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Active research area

Nonlinear stability analysis of fluid flows using polynomial sum of squares

In the majority of applications steady flows are better than unsteady flows. Steady flows are usually associated with smaller fuel consumption, less fatigue, and less noise. Theoretically, a steady flow always exists as a solution of the governing equations. However, in engineering applications fluid flows are usually unsteady, or even turbulent, because the corresponding steady flow is unstable, that is, if disturbed it will never return to the steady state. In the bulk of the work on flow stability the flow velocity is represented as a sum of the steady solution and a small perturbation, and the nonlinear terms are neglected. The resulting linear problem is much easier to solve. However, steady flows are often stable with respect to infinitesimally small perturbations but unstable with respect to finite perturbations. Moreover, the finite amplitude required to destabilise the flow is often small. Hence, it is the stability with respect to finite disturbances, which represents major practical interest.

$$\frac{da_i}{dt} = f_i(a_1, \ldots, a_n)$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial a_i} f_i < 0, \quad \mathbf{a} \neq 0$$

$$V > 0, \quad \mathbf{a} \neq 0$$

$$V(0) = 0$$

Lyapunov function

For this system $V(a_1, \ldots, a_n)$ is a Lyapunov function. If V with such properties exists, then it will decay monotonicly with time because of the first inequality. The solution will approach zero for any initial condition, that is zero will be a solution stable with respect to finite perturbations.

Stability of a dynamical system with respect to finite perturbations can be established by finding a Lyapunov function. However, there is no general systematic method for constructing Lyapunov functions -- the discovery of such a function is dependent on the ingenuity and creativity of the investigator. Fortunately, in 2000 a breakthrough in control theory, made in a PhD thesis by P.Parrilo, has provided a constructive method for generating Lyapunov functions for systems whose dynamics can be described by polynomial functions. This method is based on sum-of-squares (SOS) optimization, which reduces the problem of finding a Lyapunov function for a polynomial system to one of constructing a polynomial function that satisfies a selection of algebraic conditions. Using a number of key results from semialgebraic geometry, in particular the Positivestellensatz theorem, the resulting problem can be reformulated as an optimization problem in the form of a Semi-Definite Programme (SDP). This is particularly promising since SDP optimization problems are convex and tractable (e.g. solvable in a number of operations that is a polynomial function of the problem size). A variety of well-supported software codes for solving such problems are freely available.

It is well known that the Galerkin approximation of the incompressible Navier-Stokes equations leads to an ODE with quadratic non-linearity. The SOS approach can be applied to such a system since the dynamics are represented by polynomials. SOS was applied to a hydrodynamic-type ODE system of the 9th order. A method of exploiting in the SOS approach the energy-invariance property of the bilinear component of the systems of hydrodynamic type was developed. This allowed the construction of a Lyapunov function for the value of the Reynolds number about seven times larger than the largest value for which nonlinear stability of this system can be proven by the standard energy stability approach. This demonstrates the strong potential of applying SOS in fluid dynamics.

Accurate approximation of a fluid flow requires considering systems with a large number of Galerkin modes. However, application of conventional SOS methods to very large systems is problematic, since the SDP problems to be solved in such cases are prohibitively large. A method of reducing the order of the system in SOS studies, based on representing the higher modes with their energy only, has been developed. The method is guaranteed always to give results at least as good as the standard energy stability analysis. Moreover, it has also been proven mathematically that if the flow remains globally stable for the Reynolds number range beyond that given by the energy stability analysis, then there exists a polynomial Lyapunov function giving a better stability bound than the energy stability theory.

Polynomial Lyapunov function

If both f and V are polynomial then verifying that V is a Lyapunov function reduces to verifying that V and $-\frac{\partial V}{\partial a_i} f_i$ are positive-definite. This is too difficult. However, if a polynomial is a sum of squares of other polynomials then it is positive-definite. Hence, one can search for such V that V and $-\frac{\partial V}{\partial a_i} f_i$ are sum of squares.

Polynomial being a SOS is equivalent to a matrix being positive-definite

Monomials are terms like $a_1, a_2, \ldots, a_1a_2, a_1a_3, a_2a_3$ and so on. Introducing a vector of monomials $\mathbf{M} = (M_1, \ldots, M_N)$, a polynomial can be represented as a sum of monomials $P = Q_{ij}M_iM_J$. If the matrix Q_{ij} is positive-definite, P can be represented as a sum of squares.

More information: P.Goulart and S. Chernyshenko. Global Stability Analysis of Fluid Flows using Sum-of-Squares. 2011. arXiv:1101.1043v2 [math.OC]