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https://fmfpamu2024.in/

ALIGARH MUSLIM UNIVERSITY

The great problem of turbulence: new developments in the new millennium

(a review of the work of many)

by Sergei Chernyshenko

Imperial College London

The talk

I. The great problem of turbulence

II. SOS: the revolution at the start of the millennium

III. Auxiliary functionals

IV. Tying the three: uncertain system method

V. Examples:

NSE stability and Kuramoto-Sivashinsky bounds

VI. Two challenges

- Speed
- Unstable solutions

VII. Glimpse of the future: optimal unstable orbits

VIII. Conclusion

The great problem of turbulence is to get answer we really need at a small fraction of the total solution cost

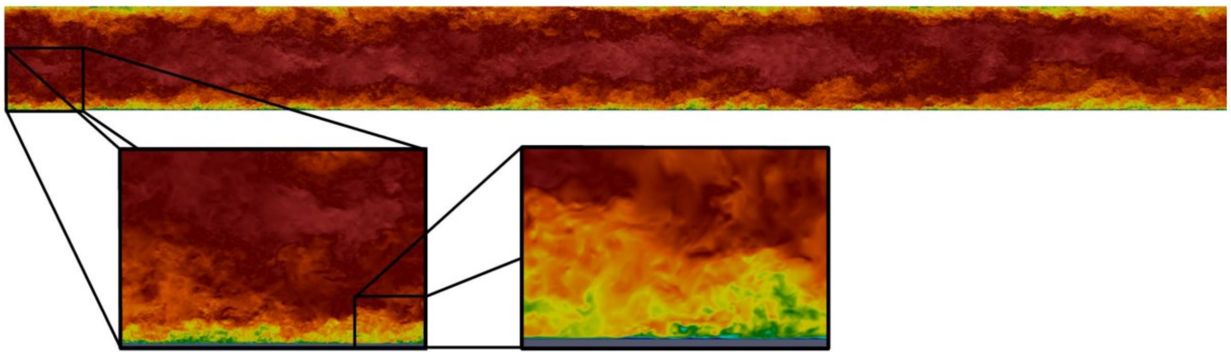


Figure 7: Instantaneous streamwise velocity component over the entire streamwise length of the simulated channel. Zooming in on the flow highlights the multi-scale nature of the turbulence.

Channel flow calculation that required 260,000,000 CPU hours and resolved all the minute detail.

[Lee, Malaya, & Moser 2013]

Or: to buy the bolt without the factory

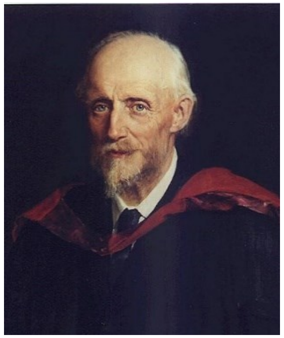
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\overline{\Phi[\mathbf{u}]} = ?$$

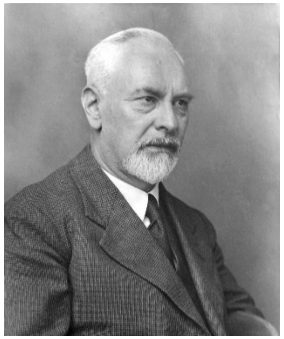
Acknowledgements: Lee, MK, Malaya, N & Moser, R. (2013). Petascale direct numerical simulation of turbulent channel flow on up to 786K cores. International Conference for High Performance Computing, Networking, Storage and Analysis, SC. 10.1145/2503210.2503298.
https://upload.wikimedia.org/wikipedia/commons/8/88/Non-preload_bolt_assembly_%28EN_15048%29.png
Photo by Batuhan Kocabaş: https://www.pexels.com/photo/smoke-in-a-factory-in-a-city-19750125/

A bit of history. The most known road to solving the problem of turbulence: RANS and `closures’



$$\frac{\partial \langle u_i \rangle \langle u_j \rangle}{\partial x_j} = -\frac{\partial \langle p \rangle}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 \langle u_i \rangle}{\partial x_j \partial x_j} - \frac{\partial \langle u_i' u_j' \rangle}{\partial x_j}$$

Osborne Reynolds, 1895



$$u'v' = -l^2 \left| \frac{dU}{dy} \right| \frac{dU}{dy}$$

Ludwig Prandtl 1925

$$\frac{\partial \langle u_i' u_j' \rangle}{\partial t} = -\frac{\partial \langle u_i' u_j' u_k' \rangle}{\partial x_k} + \dots$$

Chou Pei-Yuan 1945

RANS is an industry workhorse now. But it is not rigorous.

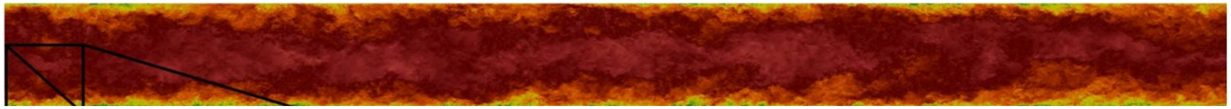
Picture attributions: Reynolds – Public domain, Prandtl - DLR, CC-BY 3.0 , Peiyuan - courtesy of Lai Times

A bit of history. Bounds for average energy dissipation rate have indeed been derived:

$$\overline{\Phi} = ? \quad \rightarrow \quad L \leq \overline{\Phi} \leq B, \quad L, B = ?$$

Howard (1972), Busse (1978)
…
Doering & Constantin (1994), Constantin & Doering (1995), Background flow method – the current workhorse of bounding time averages
…
…

For channel flow:



$$\frac{12}{Re} \leq C_f \leq \frac{27}{32\sqrt{2}} \left(1 + \frac{4\sqrt{2}}{Re}\right)^2$$

Re	Lower bound x 10^3	Experiment x 10^3	Upper bound x 10^3
2 x 10^3	6	11.1	600
10^4	1.2	7.43	597
10^5	0.12	4.18	597

But the existing bounds are not good enough. Ability to trade off the error for the cost is needed

$$\overline{\Phi} = ? \quad \rightarrow \quad L \le \overline{\Phi} \le B, \quad L, B = ?$$

Buildings commonly use a factor of safety of 2.0. Pressure vessels use 3.5 to 4.0, automobiles use 3.0, and aircraft and spacecraft use 1.2 to 3.0.

From Wikipedia

For currently available bounds for a channel flow

Re	Experiment / Lower bound	Upper bound / Experiment
2×10^3	1.9	54
10^4	6.2	82
10^5	35	140

Start of millennium: a breakthrough in semi-algebraic geometry - SOS

$$P_2 = ax^2 + 2bxy + cy^2 \geq 0 \ ?$$

$$P_2 = q_{ij}x_ix_j = \mathbf{x}^T Q \mathbf{x} = \sum_i \lambda_i \tilde{x}_i^2 = \sum_i \left(\sqrt{\lambda_i} A_{ij} x_j\right)^2$$

$$P_2 \geq 0 \ \Leftrightarrow \ Q \succeq 0 \ \Leftrightarrow \ P_2 = \text{SOS}$$

… but NP-hard for higher degrees ☹

Nesterov 2000, Parrilo 2000, Lasserre 2001

Start of millennium: a breakthrough in semi-algebraic geometry - SOS

$$P_2 = ax^2 + 2bxy + cy^2 \geq 0 \ ?$$

$$P_2 = q_{ij}x_ix_j = \mathbf{x}^T Q \mathbf{x} = \sum_i \lambda_i \tilde{x}_i^2 = \sum_i \left(\sqrt{\lambda_i} A_{ij} x_j \right)^2$$

$$P_2 \geq 0 \ \Leftrightarrow \ Q \succeq 0 \ \Leftrightarrow \ P_2 = \mathrm{SOS}$$

$$P_3 = ax^3 + bx^2y + cxy^2 + dy^3 \geq 0? \quad x^2 = u, \ y^2 = w$$

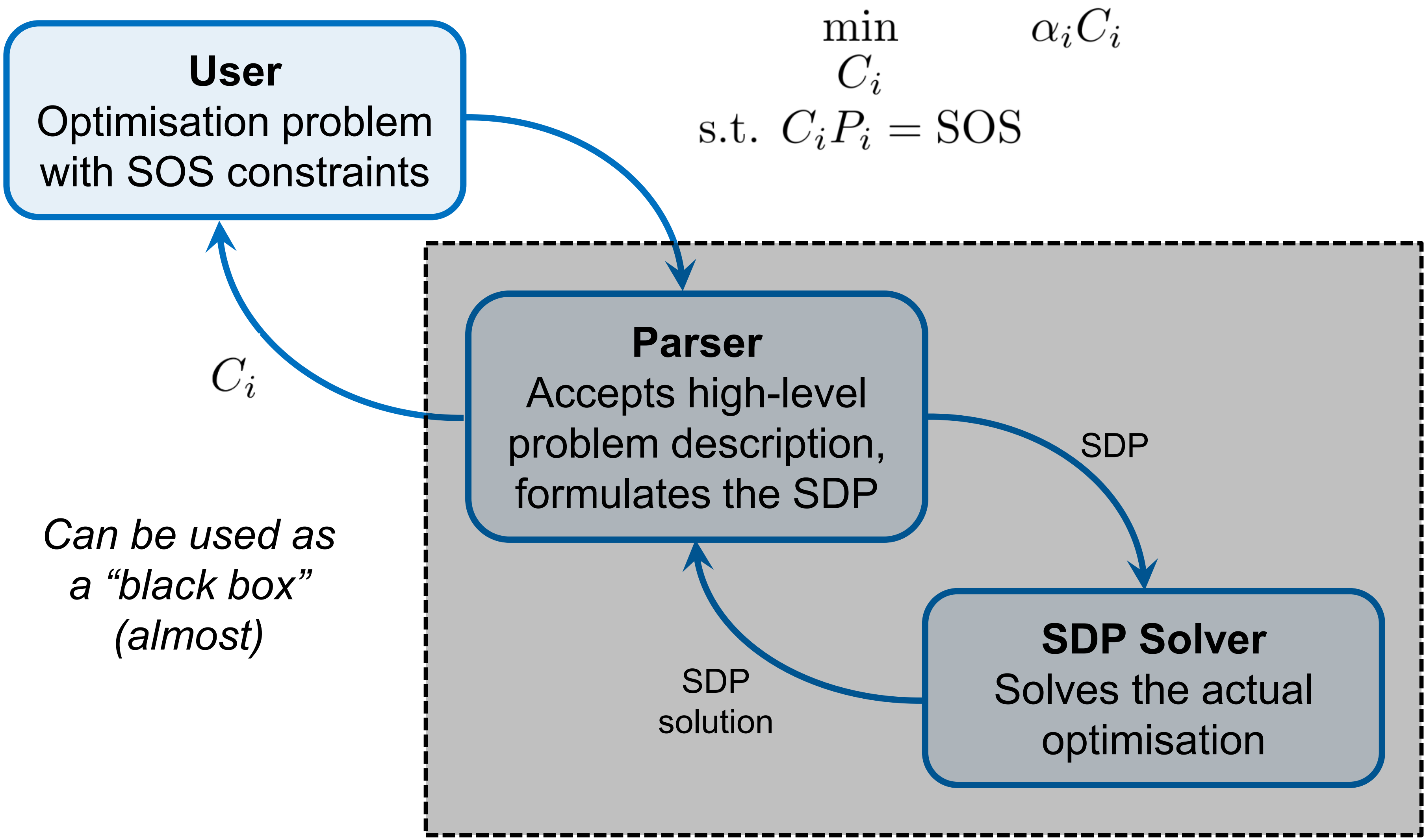
$$P_3 = axu + buy + cxw + dyw = \tilde{P}_2(x, y, u, w) \geq 0 \ ?$$

$$P = \mathrm{SOS} \Leftrightarrow \ Q \succeq 0 \ \Rightarrow \quad \text{Semi-Definite Programming (SDP)}$$

$$\boxed{\text{If } P \geq 0 \text{ then } \underline{\text{very often}} \ P = \mathrm{SOS}}$$

Nesterov 2000, Parrilo 2000, Lasserre 2001

The good news: available software makes it simple



Can be used as a "black box" (almost)

Acknowledgements: the slide courtesy of Giovanni Fantuzzi

Common software toolboxes

Parsers

- YALMIP (https://yalmip.github.io/)
- SOSTOOLS (http://www.cds.caltech.edu/sostools/)
- GloptiPoly (http://homepages.laas.fr/henrion/software/gloptipoly3/)

Solvers

- SDPT3 (free, https://github.com/sqlp/sdpt3
- Mosek (free for academia, https://www.mosek.com/)
- SeDuMi (free, https://github.com/sqlp/sedumi)
- CDCS (free, https://github.com/oxfordcontrol/CDCS)
- SCS (free, https://github.com/cvxgrp/scs)

… and many more!

→ Since 2000 SOS technique was spreading like fire

Acknowledgements: the slide courtesy of Giovanni Fantuzzi

Jump ahead of the rest of the talk - a seed of an idea:

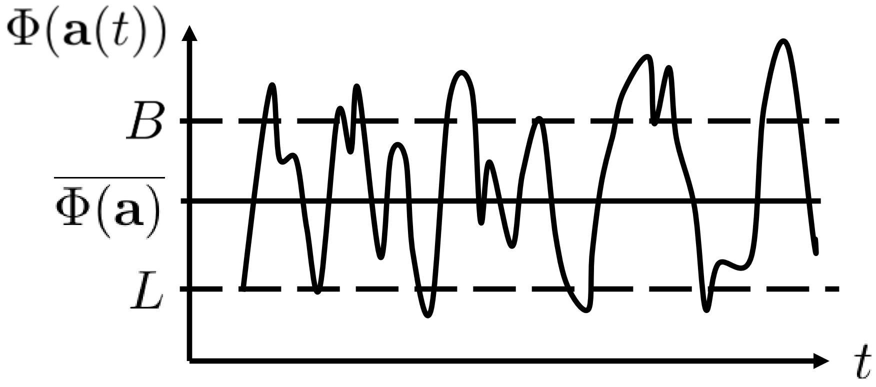
Can we reduce the problem of turbulence to the question of something being always positive?

Can we then reduce this something to a polynomial?

And can we then check the positiveness of this polynomial using the start-of-the millennium breakthrough in SOS?

Auxiliary function method

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \overline{\Phi(\mathbf{a}(t))} = ?$$



Chernyshenko et al. 2014

Auxiliary function method

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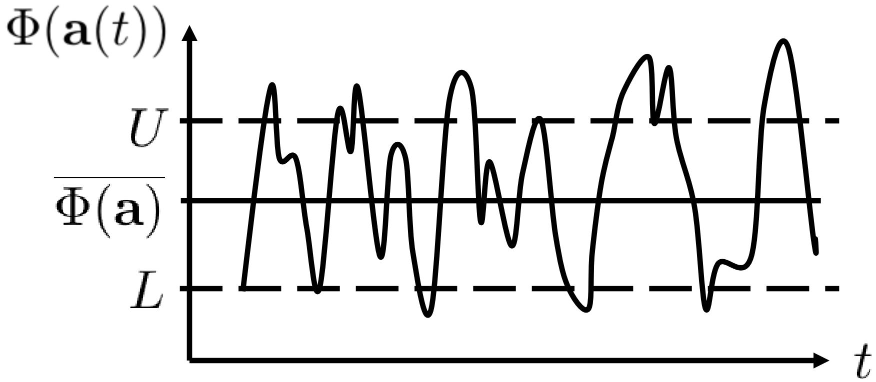

$$\overline{D(\mathbf{a}(t))} = 0, \ \Phi(\mathbf{a}) + D(\mathbf{a}) \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B$$

$$V < \infty \quad \Rightarrow \quad \overline{\frac{dV}{dt}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{dV}{dt} dt = \lim_{T\to\infty} \frac{V|_{t=T} - V|_{t=0}}{T} = 0$$

$$\Rightarrow \quad \overline{\Phi + \frac{dV}{dt}} = \overline{\Phi}$$

$$\Phi(\mathbf{a}) + \frac{dV}{dt} \leq B \qquad \Rightarrow \quad \overline{\Phi} \leq B$$

Chernyshenko et al. 2014

Auxiliary function method

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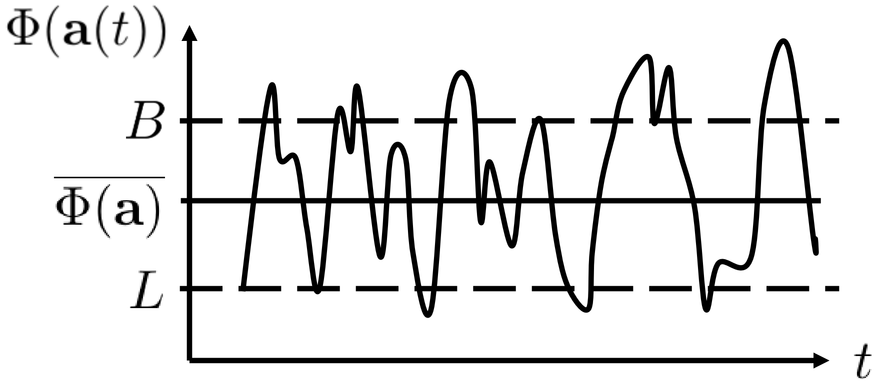

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$$\Phi(\mathbf{a}) + \frac{dV}{dt} \leq B \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B$$

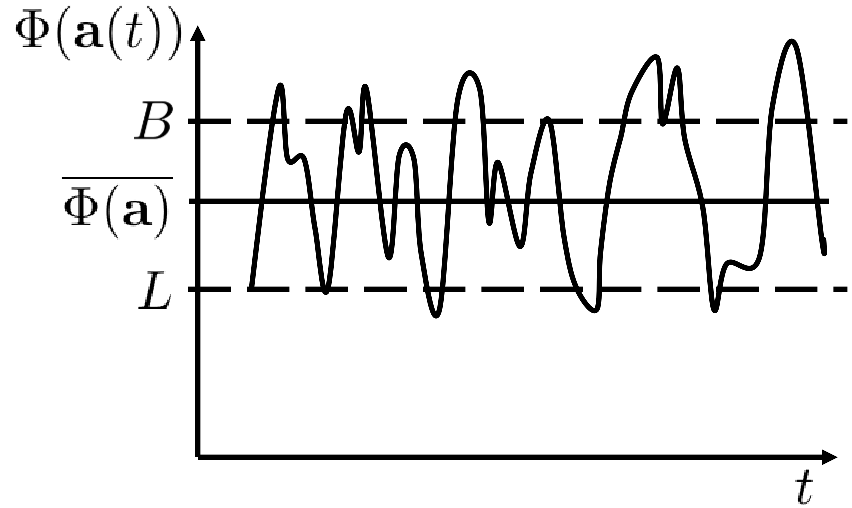
$$V(\mathbf{a}(t)) \Rightarrow \frac{dV}{dt} = \mathbf{f} \cdot \nabla V \quad \Rightarrow \quad \boxed{B - \Phi(\mathbf{a}) - \mathbf{f} \cdot \nabla V \geq 0 \quad \forall \mathbf{a} \quad \Rightarrow \quad \overline{\Phi} \leq B}$$

Chernyshenko et al. 2014

Chernyshenko 2022

Auxiliary function(al)s provide arbitrarily sharp bounds on time averages

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \overline{\Phi(\mathbf{a}(t))} = ?$$



$$L = \max_{V(\mathbf{a}),C} C$$

$$\text{s.t. } \mathbf{f} \cdot \nabla V + \Phi(\mathbf{a}) - C \geq 0$$

Tobasco *et al.* 2018, Rosa & Temam 2020

The largest lower bound problem
is dual
to the smallest time average problem.

$$L = \min_{\mathbf{a}_0} \overline{\Phi}(\mathbf{a}(t))$$

$$\text{s.t.} \quad \frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \mathbf{a}(0) = \mathbf{a}_0$$

Lasserre 2001, Korda et al. 2021

A promise of a trade-off between accuracy and cost

Tying the three together: the plan

- Use auxiliary functions to bound time averages of turbulent flows
- Reduce NSE to ODE with polynomial RHS by Galerkin expansion
- Replace ‘positive-semidefinite’ with ‘sum of squares’

$$L = \max_{V(\mathbf{a}),C} C \quad\quad \Longrightarrow \quad\quad L = \max_{V(\mathbf{a}),C} C$$

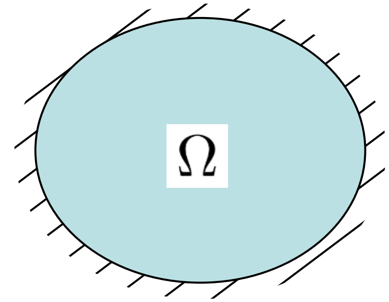
$$\text{s.t. } \mathbf{f} \cdot \nabla V + \Phi(\mathbf{a}) - C \geq 0 \quad\quad\quad \text{s.t. } \mathbf{f} \cdot \nabla V + \Phi(\mathbf{a}) - C \in \Sigma$$

- Make it rigorous for PDE (by the uncertain system method)

Polynomial systems are generated from Navier-Stokes equations by truncated Galerkin expansion

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u} + \mathbf{b},$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}|_{\partial\Omega} = 0$$



$$\text{Basis: } \{\mathbf{u}_i(\mathbf{x})\}, i = 1, \ldots \infty, \ \nabla\cdot\mathbf{u}_i = 0, \ \ \mathbf{u}_i|_{\partial\Omega} = 0$$

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{k} a_j(t)\mathbf{u}_j(\mathbf{x})$$

$$\frac{da_i}{dt} = f_i(\mathbf{a}) = N_{ijk}a_j a_k + L_{ij}a_j + b_i$$

Uncertain system method rigorously reduces Navier-Stokes to a finite-dimensional, but uncertain, system

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u} + \mathbf{b},$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}|_{\partial\Omega} = 0$$

$$\mathbf{u}(\mathbf{x},t) = \sum_{j=1}^{k} a_j(t)\mathbf{u}_j(\mathbf{x}) + \mathbf{u}_s(\mathbf{x},t)$$

$$q^2 = ||\mathbf{u}_s||^2/2$$

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}) + \boldsymbol{\Theta}$$

$$\frac{dq^2}{dt} = -\mathbf{a}\cdot\boldsymbol{\Theta} + \Gamma + \chi$$

$$\|\boldsymbol{\Theta}\|^2 \le p(\mathbf{a}, q^2)$$

$$\Gamma \le \kappa q^2$$

$$\chi^2 \le r(\mathbf{a}, q^2)$$

1. Choosing $\mathbf{u}_j$ and k makes $\chi = 0$ and $\kappa < 0$
2. Polynomial p is quadratic
3. $V[\mathbf{u}] = V(\mathbf{a}, q^2)$
4. The optimization constraint is reducible to SOS

Goulart & Chernyshenko 2012

Before looking at examples: Lyapunov functional is (almost) a special case of auxiliary functional

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}), \quad \mathbf{f}(0) = 0$$

Lyapunov function

$$V(0) = 0, \quad V(\mathbf{a}) > 0 \quad \forall \mathbf{a} \neq 0$$

$$\mathbf{f} \cdot \nabla V < 0 \quad \forall \mathbf{a} \neq 0$$

$$B = \min_{V(\mathbf{a}), C} C$$

$$\text{s.t. } V(\mathbf{a}) > 0, \mathbf{f} \cdot \nabla V - C < 0$$

$$B = 0 \Rightarrow \text{stable}$$

Auxiliary function

$$B = \min_{V(\mathbf{a}), C} C$$

$$\text{s.t. } \Phi(\mathbf{a}) + \mathbf{f} \cdot \nabla V - C \leq 0$$

$$\Rightarrow \quad \overline{\Phi} \leq B$$

Theorem: under mild conditions V is bounded from below

Goluskin and Fantuzzi 2019

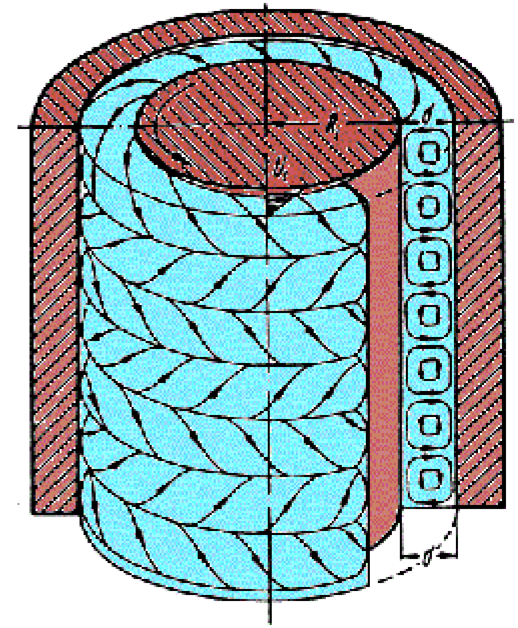
1. Stability of double-periodic rotating Couette flow

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + v = \Omega v + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\Omega u - \frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right),$$

$$\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$



$\mathbf{u} = (u, v, w)$ is the velocity perturbation and $\bar{\mathbf{u}} = (y, 0, 0)$ is the equilibrium flow

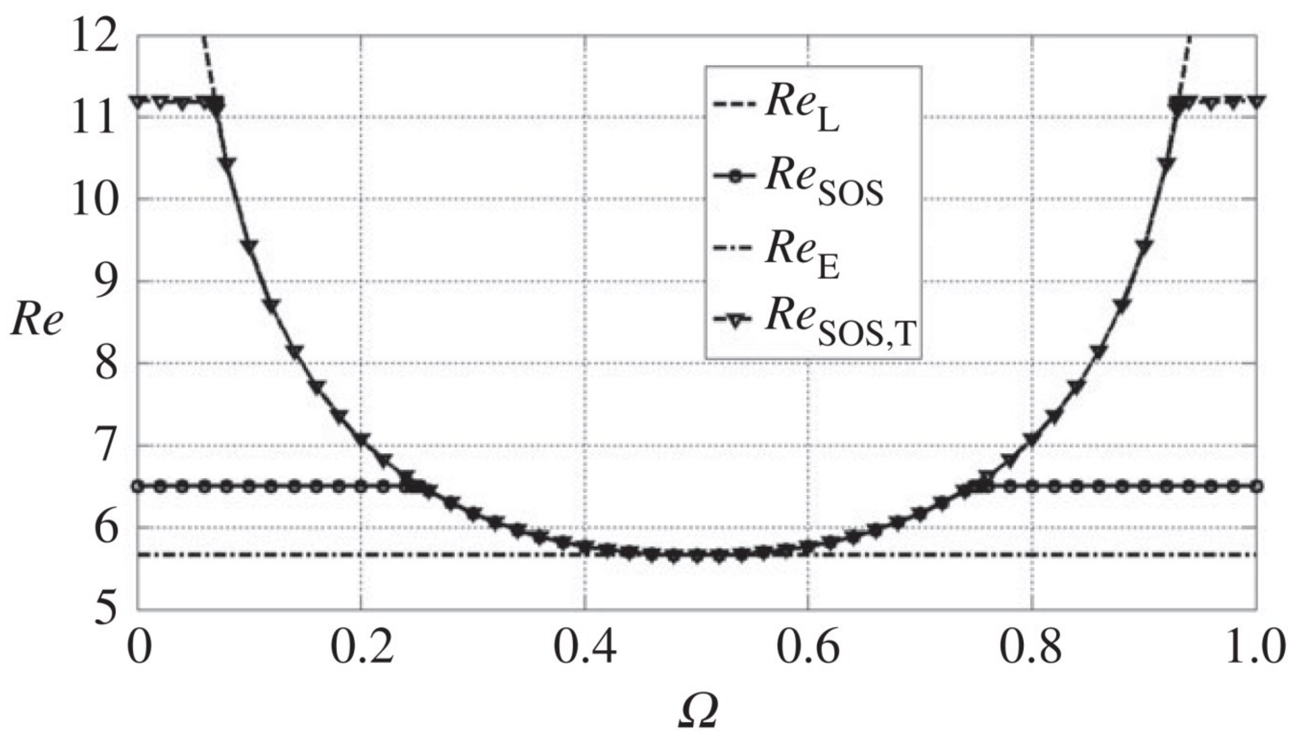
$$\mathbf{u}(y,z) = \mathbf{u}(y+2\pi, z) = \mathbf{u}(y, z+2\pi), \quad p(y,z) = p(y+2\pi, z) = p(y, z+2\pi),$$

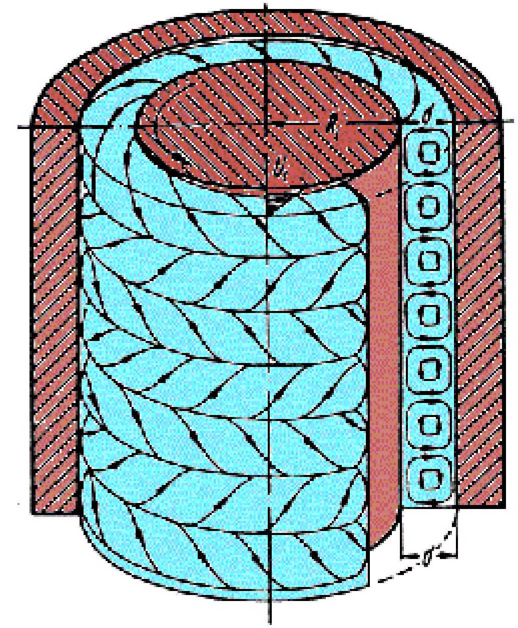
$$u(y,z) = -u(-y,z) = u(y,-z), \quad v(y,z) = -v(-y,z) = v(y,-z)$$

$$w(y,z) = w(-y,z) = -w(y,-z).$$

Huang *et al*. 2015

1. Stability of double-periodic rotating Couette flow





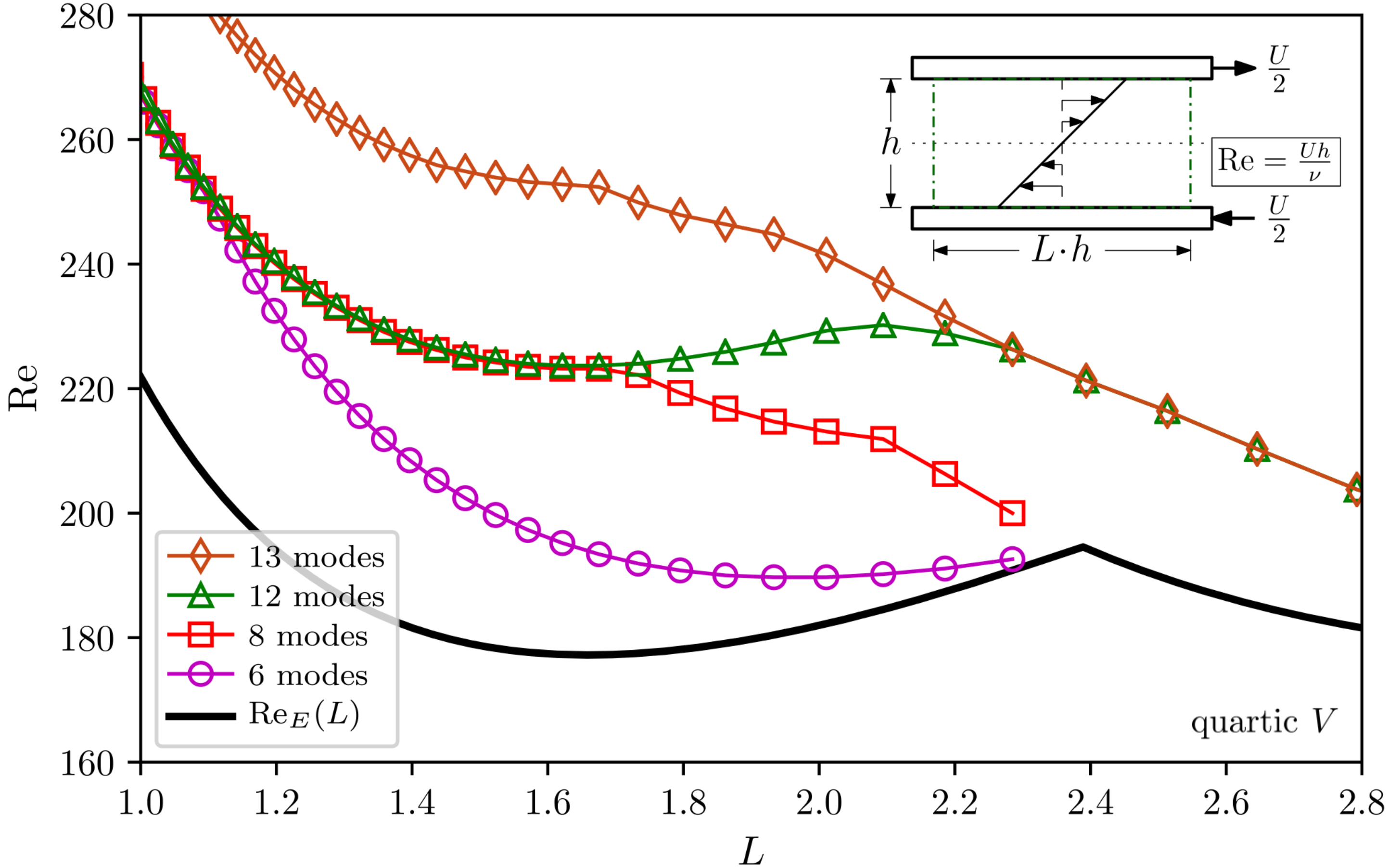
6 modes, 4th-degree V

The Lyapunov functionals obtained depend on Ω. For example, for $\Omega = 0.1$ it is

$$V(\mathbf{a}, q^2) = 7.9294(|\mathbf{a}|^2 + 2q^2)^2 - a_5(1.5894a_1a_2 + 3.1590a_2a_3 - 0.9151a_1a_4 - 0.0949a_3a_4)$$
$$- a_6(29.2233a_2^2 - 0.1480a_1a_3 + 4.8479a_3^2 + 2.6869a_2a_4 + 2.8354a_4^2 + 2.2428a_5^2 + 7.1675q^2)$$
$$+ 23.4772a_1^2 + 29.3767a_2^2 + 28.9780a_3^2 + 20.7968a_4^2 + 20.5949a_5^2 + 23.1982a_6^2 + 23.6155q^2$$
$$- 27.7365a_2a_4 + 0.0557a_1a_3.$$

Huang *et al.* 2015

2. Stability of 2D plane Couette flow



Re_E , Orr 1907

Fuentes *et al.* 2019

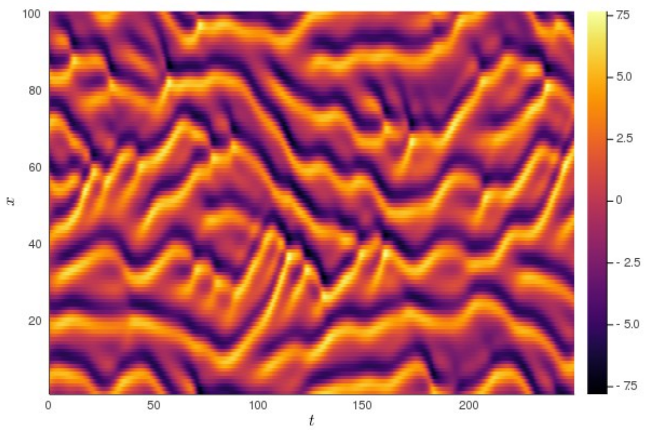
3. Bounds on mean energy in the Kuramoto–Sivashinsky equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^4 u}{\partial x^4} = 0,$$

$$u(x,t) = u(x + 2\pi L, t),$$

$$u(-x,t) = u(x,t)$$

$$\Phi = \mathcal{E} = \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} u^2\, dx$$



Kuramoto–Sivashinsky spatiotemporal evolution.

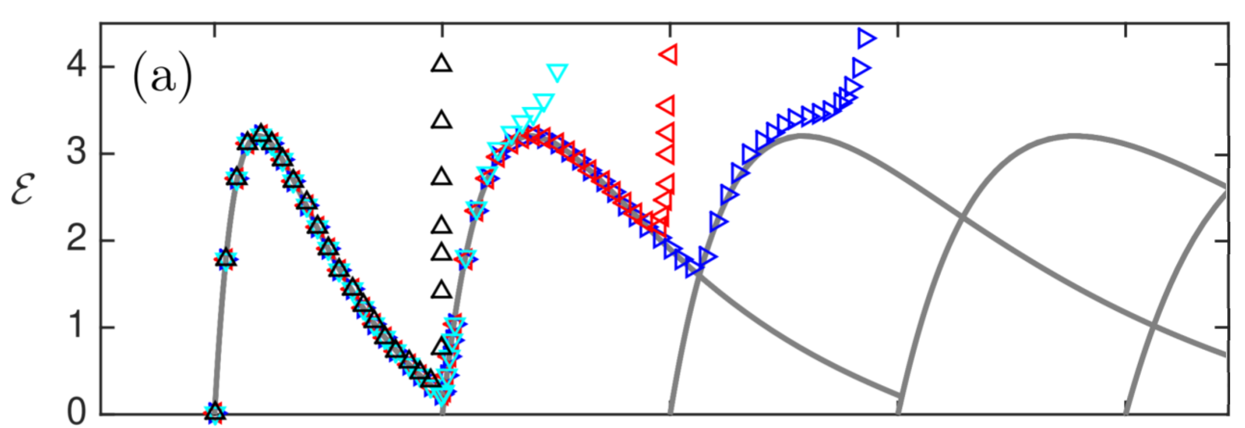
Describes instabilities of laminar flame front and of liquid film on an inclined plane, with trapped-ion instability and more.

Goluskin and Fantuzzi 2019

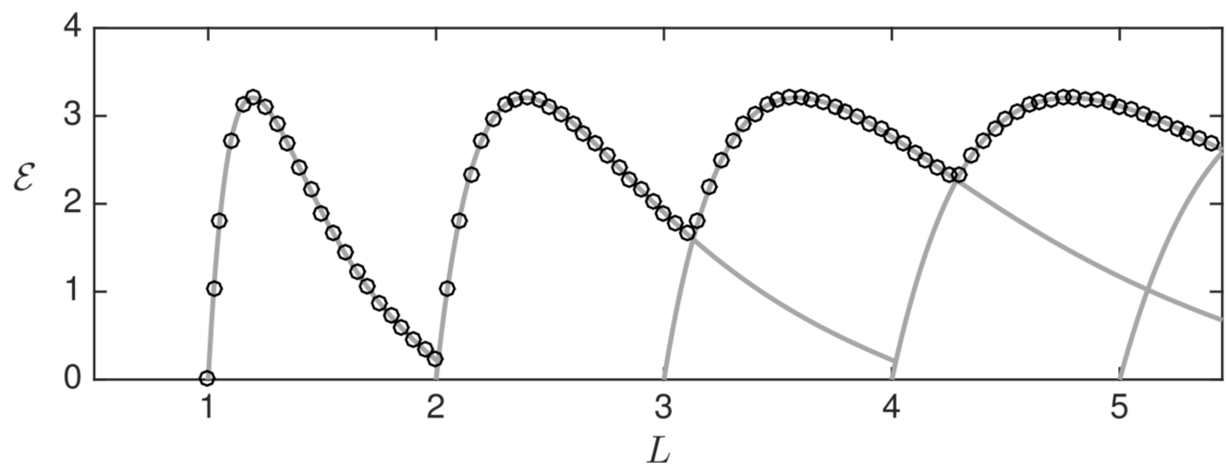
Acknowledgement: Picture by Eviatar Bach, CC0, via Wikimedia Commons, https://commons.wikimedia.org/wiki/File:Kuramoto%E2%80%93Sivashinsky_spatiotemporal_evolution.png

3. Bounds on mean energy in the Kuramoto–Sivashinsky equation

quartic V



number of modes 4 (△), 6 (▽), 8 (◁), and 10 (▷)



number of modes 24

[Goluskin and Fantuzzi 2019]

The first challenge: the problem of cost

The software is free, but there is no free lunch!

$$p(\mathbf{x}) = \mathbf{v}_d^T Q \mathbf{v}_d, \qquad Q \succeq 0$$

With $p(\mathbf{x})$ of degree $2d$ and n variables, in general,

$$Q \text{ is } \binom{n+d}{d} \times \binom{n+d}{d}$$

Example.

$$n = 10,\ d = 6 \implies Q \text{ is } 8008 \times 8008$$
$$n = 10,\ d = 8 \implies Q \text{ is } 43\,758 \times 43\,758$$
$$n = 10,\ d = 10 \implies Q \text{ is } 184\,756 \times 184\,756$$

Computations practical only for moderate n and d

There is a hope in developing the specialist software

DNS vs SOS

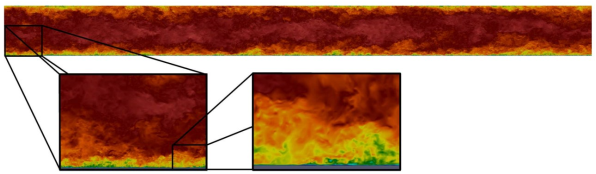
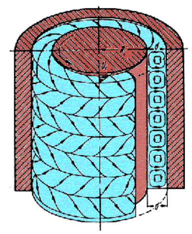


Figure 7: Instantaneous streamwise velocity component over the entire streamwise length of the simulated channel. Zooming in on the flow highlights the multi-scale nature of the turbulence.

Re_τ=5200
242000000000 degrees of freedom

Specialised NSE solver
Parallel code running on 786000 cores
260,000,000 CPU hours

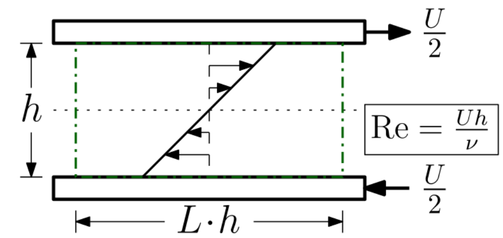
Lee, Malaya, & Moser 2013



Re ~ 100
~ 10 degrees of freedom

General-purpose SOS software
Serial code
Less or ~ 100 CPU hours

Huang *et al*. 2015



Re ~ 280
~ 13 degrees of freedom

General-purpose SOS software
Serial code

Fuentes *et al*. 2019

In 2021: auxiliary-functional formulation of the background flow method solved with about 12000 sparsely coupled variables.

Arslan *et al*. 2021 "Bounds on heat transport for convection driven by internal heating"

The second challenge: the problem of unstable solutions

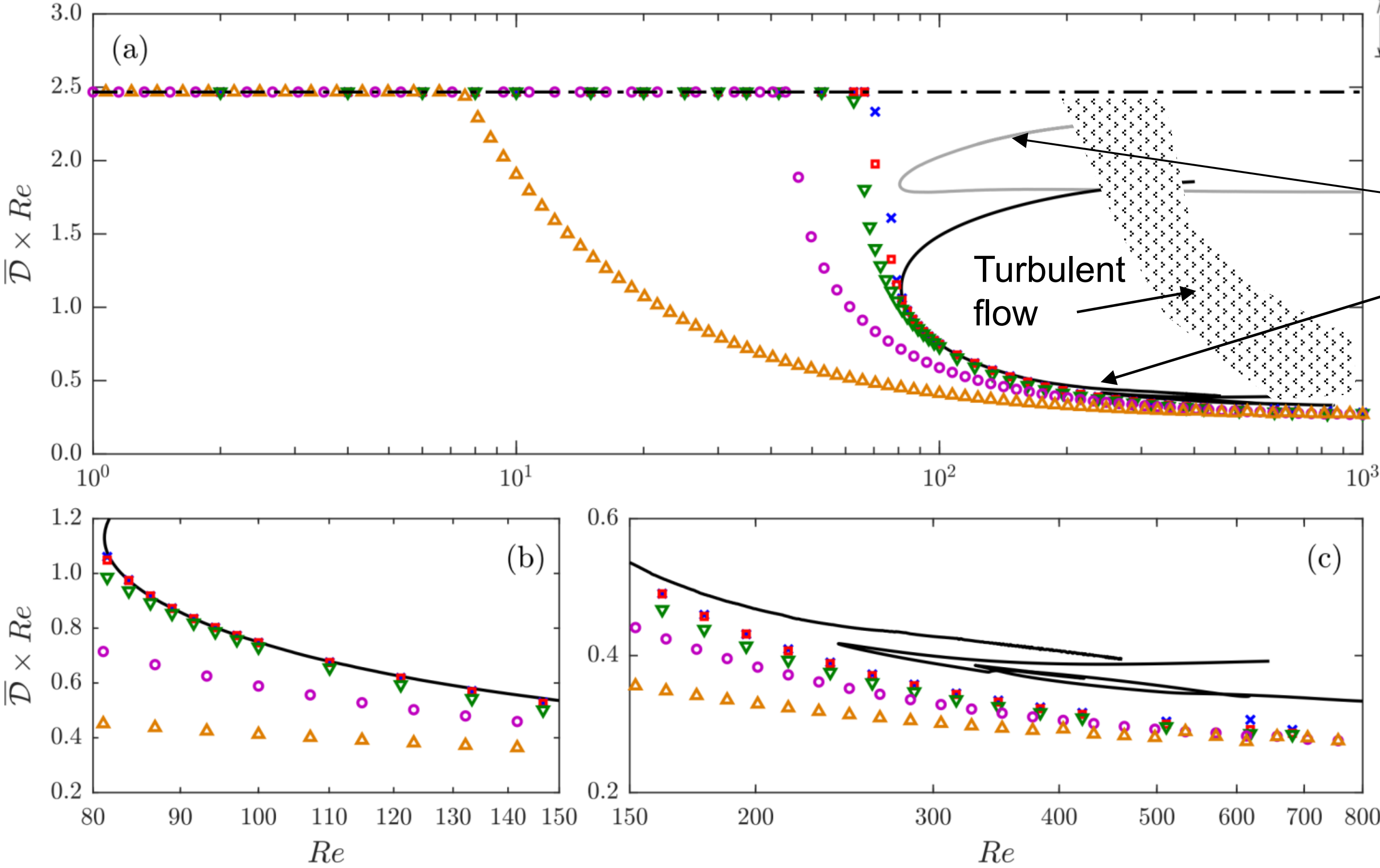
$$\frac{da}{dt} = a - a^3$$

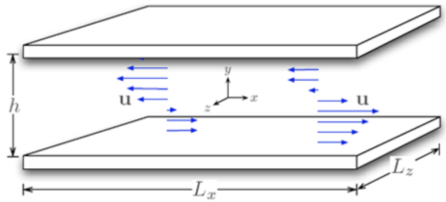
$$a = 0, \quad a = -1, \quad a = +1$$

$$\Phi = a^2, \quad 0 \leq \overline{\Phi} \leq 1$$

$$L \leq \Phi + (a - a^3)\frac{dV}{da} \leq U$$

Bounds for the 9-mode model of shear flow are constrained by the periodic orbits





Periodic orbits, Moehlis et al. 2005

Volume force

$$\left(\frac{\sqrt{2}\pi^2}{4\mathrm{Re}}\sin(\pi y/2);\, 0;\, 0\right),$$

free slip, laminar profile is independent of *Re*.

Onset of a periodic orbit at *Re*=80.54

Auxiliary function degree: 2 (△), 4 (○), 6 (▽), 8 (□) and 10 (×)

Lakshmi *et al.* 2020

Another example of turbulent flow averages sandwiched between travelling waves averages, based on DNS of a pipe flow, is Figs 1 and 2 in Kerswell & Tutty 2007

Towards a solution of the second challenge: accounting for noise in auxiliary function method

$$\frac{d\mathbf{a}}{dt} = \mathbf{f}(\mathbf{a}) + \sqrt{2\epsilon}\boldsymbol{\xi}(t)$$

$$\nabla \rho \mathbf{f} - \epsilon \nabla^2 \rho = 0$$

$$\bar{\Phi} = \langle \Phi \rangle = \int \rho \Phi \, da, \qquad \int \rho \, da = 1$$

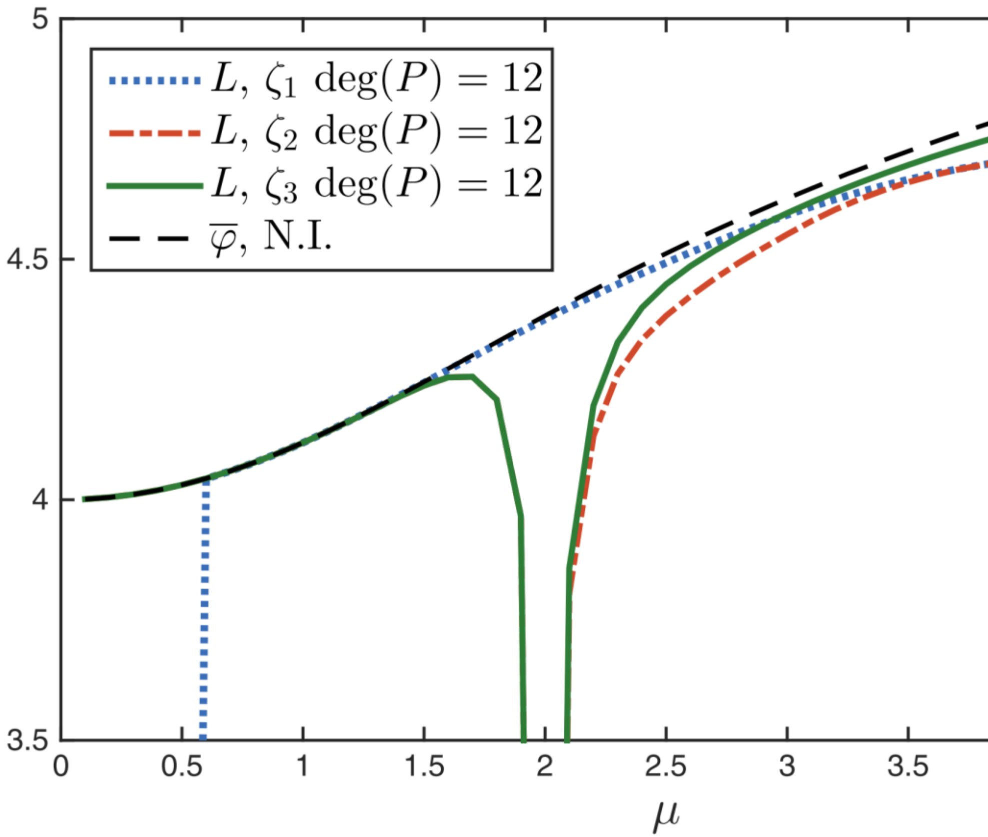
$$\Phi + \mathbf{f} \cdot \nabla V + \epsilon \nabla^2 V \lesseqgtr C \quad \forall \mathbf{a}$$

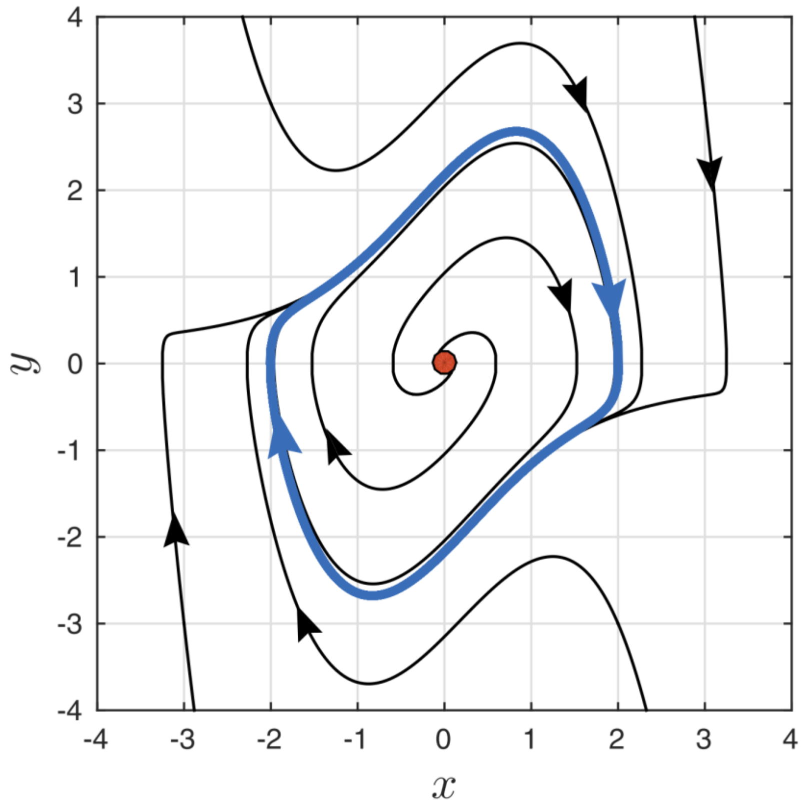
$$\Rightarrow \quad \bar{\Phi} \lesseqgtr C$$

Chernyshenko et al. 2014

See also the related works by Kuntz et al 2016, Korda et al 2021, and references therein.

Example: vanishing noise for Van der Pol oscillator

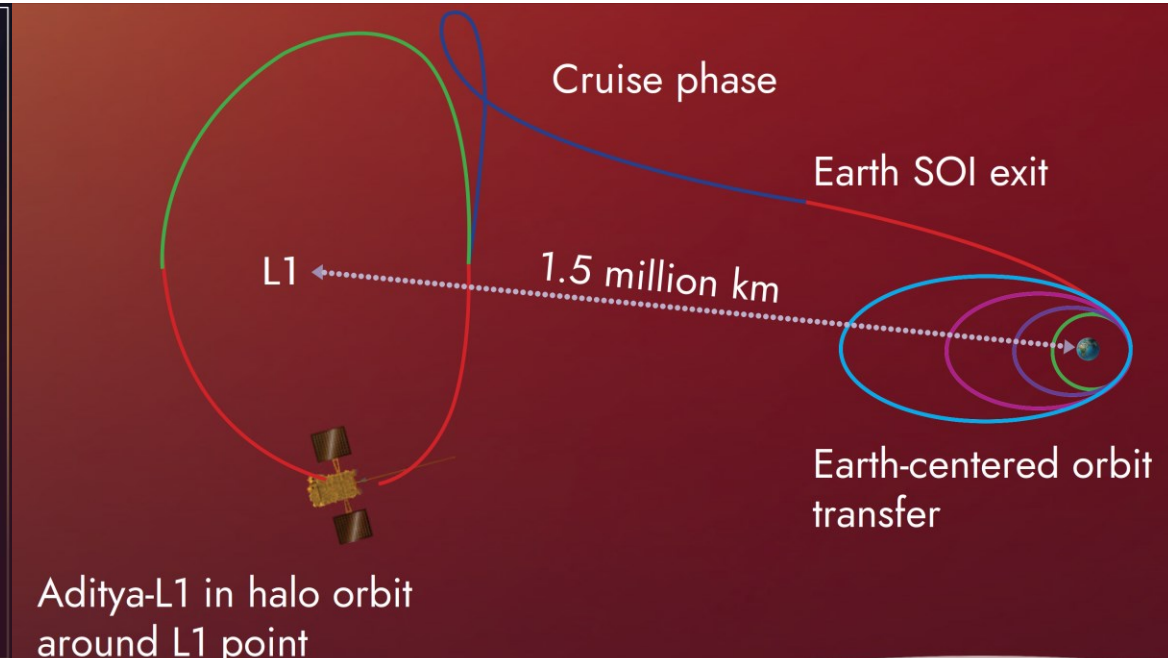




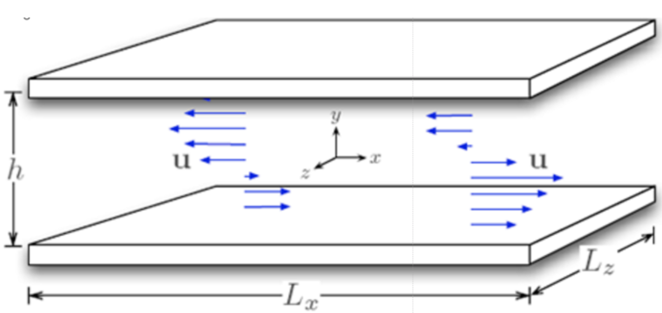
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ \mu(1 - x^2)y - x \end{bmatrix}$$

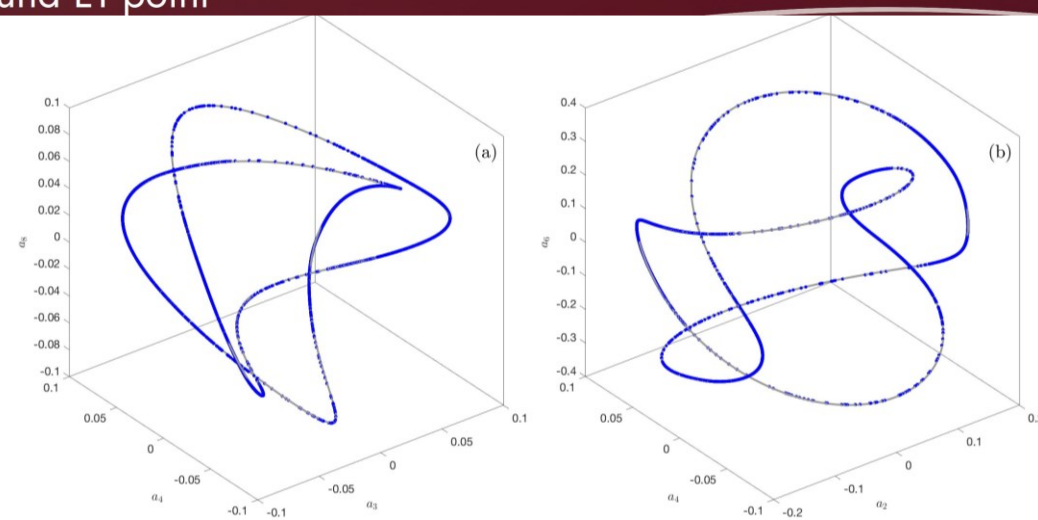
Fantuzzi *et al.* 2016

A glimpse of more: bounds for time averages give a method of finding even unstable optimal periodic orbits



Acknowledgement: Courtesy of DOS/ISRO





Laksmi *et al.* 2021

Summary and Conclusion

- The great problem of turbulence: to buy bolts without factories. We seek bounds for time averages
- The SOS breakthrough: P=SOS is feasible
- Auxiliary function: whose Lie derivative is added to an observable to gain a benefit
- Uncertain system: semi-truncated Galerkin plus the energy of the residual, with rigorous bounds of the right-hand-side
- Uncertain system gives a rigorous polynomial formulation
- Computational costs are high but are reducing
- Unstable solutions need to be eliminated by adding a vanishing noise
- Beyond the present talk: there are numerous other applications

New method came to fluid dynamics. It is different. It is versatile. It is full of intrinsic beauty. And it is promising a lot.

A copy of this presentation with clickable links is available at
https://www.imperial.ac.uk/aeronautics/fluiddynamics/FMFP2024_Chernyshenko_web.pdf
A simple introduction to auxiliary functions is here:
https://www.imperial.ac.uk/aeronautics/fluiddynamics/IntroToAuxiliaryFunctions.pdf