$$V < \infty \quad \Rightarrow \quad \overline{\frac{dV}{dt}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{dV}{dt}\, dt = \lim_{T\to\infty} \frac{V|_{t=T} - V|_{t=0}}{T} = 0$$

$$\Rightarrow \quad \overline{\Phi + \frac{dV}{dt}} = \overline{\Phi}$$

$$\Phi(\mathbf{a}) + \frac{dV}{dt} \leq B \qquad \Rightarrow \qquad \overline{\Phi} \leq B$$

Chernyshenko et al. 2014

Reminder. Auxiliary function method of bounding time averages

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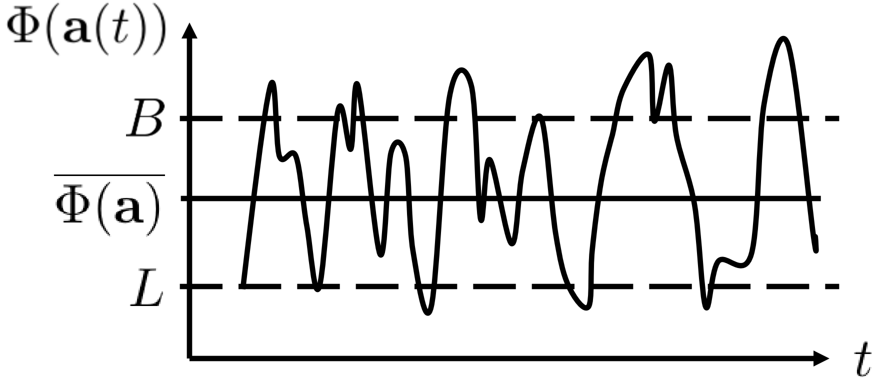

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$$\Phi(\mathbf{a}) + \frac{dV}{dt} \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B$$

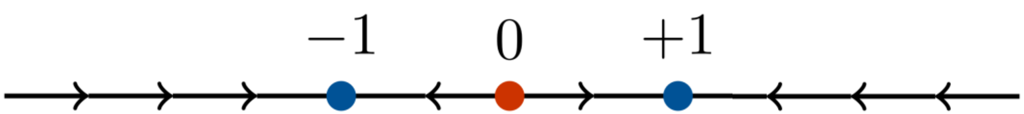
$$V(\mathbf{a}(t)) \Rightarrow \frac{dV}{dt} = \mathbf{f} \cdot \nabla V \quad \Rightarrow \quad \boxed{\Phi(\mathbf{a}) + \mathbf{f}\cdot\nabla V \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B}$$

Lie derivative

Chernyshenko et al. 2014

The problem of unstable solutions

$$\frac{da}{dt} = a - a^3$$

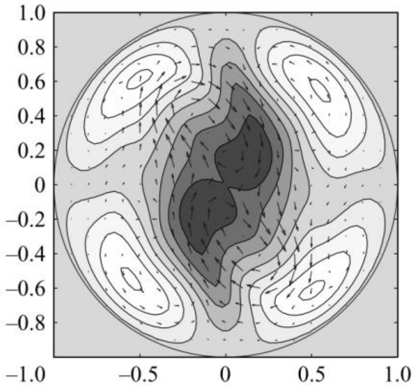


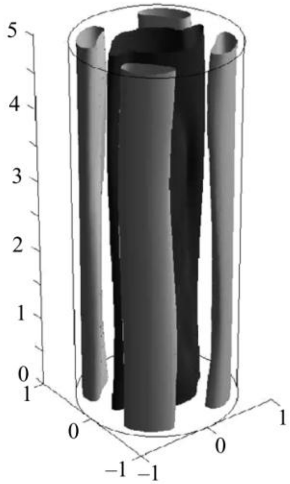
$$a = 0, \quad a = -1, \quad a = +1$$

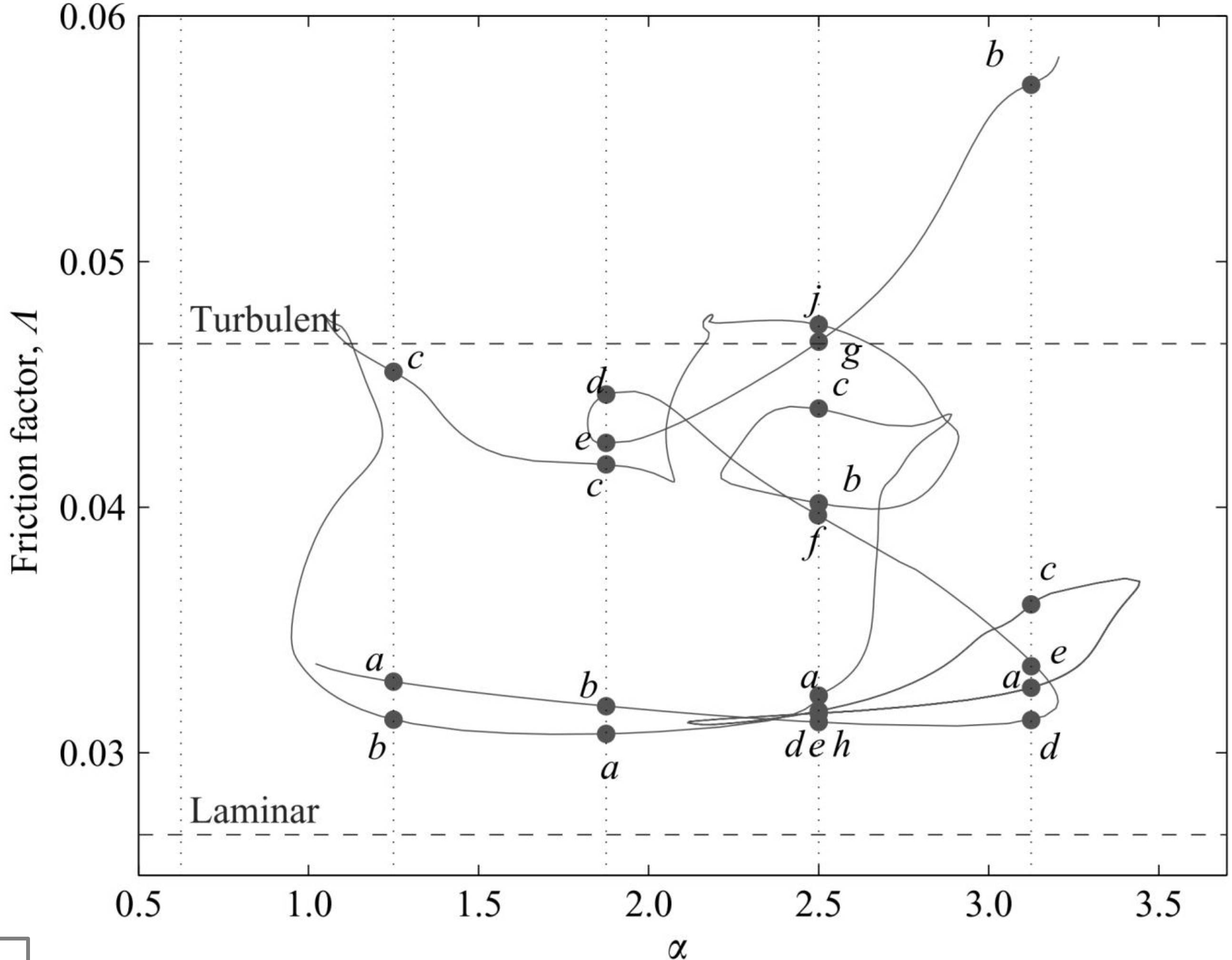
$$\Phi = a^2, \quad 0 \le \overline{\Phi} \le 1$$

$$L \le \Phi + (a - a^3)\frac{dV}{da} \le U$$

For pipe flow bounds are constrained by travelling wave solutions (Re=2400)







Kerswell & Tutty 2007, Fig 1

Accounting for noise in auxiliary function method

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}) + \sqrt{2\epsilon}\boldsymbol{\xi}(t)$$

$$\nabla \rho \mathbf{f} - \epsilon \nabla^2 \rho = 0$$

$$\bar{\Phi} = \langle \Phi \rangle = \int \rho \Phi \, da, \qquad \int \rho \, da = 1$$

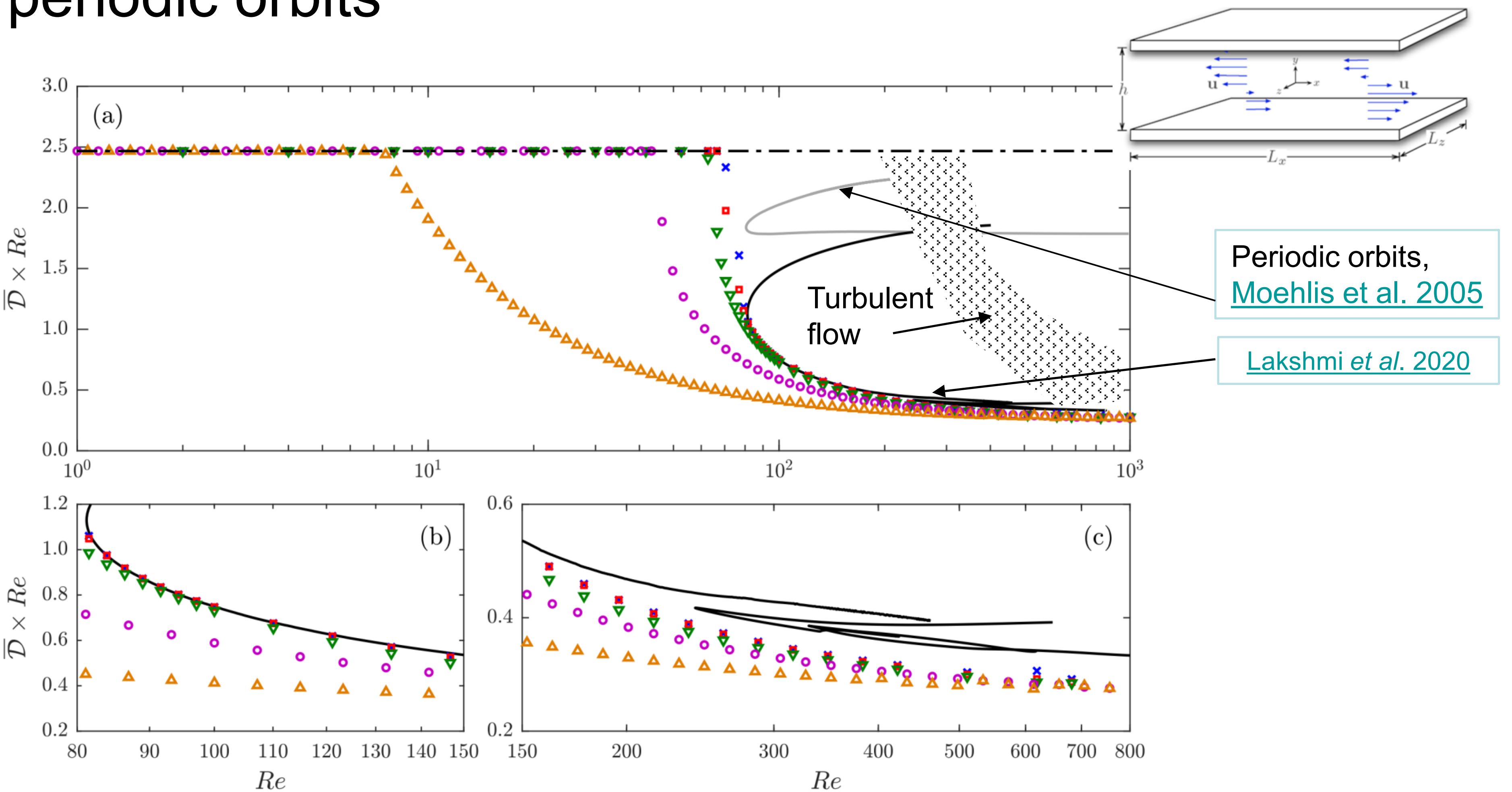
$$\Phi + \mathbf{f} \cdot \nabla V + \epsilon \nabla^2 V \lessgtr C \quad \forall \mathbf{a}$$

$$\Rightarrow \quad \bar{\Phi} \lessgtr C$$

Chernyshenko et al. 2014

See also the related works by Fantuzzi *et al*. 2016, Kuntz et al 2016, Korda et al 2021, and references therein.

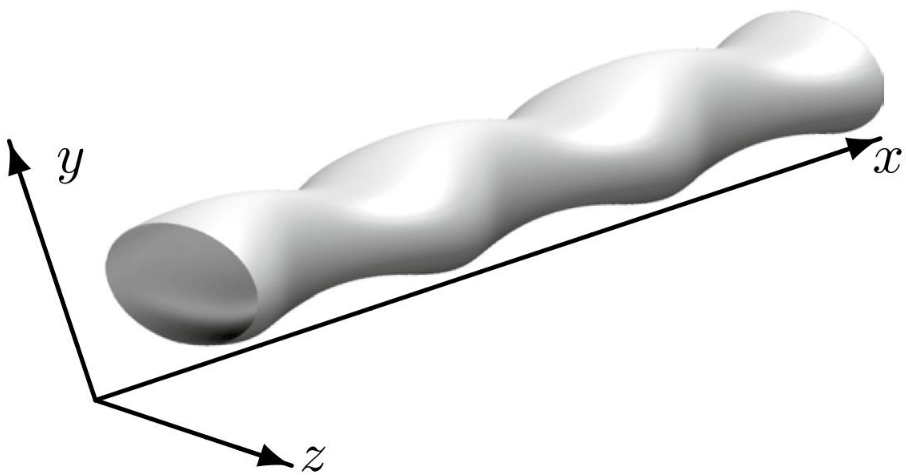
Bounds for the 9-mode system are constrained by the periodic orbits



Auxiliary function degree: 2 (△), 4 (○), 6 (▽), 8 (□) and 10 (×)

(Turbulent > laminar or laminar > turbulent?)

Is there Re-independent bound for energy dissipation?



$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \nu \nabla^2 \mathbf{u} + g\mathbf{e}_x,$$

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{\text{wall}} = 0,$$

$$\mathbf{u}(x+L) = \mathbf{u}(x), \quad p(x+L) = p(x).$$

Inner product and norm

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle_\Omega = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\Omega, \quad ||\mathbf{u}||^2 = \langle \mathbf{u}, \mathbf{u} \rangle$$

$$\langle \mathbf{u}, \mathbf{v} \cdot \nabla \mathbf{w} \rangle = -\langle \mathbf{w}, \mathbf{v} \cdot \nabla \mathbf{u} \rangle \text{ and } \langle \mathbf{u}, \nabla^2 \mathbf{v} \rangle = -\langle \nabla \mathbf{u}, \nabla \mathbf{v} \rangle$$

Energy conservation

$$\frac{1}{2} \frac{d||\mathbf{u}||^2}{dt} = -\nu ||\nabla \mathbf{u}||^2 + \langle \mathbf{u}, g\mathbf{e}_x \rangle$$

Dissipation Production

$$\nu \overline{||\nabla \mathbf{u}||^2} > D_L \ ?$$

The simplest auxiliary functional is linear:

$$\boxed{V[\mathbf{u}] = \langle \mathbf{b}, \mathbf{u} \rangle} \qquad \mathbf{b}|_{\text{wall}} = 0, \ \ \mathbf{b}(x+L) = \mathbf{b}(x), \text{ and } \nabla \cdot \mathbf{b} = 0.$$

$$\mathcal{L}V[\mathbf{u}(t)] = \frac{dV[\mathbf{u}(t)]}{dt}$$

$$\left\langle \mathbf{b}, \frac{\partial \mathbf{u}}{\partial t} \right\rangle$$

$$\nu||\nabla \mathbf{u}||^2 + \mathcal{L}\langle \mathbf{b}, \mathbf{u} \rangle = \nu||\nabla \mathbf{u}||^2 + \left\langle \mathbf{b}, -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x \right\rangle \overset{?}{>} D_L \ \forall \mathbf{u}$$

$$\nu||\nabla \mathbf{u}||^2 + \left\langle \mathbf{u}, \mathbf{u}\cdot\nabla\mathbf{b} \right\rangle - \nu\langle \nabla\mathbf{b}, \nabla\mathbf{u} \rangle + \langle \mathbf{b}, g\mathbf{e}_x \rangle \overset{?}{>} D_L \ \forall \mathbf{u}$$

Good guy | Bad guy | Bad guy | Good guy

Constantin & Doering (1995)

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Good guy Bad guy Bad guy Good guy

$$= \nu\left(||\nabla\mathbf{b} - \nabla\mathbf{u}||^2 - ||\nabla\mathbf{b}||^2 - ||\nabla\mathbf{u}||^2\right)/2$$

Good Bad Bad

Constantin & Doering (1995)

The simplest auxiliary functional is linear:

$$\boxed{V[\mathbf{u}] = \langle \mathbf{b}, \mathbf{u} \rangle}$$

$\mathbf{b}|_{\text{wall}} = 0$, $\mathbf{b}(x+L) = \mathbf{b}(x)$, and $\nabla \cdot \mathbf{b} = 0$.

$$\mathcal{L}V[\mathbf{u}(t)] = \frac{dV[\mathbf{u}(t)]}{dt}$$

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$$\nu||\nabla \mathbf{u}||^2 + \left\langle \mathbf{u}, \mathbf{u}\cdot\nabla\mathbf{b} \right\rangle - \nu\langle \nabla\mathbf{b}, \nabla\mathbf{u}\rangle + \langle \mathbf{b}, g\mathbf{e}_x\rangle \overset{?}{>} D_L \;\; \forall \mathbf{u}$$

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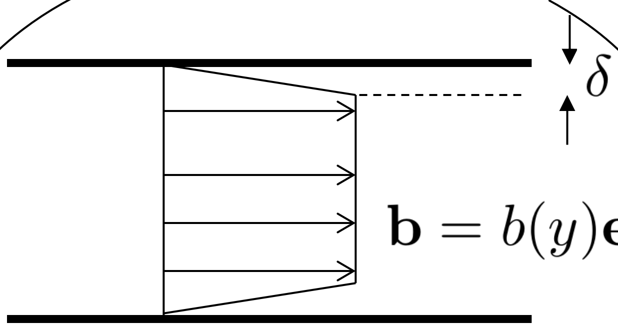
Good | Bad | Bad



$$\mathbf{b} = b(y)\mathbf{e}_x$$

$$\delta \sim \nu, \quad \mathbf{b} \sim 1$$

Poincaré $\Rightarrow$ $$\left|\left\langle \mathbf{u}, \mathbf{u}\cdot\nabla\mathbf{b}\right\rangle\right| \le C\delta^2||\nabla\mathbf{u}||^2\left|\left|\nabla\mathbf{b}\right|\right|_\infty$$

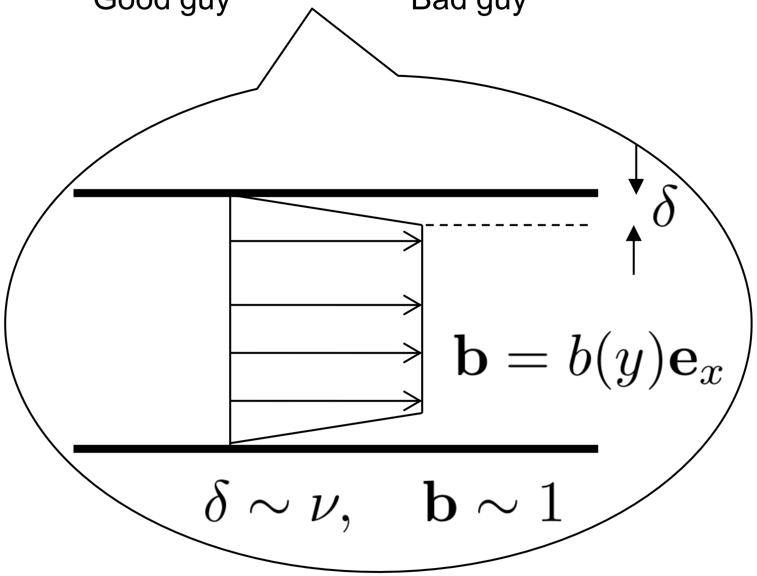
$$\boxed{\overline{\nu||\nabla\mathbf{u}||^2} \ge \langle \mathbf{b}, g\mathbf{e}_x\rangle - \frac{\nu}{2}||\nabla\mathbf{b}||^2}$$

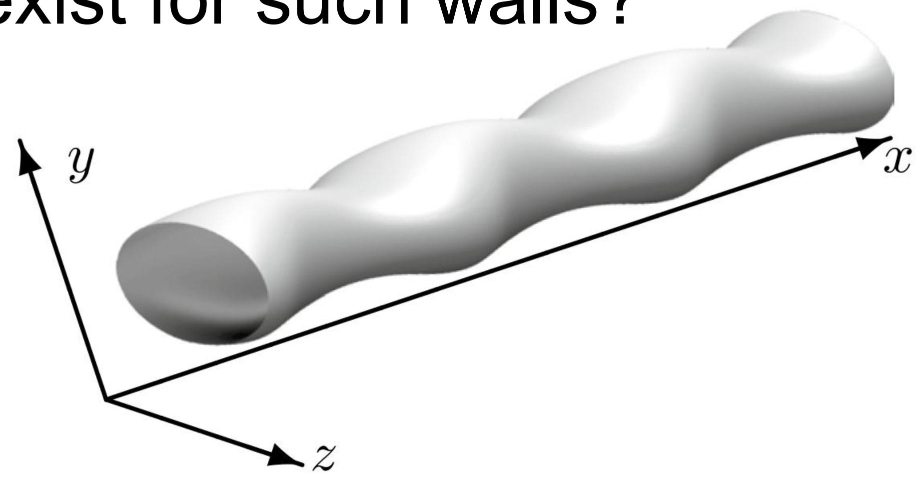
Constantin & Doering (1995)

Are curvy walls just harder to study, or do Re-independent bounds not exist for such walls?

$$\nu\|\nabla \mathbf{u}\|^2 + \langle \mathbf{u}, \mathbf{u}\cdot\nabla\mathbf{b}\rangle + \ \ldots$$

Good guy   Bad guy





$$\langle \mathbf{u}, \mathbf{u}\cdot\nabla\mathbf{b}\rangle = \langle \mathbf{u}, \mathbf{u}\cdot\left(\nabla\mathbf{b} + \nabla\mathbf{b}^\top\right)\rangle/2$$

Poincaré

$$\overline{\nu\|\nabla \mathbf{u}\|^2} \geq \langle \mathbf{b}, g\mathbf{e}_x\rangle - \frac{\nu}{2}\|\nabla\mathbf{b}\|^2$$

$$V[\mathbf{u}] \overset{?}{=} \langle \mathbf{b}, \mathbf{u}\rangle$$

$$L = \max_{V(\mathbf{a}),C} C$$

$$\text{s.t.}\ \ \mathbf{f}\cdot\nabla V + \Phi(\mathbf{a}) - C \geq 0$$

Theorem: Auxiliary functions provide arbitrarily sharp bounds on time averages.

Tobasco *et al.* 2018, Rosa & Temam 2020

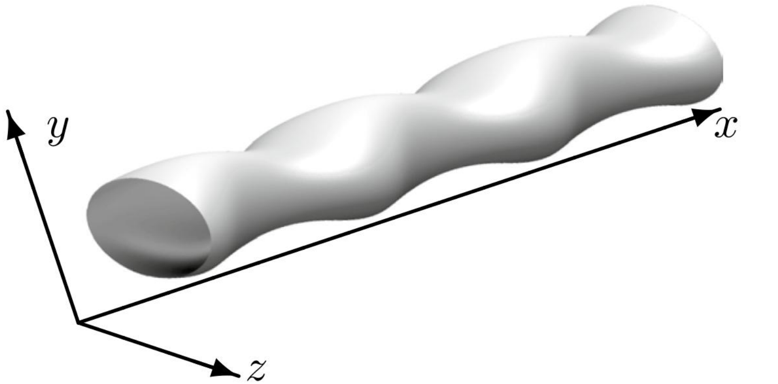
Works on Re-independent bounds for walls of general shape are rare

Wang (1997) – the first ever!

Goluskin & Doering (2016) – convection

Tilgner (2021) – assumes finite TKE

Chernyshenko (2025) – below



$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x$$

$$\frac{d\langle \mathbf{e}_x, \mathbf{u}\rangle}{dt} = F_b - F_p - F_f, \quad \overline{F_b} = \overline{F_p} + \overline{F_f}$$

$$F_b = g\langle \mathbf{e}_x, \mathbf{e}_x\rangle, \quad F_p = \int_{\text{wall}} p\mathbf{e}_x\cdot\mathbf{n}\,d\sigma$$

$$\text{and } F_f = -\nu\int_{\text{wall}} (\mathbf{e}_x\times\boldsymbol{\omega})\cdot\mathbf{n}\,d\sigma$$

$\overline{F_f}$ is given, $\overline{F_b}$ is unknown

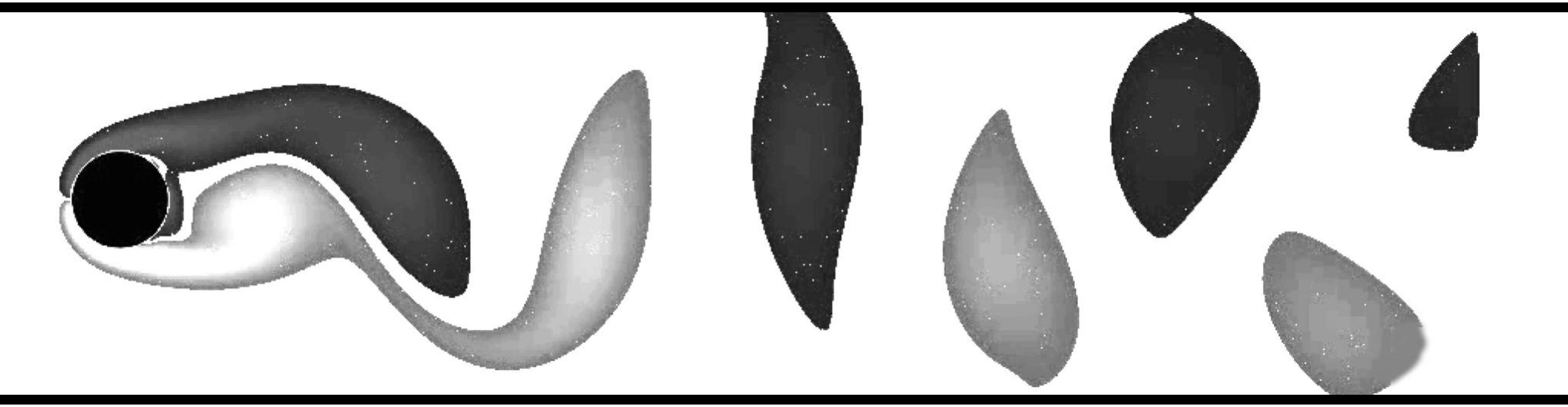
↓

Background flow zero in the bulk is used

↓

The bound obtained

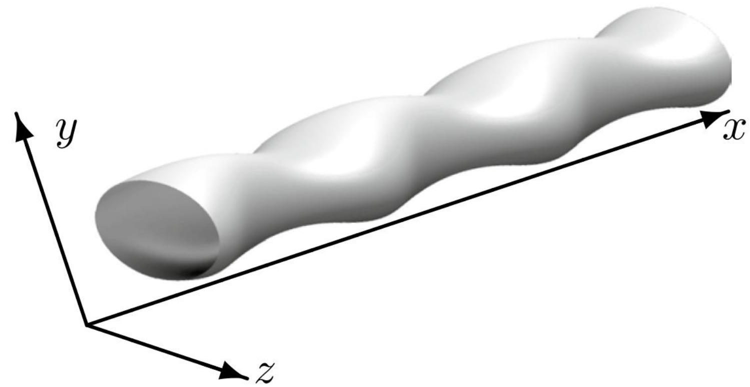
Can friction tend to zero but the pressure drag remain finite as Re tends to infinity? Probably yes …



Chernyshenko (2025)

Steady flow should be easier to study, but …

$$-\mathbf{u}\cdot\nabla\mathbf{u}-\nabla p+\nu\nabla^2\mathbf{u}+g\mathbf{e}_x=0,$$
$$\nabla\cdot\mathbf{u}=0,\ \ \mathbf{u}|_{\mathrm{wall}}=0,$$
$$\mathbf{u}(x+L)=\mathbf{u}(x),\ \ p(x+L)=p(x)$$



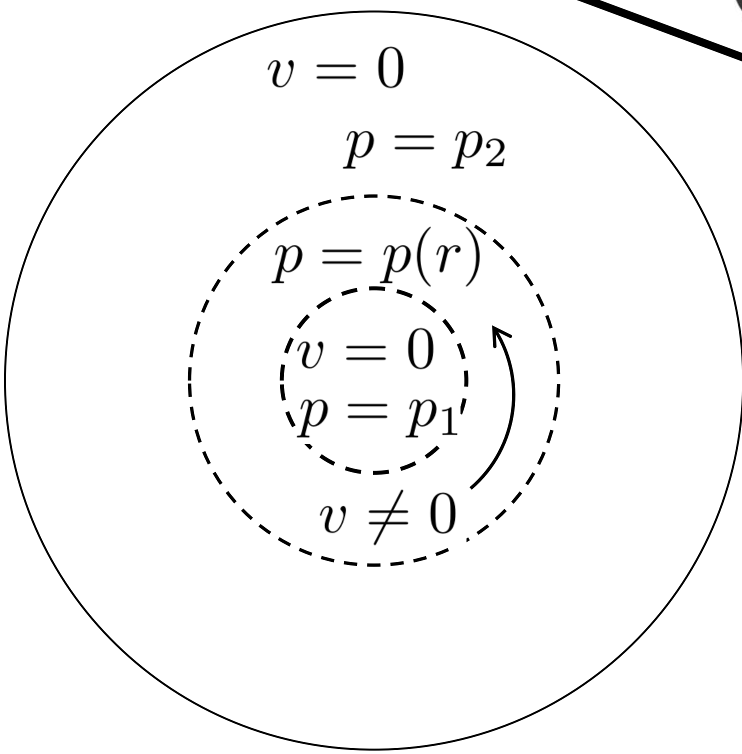
$$\nu\|\nabla\mathbf{w}\|^2+\left\langle\mathbf{b}_{\mathbf{w}},-\mathbf{w}\cdot\nabla\mathbf{w}-\nabla p+\nu\nabla^2\mathbf{w}+g\mathbf{e}_x\right\rangle\geq D_L\quad\forall\mathbf{w}$$
$$\Rightarrow\nu\|\nabla\mathbf{u}\|^2\geq D_L$$

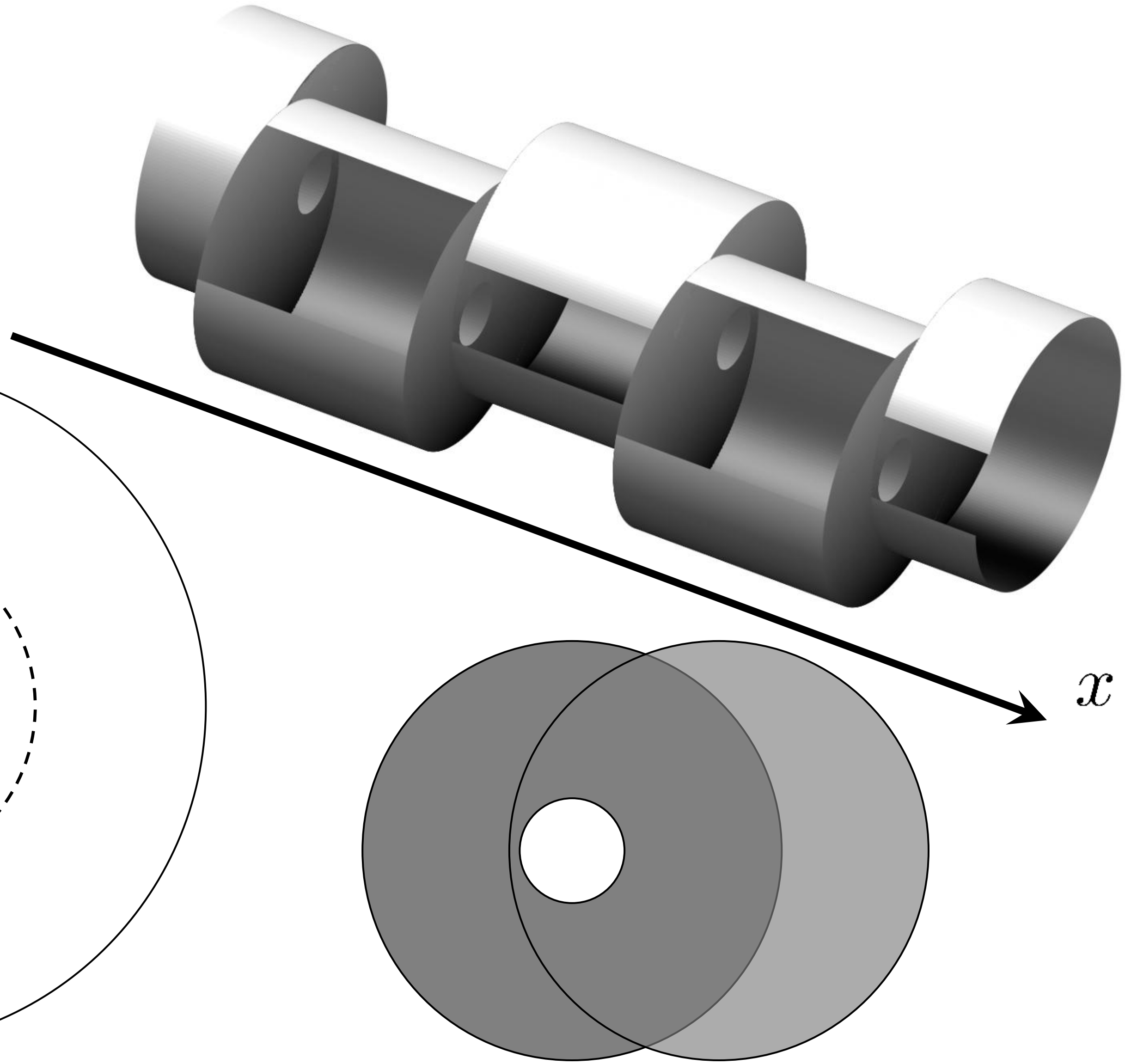
$\langle\mathbf{b}_{\mathbf{w}},-\mathbf{w}\cdot\nabla\mathbf{w}\rangle$ is not bad anymore
$\left\langle\mathbf{b}_{\mathbf{w}},\nu\nabla^2\mathbf{w}\right\rangle$ can be dealt with as before
$\langle\mathbf{b}_{\mathbf{w}},g\mathbf{e}_x\rangle$ might produce Re-independent bound

Unless $\exists\,\mathbf{w},p:-\mathbf{w}\cdot\nabla\mathbf{w}-\nabla p+g\mathbf{e}_x=0$

Chernyshenko (2025)

Inviscid pipe flow with zero flow rate but nonzero pressure drop





Can an inviscid pipe flow with zero flow rate but nonzero pressure drop be the high-Re limit?

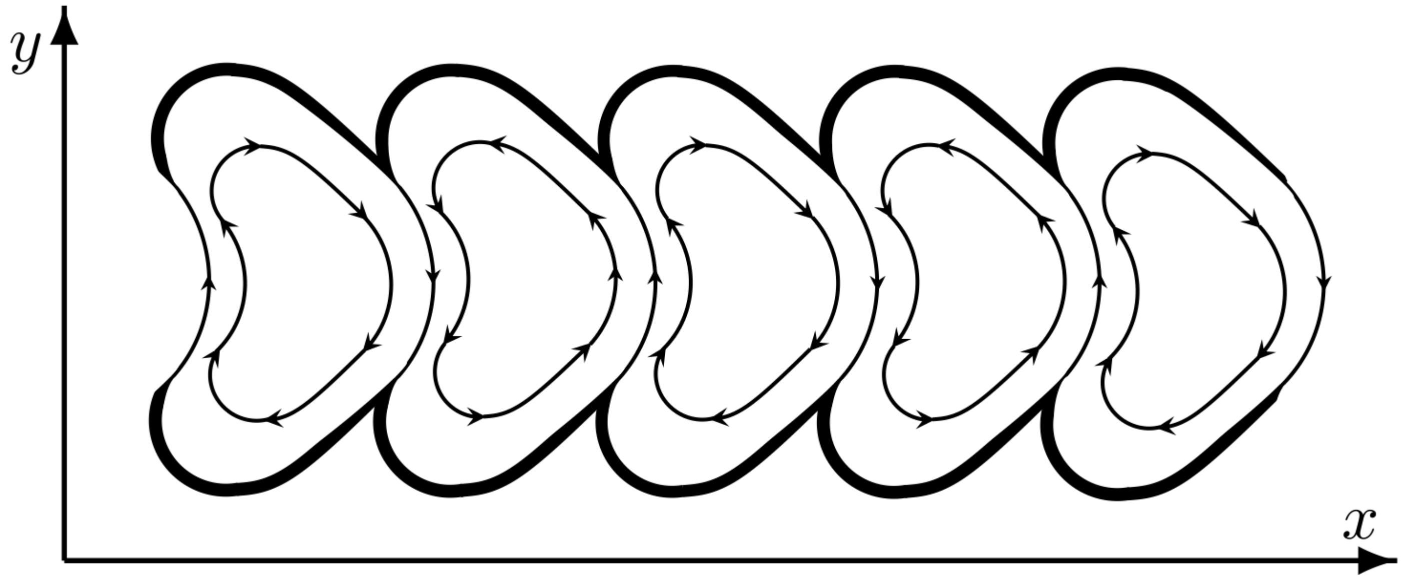


Fig. 3 Hypothetical inviscid flow. The velocity and the Bernoulli constant jump across the streamlines separating the neighbouring cells. The pressure is continuous. The body force is compensated by the curvature of the streamlines in the middle and the wall pressure. The orientation corresponds to $g < 0$, that is the body force is directed to the left.

Conjecture:

As Re tends to infinity, the gap between the real turbulent flow and the most different, but unstable, solution also tends to infinity.

Conclusion:

If you are working in application of LMI to PDE, keep an eye on whether your approach will also work for stochastic PDE.

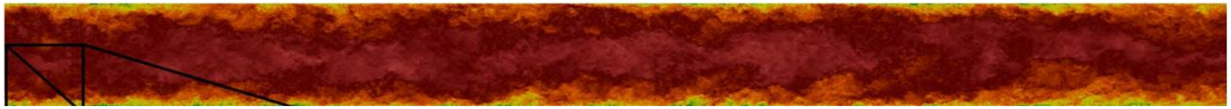
Appendices

Stepping back to move forward: bounds for time averages

$$\overline{\Phi} = ? \quad \rightarrow \quad L \leq \overline{\Phi} \leq B, \quad L, B = ?$$

Howard (1972), Busse (1978)
…
Constantin & Doering (1995), Background flow method – the current workhorse of bounding time averages
…
…

For channel flow:



$$\frac{12}{Re} \leq C_f \leq \frac{27}{32\sqrt{2}} \left(1 + \frac{4\sqrt{2}}{Re}\right)^2$$

Re	Lower bound x 10^3	Experiment x 10^3	Upper bound x 10^3
2 x 10^3	6	11.1	600
10^4	1.2	7.43	597
10^5	0.12	4.18	597

Until recently we had only pen and paper bounds and numerical simulations, but nothing in between

Buildings commonly use a factor of safety of 2.0. Pressure vessels use 3.5 to 4.0, automobiles use 3.0, and aircraft and spacecraft use 1.2 to 3.0.

From Wikipedia

For currently available bounds for a channel flow

Re	Experiment / Lower bound	Upper bound / Experiment
2×10^3	1.9	54
10^4	6.2	82
10^5	35	140

Solution of the problem of turbulence should provide the ability to trade the accuracy for the cost

Uncertain system method rigorously reduces Navier-Stokes to a finite-dimensional, but uncertain, system

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u} + \mathbf{b},$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}|_{\partial\Omega} = 0$$

$$\mathbf{u}(\mathbf{x},t) = \sum_{j=1}^{k} a_j(t)\mathbf{u}_j(\mathbf{x}) + \mathbf{u}_s(\mathbf{x},t)$$

$$q^2 = ||\mathbf{u}_s||^2/2$$

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}) + \boldsymbol{\Theta}$$

$$\frac{dq^2}{dt} = -\mathbf{a}\cdot\boldsymbol{\Theta} + \Gamma + \chi$$

$$\|\boldsymbol{\Theta}\|^2 \le p(\mathbf{a}, q^2)$$

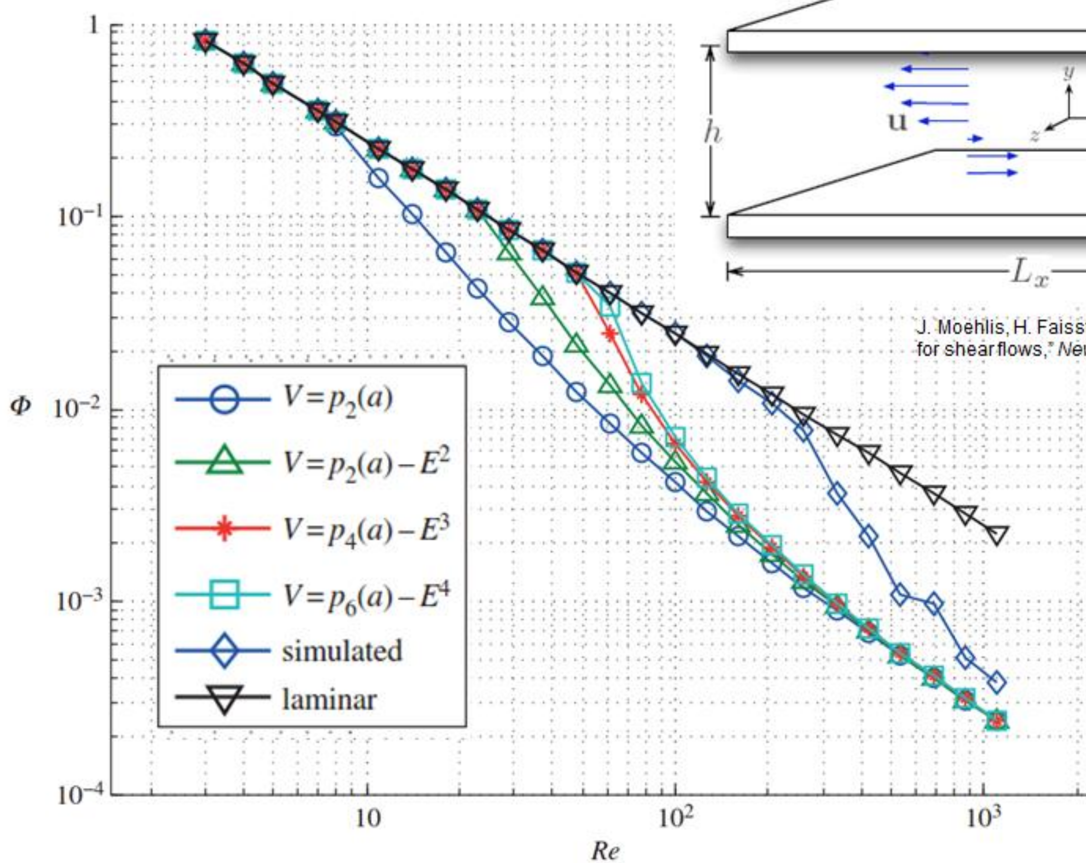
$$\Gamma \le \kappa q^2$$

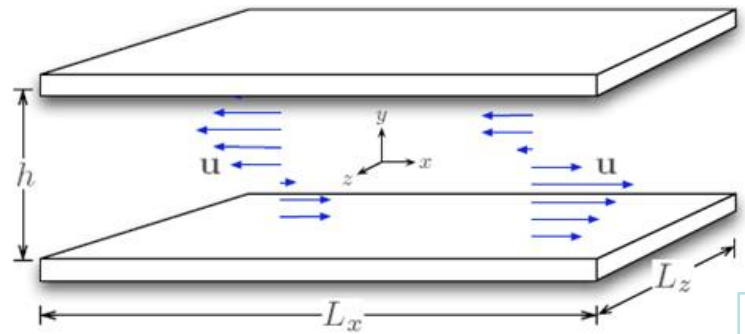
$$\chi^2 \le r(\mathbf{a}, q^2)$$

1. Choosing $\mathbf{u}_j$ and k makes $\chi = 0$ and $\kappa < 0$
2. Polynomial p is quadratic
3. $V[\mathbf{u}] = V(\mathbf{a}, q^2)$
4. The optimization constraint is reducible to SOS

Goulart & Chernyshenko 2012

The second problem. Example: bounds for dissipation in Moehlis-Faisst-Eckhardt 9-mode model





Moehlis *et al.* 2004

J. Moehlis, H. Faisst, and B. Eckhardt, "A low dimensional model for shear flows," *New Journal of Physics*, vol. 6, no. 56, 2004.

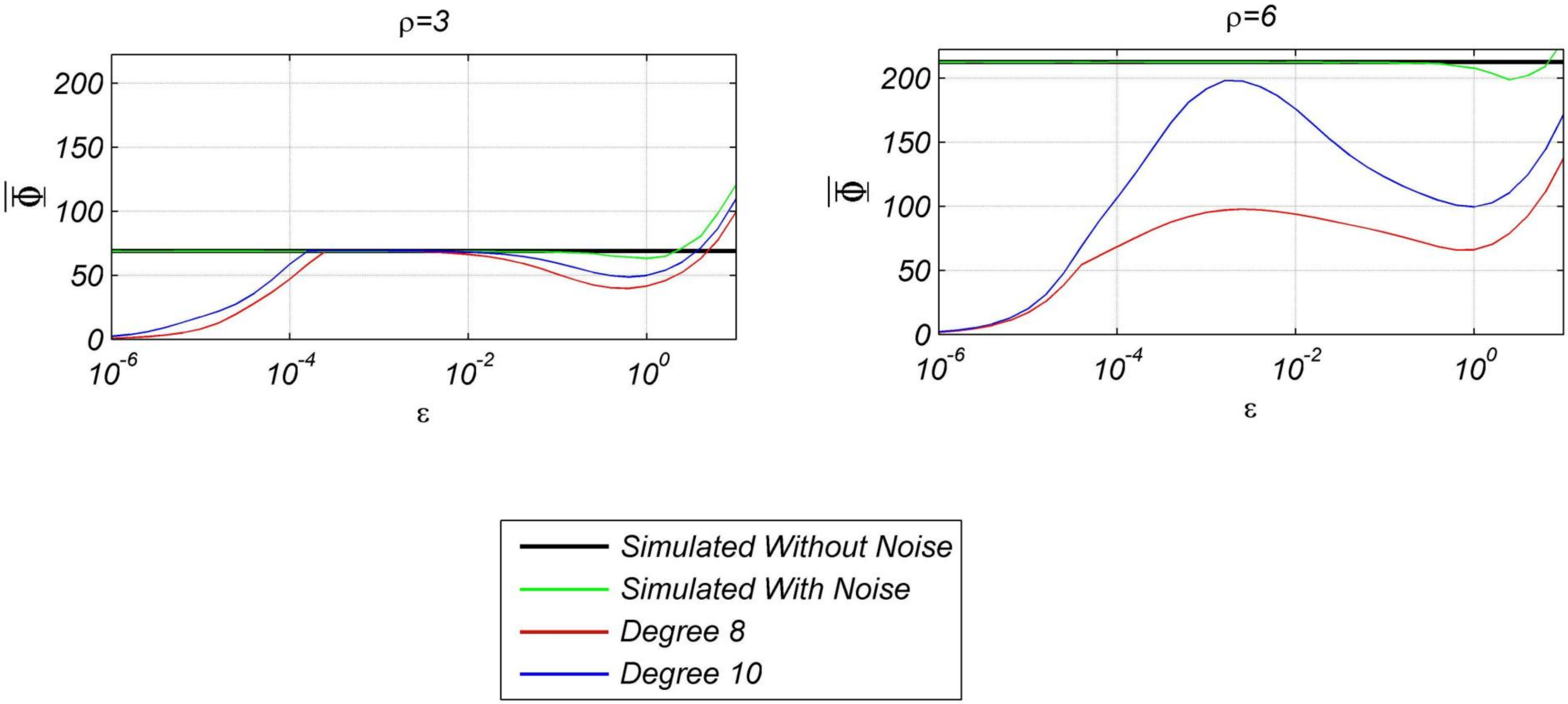
Volume force

$$\left(\frac{\sqrt{2}\pi^2}{4\mathrm{Re}}\sin(\pi y/2);\, 0;\, 0\right),$$

free slip, laminar profile is independent of *Re*.

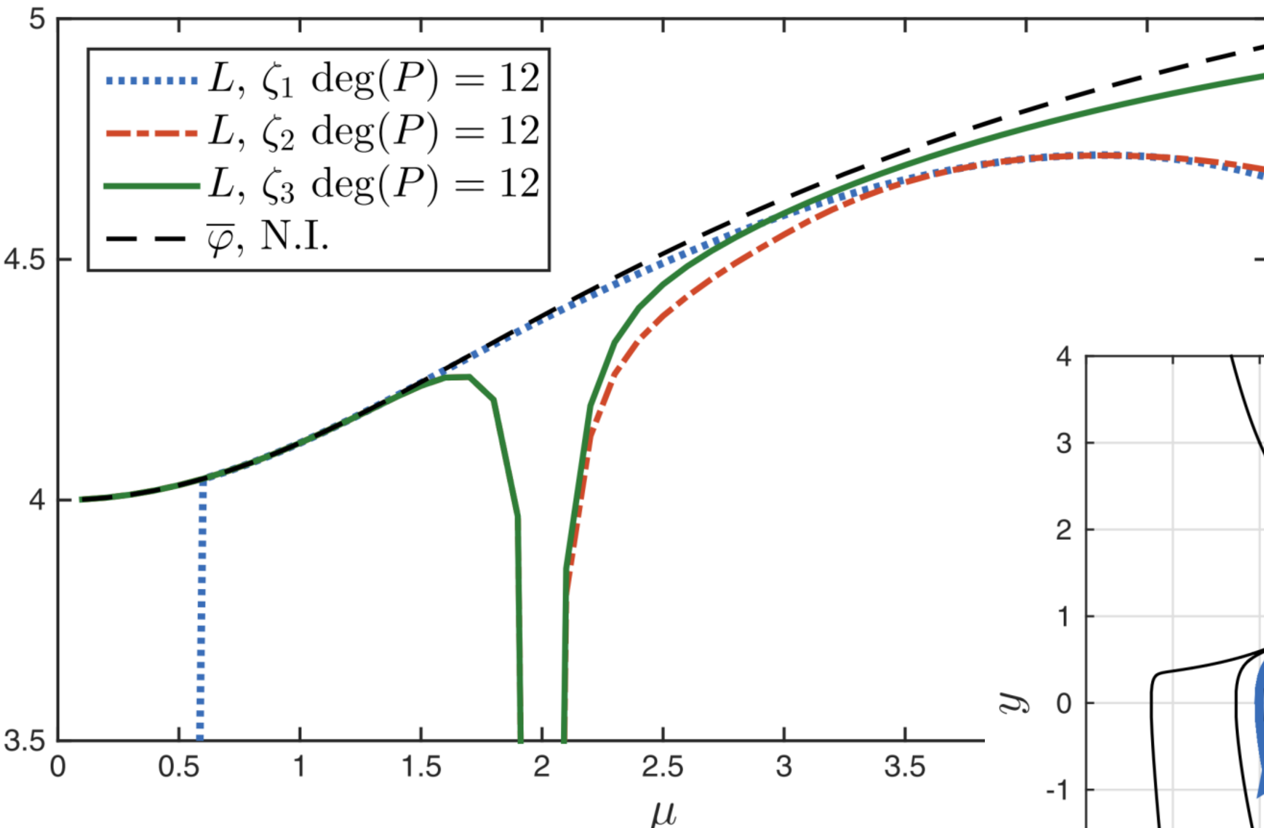
Chernyshenko et al. 2014

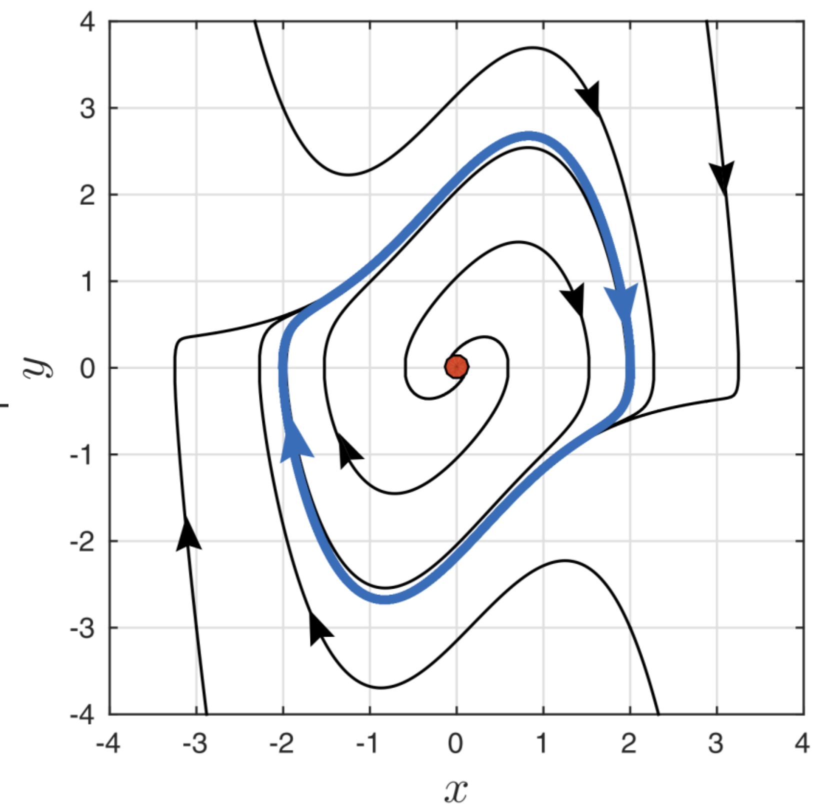
Example: Lower bounds for Lorenz system



M. Baber (2014) Bounds On Time Averaged Properties Of Bounded Chaotic Systems Using Semidefinite Programming. MSc thesis, Department of Aeronautics, Imperial College London.

Example: vanishing noise for Van der Pol oscillator





$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ \mu(1 - x^2)y - x \end{bmatrix}$$

Fantuzzi *et al.* 2016

A problem related to bounding time averages for turbulent flows:

Is such an inviscid flow possible: the channel shape is periodic in horizontal direction, the velocity distribution is also periodic, but the pressure is not, it is growing from cell to cell?