Imperial College London

Department of Aeronautics

Aeronautics Mathematics Aptitude Test (AMAT)

Sample Paper
[30 minutes]

There are **30 questions** in the AMAT.
The solutions for each question is provided at the end of the question paper.

Disclaimers:

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2. *The marks allocated is shown for each question. For every incorrect answer, 0.5 marks will be deducted from the total score. No marks are given for each unanswered question. This marking scheme is the same for the actual AMAT.*

The questions are based on the standard A-level Maths syllabus. They are designed to test problem-solving abilities. There are a handful of A-level Further Maths questions. Where possible, adequate information is provided such that all candidates are able to attempt these questions.

Instructions

1. You are not expected to answer all questions, unanswered questions are not penalised. Each incorrect answer deducts half a mark from your result.
2. The test consists of 30 questions — 20 are worth one mark, 10 are worth two marks. You will be able to see how many marks each question is worth during the test.
3. All questions are multiple choice. Each question presents four choices, only one is correct.
4. You are allowed to use A4 papers for working out your answers.
5. This is a closed-book examination, any access to resources and calculators are not allowed.

1. A sequence a_n is defined by [1 mark]

$$a_1 = \frac{1}{2}\left(1-\sqrt{5}\right)$$
$$a_{n+1} = 1 + \frac{1}{a_n}$$

What is a_{100}?

A. 1
B. $\frac{1}{2}\left(1+\sqrt{5}\right)$
C. $\frac{1}{2}\left(1-\sqrt{5}\right)$
D. $\left(\frac{1+\sqrt{5}}{2}\right)^{100}$

2. Two sequences a_n and b_n are defined by, [1 mark]

$$a_1 = 1 \qquad a_{n+1} = 2a_n \tag{1}$$
$$b_1 = \frac{1}{4} \qquad b_{n+1} = 3b_n \tag{2}$$

for $n \geq 1$. Find the smallest n for which $a_n < b_n$.

A. 4
B. 5
C. 6
D. 7

3. The polynomial [1 mark]

$$3x^3 + 31x^2 + 89x + 85$$

can be expressed in the form

$$A(x+B)^3 + C(x+1)^2 \tag{1}$$

Find A, B and C.

A. $A = 3, B = 3, C = 2$
B. $A = 2, B = -3, C = 1$
C. $A = 3, B = 3, C = 4$
D. $A = 2, B = 2, C = 3$

4. Suppose that [1 mark]

$$x = 2\cos(t)\sin(t) \qquad y = 1 + \cos(2t)$$

Which of the following is true?

A. $x^2 + y^2 = 1$

B. $x^2 + y^2 = 2y$

C. $x^2 + 2y^2 - 2y = 0$

D. $x^2 = 1 + y^2$

5. A straight line passes through the points $(0, 0)$ and $(1, 5)$. A second straight line passes through the points $(2, 0)$ and $(-3, c)$. Find the value of c for which the two lines are perpendicular. [1 mark]

A. -1

B. -2

C. 2

D. 1

6. The equation [1 mark]

$$x^2 - 12x + y^2 - 10y + 60 = 0$$

describes a circle with radius r and centre c. Find r and c.

A. $r = 6, c = (1, 2)$

B. $r = 1, c = (5, 6)$

C. $r = 1, c = (6, 5)$

D. $r = 6, c = (2, 1)$

7. For what range of values of c do the curves, [1 mark]

$$y = x^2 + 2x + c \tag{1}$$
$$y = 4x + 3 \tag{2}$$

never intersect?

A. $c < -4$

B. $c > 2$

C. $c < -2$

D. $c > 4$

8. The function [2 marks]

$$\cos(x)\sin\left(x+\frac{\pi}{2}\right)$$

is equal to which of the following?

   A. $\cos^2(x)$
   B. $\cos(x)\sin(x)$
   C. $-\cos(x)\sin(x)$
   D. $\sin^2(x)$

9. Which of the following is the expression [1 mark]

$$\frac{\sqrt{3}}{\sqrt{6}-\sqrt{3}}$$

is equal to?

   A. $1+\sqrt{2}$
   B. $1-\sqrt{3}$
   C. $3+\sqrt{18}$
   D. $1-\sqrt{2}$

10. Consider the polynomial [1 mark]

$$p(x) = x^3 - 11x^2 + 23x + 35$$

Given that $p(-1) = 0$, find the other two solutions to $p(x) = 0$.

   A. $x = 1, 7$
   B. $x = 5, -7$
   C. $x = 5, 7$
   D. $x = -3, 5$

11. Find the derivative of, [2 marks]

$$e^{-e^{-x}}$$

   A. $e^{-(x+e^{-x})}$
   B. $-e^{-x}e^{-e^{-x}}$
   C. $e^{e^{-x}}$
   D. $e^{x+e^{-x}}$

12. Suppose that [2 marks]

$$x = 2\tan(y)$$

Find $\frac{dy}{dx}$.

A. $\frac{2}{1+x^2}$
B. $2\sec^2(x)$
C. $\frac{1}{2}(1+\tan^2(x))$
D. $\frac{2}{4+x^2}$

13. Find the derivative of [2 marks]

$$\frac{1+x}{2+x^2}$$

A. $\frac{2-2x-x^2}{4+4x+x^2}$
B. $\frac{2-x^2}{(2+x^2)^2}$
C. $2\frac{(1-x-x^2)}{(2+x^2)^2}$
D. $\frac{2-2x-x^2}{4+4x^2+x^4}$

14. Find the derivative of [2 marks]

$$\frac{x}{\sqrt{1-x^2}}$$

A. $\frac{-x}{(1-x^2)^{\frac{3}{2}}}$
B. $\frac{1}{\sqrt{1-x^2}}$
C. $\frac{1-2x^2}{(1-x^2)^{\frac{3}{2}}}$
D. $\frac{1}{(1-x^2)^{\frac{3}{2}}}$

15. Find the derivative of [2 marks]

$$\ln(\ln(\ln x))$$

A. $\frac{1}{\ln(x^x)\ln(\ln x)}$
B. $\frac{1}{(\ln x)\ln(\ln x)}$
C. $\frac{1}{x+\ln x+\ln(\ln(x))}$
D. $\frac{1}{\ln(x\ln x)}$

16. A sequence a_n is defined by, [2 marks]

$$a_1 = \frac{1}{2} \qquad a_{n+1} = \frac{1}{2}a_n$$

What is

$$\sum_{n=1}^{19} a_n$$

A. $2(1-2^{-20})$
B. $1-2^{-20}$
C. $2(1-2^{-19})$
D. $1-2^{-19}$

17. Suppose that $x = \cos(2t)$ and $y = \frac{1}{2}\sin(2t)$. Which of the following is true? [1 mark]

A. $\frac{1}{2}x^2 + y^2 = 1$
B. $x^2 + 4y^2 = 1$
C. $x^2 + 2y^2 = 1$
D. $x^2 + y^2 = 1$

18. The circle [1 mark]

$$x^2 + y^2 = 1 \tag{1}$$

and the line

$$y = x + 1 \tag{2}$$

intersect at two points. Find the coordinates (x, y) of these points.

A. $\left(\pm 2^{-\frac{1}{2}}, \pm 2^{-\frac{1}{2}}\right)$
B. $(0,1),(1,0)$
C. $(0,1),(-1,0)$
D. $(0,-1),(1,0)$

19. For what value of c do the curves [1 mark]

$$y = x^2 + 2x + c \tag{1}$$

and

$$y = 4(x+1) \tag{2}$$

intersect at exactly one point?

A. $c = 1$
B. $c = 2$
C. $c = 4$
D. $c = 5$

20. The expression [1 mark]

$$\left(a^{\frac{1}{3}}b^{\frac{1}{2}}\right)^6 b^{-1} + a^2 b$$

is equal to?

A. $a^2b^2(1+b)$
B. $a^2b + ab^2$
C. $a^{\frac{19}{3}}b^{\frac{11}{2}} + a^2b$
D. $a^2b(1+b)$

21. The expression $\left(b^{\frac{1}{2}}a^3\right)^4 a^{-8} + (ab)^2$ is equal to: [1 mark]

A. $a^2b^2(1+a^2)$
B. $b^2a^{-1} + a^2b^2$
C. $2b^2a^2$
D. $ab(1+a^2)$

22. Suppose that $x = \sin(y)$. Find $\frac{dy}{dx}$. [2 marks]

A. $\frac{1}{1+x^2}$
B. $\frac{1}{\sqrt{1+x^2}}$
C. $\frac{1}{\cos(\sin(x))}$
D. $\frac{1}{\sqrt{1-x^2}}$

23. Find the derivative of [2 marks]

$$\frac{1}{\ln(\ln(x))}$$

.

A. $\frac{-1}{x\ln(x\ln x)}$
B. $\frac{1}{x\ln(x\ln x)}$
C. $\frac{-1}{\ln(x^x)\ln(\ln x)^2}$
D. $\frac{1}{\ln(x^x)\ln x}$

24. A straight line passes through the points $(0, 0)$ and $(1, 5)$. A second straight line passes through the points $(2, 0)$ and $(4, c)$. Find the value of c for which the two lines are parallel. [1 mark]

    A. 5
    B. 10
    C. -5
    D. 20

25. The circle [1 mark]

$$x^2 + y^2 = 1 \tag{1}$$

and the line

$$y = x + c \tag{2}$$

intersect at exactly one point. What are the possible values of c?

    A. $\sqrt{2}, -\sqrt{2}$
    B. $-1, 1$
    C. $1, \sqrt{2}$
    D. $-2, 2$

26. The polynomial [2 marks]

$$2x^4 + 16x^3 + 51x^2 + 70x + 35$$

can be expressed in the form

$$A(x + B)^4 + C(x + 1)^2$$

Find A, B and C.

    A. $A = 3, B = 2, C = 1$
    B. $A = 2, B = -3, C = 1$
    C. $A = 3, B = 1, C = -2$
    D. $A = 2, B = 2, C = 3$

27. Two sequences a_n and b_n are defined by: [1 mark]

$$a_1 = 1 \qquad a_{n+1} = a_n + 3 \tag{1}$$
$$b_1 = 20 \qquad b_{n+1} = b_n + 1 \tag{2}$$

for $n \geq 1$. Find the smallest n for which $a_n > b_n$.

    A. 7

B. 9
C. 11
D. 13

28. Consider the polynomial: [1 mark]

$$p(x) = x^3 + x^2 - 14x - 24$$

Given that $p(-2) = 0$, find the other two solutions to $p(x) = 0$.

A. $x = -3, 2$
B. $x = -3, -4$
C. $x = -2, 1$
D. $x = -3, 4$

29. The equation [1 mark]

$$x^2 - 6x + y^2 - 4y - 3 = 0$$

describes a circle with radius r and centre c. Find r and c.

A. $r = 3, c = (1, 3)$
B. $r = 3, c = (3, 1)$
C. $r = 4, c = (3, 2)$
D. $r = 4, c = (2, 3)$

30. A sequence a_n is defined by, [1 mark]

$$a_1 = 1 \qquad a_{n+1} = \frac{1}{3}a_n$$

Evaluate

$$\sum_{n=1}^{\infty} a_n$$

A. $\frac{2}{3}$
B. 1
C. $\frac{3}{2}$
D. 2

Question	Points	Score
1	1	
2	1	
3	1	
4	1	
5	1	
6	1	
7	1	
8	2	
9	1	
10	1	
11	2	

Question	Points	Score
12	2	
13	2	
14	2	
15	2	
16	2	
17	1	
18	1	
19	1	
20	1	
21	1	
22	2	

Question	Points	Score
23	2	
24	1	
25	1	
26	2	
27	1	
28	1	
29	1	
30	1	
Total:	40	

End of AMAT

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Sample Paper Solutions
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1. A sequence a_n is defined by [1 mark]

$$a_1 = \frac{1}{2}\left(1-\sqrt{5}\right)$$
$$a_{n+1} = 1 + \frac{1}{a_n}$$

What is a_{100}?

A. 1
B. $\frac{1}{2}\left(1+\sqrt{5}\right)$
C. $\frac{1}{2}\left(1-\sqrt{5}\right)$
D. $\left(\frac{1+\sqrt{5}}{2}\right)^{100}$

Solution:

$$\begin{aligned} a_2 &= 1 + \frac{2}{1-\sqrt{5}} \\ &= \frac{3-\sqrt{5}}{1-\sqrt{5}} \\ &= \frac{(3-\sqrt{5})(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} \\ &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

Since, $a_2 = a_1$, it can be concluded that,

$$a_{100} = a_1$$

2. Two sequences a_n and b_n are defined by, [1 mark]

$$a_1 = 1 \qquad a_{n+1} = 2a_n \tag{1}$$
$$b_1 = \frac{1}{4} \qquad b_{n+1} = 3b_n \tag{2}$$

for $n \geq 1$. Find the smallest n for which $a_n < b_n$.

A. 4
B. 5

C. 6
D. 7

Solution: For sequence 1, the pattern is as follows,

$$1, 2, 4, 8, 16, ...$$

For sequence 2, the pattern is as follows,

$$\frac{1}{4}, \frac{3}{4}, \frac{9}{4}, \frac{27}{4}, \frac{81}{4}, ...$$

Therefore, since $\frac{81}{4} > 16$,

$$n = 5$$

3. The polynomial [1 mark]

$$3x^3 + 31x^2 + 89x + 85$$

can be expressed in the form

$$A(x + B)^3 + C(x + 1)^2 \tag{1}$$

Find A, B and C.

A. $A = 3, B = 3, C = 2$
B. $A = 2, B = -3, C = 1$
C. $A = 3, B = 3, C = 4$
D. $A = 2, B = 2, C = 3$

Solution: Expand the form given in Equation 1 and equating coefficients of x^3,

$$3x^3 = Ax^3$$
$$A = 3$$

This eliminates answers B and C, $\therefore B = 3$. Substituting these values into $AB^3 + C = 85$, results in,

$$C = 4$$

4. Suppose that [1 mark]

$$x = 2\cos(t)\sin(t) \qquad y = 1 + \cos(2t)$$

Which of the following is true?

A. $x^2 + y^2 = 1$

B. $x^2 + y^2 = 2y$

C. $x^2 + 2y^2 - 2y = 0$

D. $x^2 = 1 + y^2$

Solution: Since $x = \sin(2t)$ and $y - 1 = \cos(2t)$, this can be substituited into the trigonometric relation, $\sin x^2 + \cos x^2 = 1$ to yield,

$$x^2 + (y-1)^2 = 1$$

This can be simplified and rearranged to,

$$x^2 + y^2 = 2y$$

5. A straight line passes through the points $(0, 0)$ and $(1, 5)$. A second straight line passes through the points $(2, 0)$ and $(-3, c)$. Find the value of c for which the two lines are perpendicular. [1 mark]

A. -1

B. -2

C. 2

D. 1

Solution: The gradient of the first line is,

$$g_1 = \frac{\Delta y}{\Delta x} = 5 \tag{1}$$

The gradient of the second lines is,

$$g_2 = -\frac{1}{5} = \frac{c}{-5} \tag{2}$$

The two lines are perpendicular if $g_1 = -1/g_2$, therefore,

$$c = 1 \tag{3}$$

6. The equation [1 mark]

$$x^2 - 12x + y^2 - 10y + 60 = 0$$

describes a circle with radius r and centre c. Find r and c.

A. $r = 6, c = (1, 2)$
B. $r = 1, c = (5, 6)$
C. $r = 1, c = (6, 5)$
D. $r = 6, c = (2, 1)$

Solution: By completing the square, the equation can be expressed as,

$$(x - 6)^2 + (y - 5)^2 = 1$$

Therefore,

$$c = (6, 5) \qquad r = 1$$

7. For what range of values of c do the curves, [1 mark]

$$y = x^2 + 2x + c \tag{1}$$
$$y = 4x + 3 \tag{2}$$

never intersect?

A. $c < -4$
B. $c > 2$
C. $c < -2$
D. $c > 4$

Solution: Equating equations 1 and 2,

$$x^2 + 2x + c = 4x + 3$$
$$x^2 - 2x + c - 3 = 0$$

For no real solutions, the determinant must be negative,

$$b^2 - 4ac < 0$$
$$\therefore c > 4$$

8. The function [2 marks]

$$\cos(x)\sin\left(x + \frac{\pi}{2}\right)$$

is equal to which of the following?

A. $\cos^2(x)$
B. $\cos(x)\sin(x)$
C. $-\cos(x)\sin(x)$
D. $\sin^2(x)$

Solution: Either through graph transformation or addition formula, it can be observed that $\sin\left(x+\frac{\pi}{2}\right)=\cos(x)$. Hence, $\cos(x)\sin\left(x+\frac{\pi}{2}\right)=\cos^2(x)$.

9. Which of the following is the expression [1 mark]

$$\frac{\sqrt{3}}{\sqrt{6}-\sqrt{3}}$$

is equal to?

A. $1+\sqrt{2}$
B. $1-\sqrt{3}$
C. $3+\sqrt{18}$
D. $1-\sqrt{2}$

Solution:

$$\begin{aligned}\frac{\sqrt{3}}{\sqrt{6}-\sqrt{3}} &= \frac{\sqrt{3}}{\sqrt{2\times 3}-\sqrt{3}}\\ &= \frac{\sqrt{3}}{\sqrt{2}\sqrt{3}-\sqrt{3}}\\ &= \frac{1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\\ &= \sqrt{2}+1\end{aligned}$$

10. Consider the polynomial [1 mark]

$$p(x)=x^3-11x^2+23x+35$$

Given that $p(-1)=0$, find the other two solutions to $p(x)=0$.

A. $x=1,7$

B. $x = 5, -7$

C. $x = 5, 7$

D. $x = -3, 5$

Solution: Since $p(-1) = 0$, $(x+1)$ must be a factor

$$\begin{aligned} p(x) =& (x+1)(x^2 - 12x + 35) \\ =& (x+1)(x-7)(x-5) \end{aligned}$$

Solving for $p(x) = 0$,

$$x = 5 \qquad x = 7$$

11. Find the derivative of, [2 marks]

$$e^{-e^{-x}}$$

A. $e^{-(x+e^{-x})}$

B. $-e^{-x}e^{-e^{-x}}$

C. $e^{e^{-x}}$

D. $e^{x+e^{-x}}$

Solution: Let $v = -e^{-x}$ and $y = e^v$, then,

$$\frac{dv}{dx} = e^{-x} \qquad \frac{dy}{dv} = e^v$$

Using chain rule,

$$\begin{aligned} \frac{dy}{dx} &= e^{-x+v} \\ &= e^{-(x+e^{-x})} \end{aligned}$$

12. Suppose that [2 marks]

$$x = 2\tan(y)$$

Find $\frac{dy}{dx}$.

A. $\frac{2}{1+x^2}$

B. $2\sec^2(x)$

C. $\frac{1}{2}(1+\tan^2(x))$

D. $\frac{2}{4+x^2}$

Solution: Using implicit differentiation,

$$\begin{aligned} 1 &= 2\sec^2(y)\frac{dy}{dx} \\ \sec^2(y) &= \tan^2(y)+1 \\ &= \left(\frac{x}{2}\right)^2+1 \\ \therefore \frac{dy}{dx} &= \frac{2}{x^2+4} \end{aligned}$$

13. Find the derivative of [2 marks]

$$\frac{1+x}{2+x^2}$$

A. $\frac{2-2x-x^2}{4+4x+x^2}$

B. $\frac{2-x^2}{(2+x^2)^2}$

C. $2\frac{(1-x-x^2)}{(2+x^2)^2}$

D. $\frac{2-2x-x^2}{4+4x^2+x^4}$

Solution: Using quotient rule,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

Then,

$$u'v = 1 \times (2+x^2) \qquad v'u = 2x \times (1+x)$$

Therefore,

$$\frac{2+x^2-2x-2x^2}{(2+x^2)^2} = \frac{2-2x-x^2}{4+4x^2+x^4}$$

14. Find the derivative of [2 marks]

$$\frac{x}{\sqrt{1-x^2}}$$

A. $\frac{-x}{(1-x^2)^{\frac{3}{2}}}$

B. $\frac{1}{\sqrt{1-x^2}}$

C. $\frac{1-2x^2}{(1-x^2)^{\frac{3}{2}}}$

D. $\frac{1}{(1-x^2)^{\frac{3}{2}}}$

Solution: Using quotient rule,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

Then,

$$u'v = (1-x^2)^{\frac{1}{2}} \qquad v'u = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)(x)$$
$$= -x^2(1-x^2)^{-\frac{1}{2}}$$

Therefore,

$$\frac{(1-x^2)^{\frac{1}{2}} + x^2(1-x^2)^{-\frac{1}{2}}}{(1-x^2)} = \frac{1}{(1-x^2)^{\frac{3}{2}}}$$

15. Find the derivative of [2 marks]

$$\ln(\ln(\ln x))$$

A. $\frac{1}{\ln(x^x)\ln(\ln x)}$

B. $\frac{1}{(\ln x)\ln(\ln x)}$

C. $\frac{1}{x+\ln x+\ln(\ln(x))}$

D. $\frac{1}{\ln(x\ln x)}$

Solution: Let $u = \ln(\ln(x))$, then,

$$\frac{du}{dx} = \frac{1}{\ln x} \times \frac{1}{x}$$
$$= \frac{1}{x\ln x}$$

Let $y = \ln(u)$, then,

$$\frac{dy}{du} = \frac{1}{u}$$

Using chain rule and log rules,

$$\frac{dy}{dx} = \frac{1}{x\ln(x) \times \ln(\ln x)} = \frac{1}{\ln(x^x)\ln(\ln x)}$$

16. A sequence a_n is defined by, [2 marks]

$$a_1 = \frac{1}{2} \qquad a_{n+1} = \frac{1}{2}a_n$$

What is

$$\sum_{n=1}^{19} a_n$$

A. $2(1 - 2^{-20})$
B. $1 - 2^{-20}$
C. $2(1 - 2^{-19})$
D. $1 - 2^{-19}$

Solution: Sum of first 19 terms in the geometric series with $a = r = \frac{1}{2}$

$$\begin{aligned} S_{19} &= \frac{a(1 - r^{19})}{1 - r} \\ &= \frac{\frac{1}{2}\left(1 - (\frac{1}{2})^{19}\right)}{1 - \frac{1}{2}} \\ &= 1 - 2^{-19} \end{aligned}$$

17. Suppose that $x = \cos(2t)$ and $y = \frac{1}{2}\sin(2t)$. Which of the following is true? [1 mark]

A. $\frac{1}{2}x^2 + y^2 = 1$
B. $x^2 + 4y^2 = 1$
C. $x^2 + 2y^2 = 1$
D. $x^2 + y^2 = 1$

Solution: Squaring both equations results in,

$$x^2 = \cos^2(2t) \qquad 4y^2 = \sin^2(2t)$$

Substituting into the trigonometric relation, $\cos x^2 + \sin x^2 = 1$, yields,

$$\cos^2(2t) + \sin^2(2t) = 1$$
$$x^2 + 4y^2 = 1$$

18. The circle [1 mark]

$$x^2 + y^2 = 1 \tag{1}$$

and the line

$$y = x + 1 \tag{2}$$

intersect at two points. Find the coordinates (x, y) of these points.

A. $\left(\pm 2^{-\frac{1}{2}}, \pm 2^{-\frac{1}{2}}\right)$
B. $(0, 1), (1, 0)$
C. $(0, 1), (-1, 0)$
D. $(0, -1), (1, 0)$

Solution: Squaring Equation 2, yields,

$$y^2 = x^2 + 2x + 1$$

Substituting into Equation 1 (the circle), yields,

$$2x^2 + 2x + 1 = 1$$
$$2x^2 + 2x = 0$$
$$2x(x + 1) = 0$$

Therefore,

$$x = 0 \qquad x = -1$$

19. For what value of c do the curves [1 mark]

$$y = x^2 + 2x + c \tag{1}$$

and

$$y = 4(x + 1) \tag{2}$$

intersect at exactly one point?

A. $c = 1$
B. $c = 2$
C. $c = 4$
D. $c = 5$

Solution: Equating Equation 1 and 2, yields,

$$4x + 4 = x^2 + 2x + c$$
$$0 = x^2 - 2x + (c - 4)$$

To solve for a single root (one intersection), the discriminant $(b^2 - 4ac)$ of the quadratic equation must be zero. This yields,

$$4 - 4c + 16 = 0$$
$$c = 5$$

20. The expression [1 mark]

$$\left(a^{\frac{1}{3}} b^{\frac{1}{2}}\right)^6 b^{-1} + a^2 b$$

is equal to?

A. $a^2b^2(1 + b)$
B. $a^2b + ab^2$
C. $a^{\frac{19}{3}} b^{\frac{11}{2}} + a^2 b$
D. $a^2b(1 + b)$

Solution: Using power rules,

$$\begin{aligned} a^2b^3b^{-1} + a^2b &= a^2b^2 + a^2b \\ &= a^2b(1 + b) \end{aligned}$$

21. The expression $\left(b^{\frac{1}{2}} a^3\right)^4 a^{-8} + (ab)^2$ is equal to: [1 mark]

A. $a^2b^2(1 + a^2)$
B. $b^2a^{-1} + a^2b^2$
C. $2b^2a^2$
D. $ab(1 + a^2)$

Solution: Using power rules,

$$\begin{aligned} b^2a^{12}a^{-8} + a^2b^2 &= b^2a^4 + a^2b^2 \\ &= a^2b^2(1+a^2) \end{aligned}$$

22. Suppose that $x = \sin(y)$. Find $\frac{dy}{dx}$. [2 marks]

   A. $\frac{1}{1+x^2}$
   B. $\frac{1}{\sqrt{1+x^2}}$
   C. $\frac{1}{\cos(\sin(x))}$
   D. $\frac{1}{\sqrt{1-x^2}}$

Solution: Since $x = \sin(y)$, differentiating with respect to y yields,

$$\frac{dx}{dy} = \cos(y)$$

Using the trigonometric relation, $\cos^2(y) + \sin^2(y) = 1$,

$$\cos(y) = \sqrt{1 - \sin^2(y)}$$

Using the relation, $\frac{dy}{dx} = (\frac{dx}{dy})^{-1}$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cos(y)} \\ &= \frac{1}{\sqrt{1-\sin^2(y)}} \end{aligned}$$

By substituting, $x^2 = \sin^2(y)$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

23. Find the derivative of [2 marks]

$$\frac{1}{\ln(\ln(x))}$$

.

   A. $\frac{-1}{x\ln(x\ln x)}$

B. $\frac{1}{x \ln(x \ln x)}$

C. $\frac{-1}{\ln(x^x)\ln(\ln x)^2}$

D. $\frac{1}{\ln(x^x)\ln x}$

Solution: Using the substitution, $u = \ln(x)$,

$$y = (\ln(u))^{-1}$$

Differentiate y with respect to u,

$$\frac{dy}{du} = -\frac{1}{u}(\ln(u))^{-2}$$

Differentiate u with respect to x,

$$\frac{du}{dx} = \frac{1}{x}$$

Using the relation,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This yields,

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{ux(\ln(u))^2} \\ &= \frac{-1}{\ln(x^x)\ln(\ln(x))^2} \end{aligned}$$

24. A straight line passes through the points $(0, 0)$ and $(1, 5)$. A second straight line passes through the points $(2, 0)$ and $(4, c)$. Find the value of c for which the two lines are parallel. [1 mark]

A. 5

B. 10

C. -5

D. 20

Solution: Equation of line 1 is

$$y = 5x$$

Parallel lines have the same gradient, so equation of line 2 is

$$y = 5x + a \tag{1}$$

where a is the y-intercept of the line. Substituting the coordinates $(2, 0)$ into Equation 1, yields

$$a = -10 \qquad \begin{aligned} c &= (5 \times 4) - 10 \\ &= 10 \end{aligned}$$

25. The circle [1 mark]

$$x^2 + y^2 = 1 \tag{1}$$

and the line

$$y = x + c \tag{2}$$

intersect at exactly one point. What are the possible values of c?

A. $\sqrt{2}, -\sqrt{2}$
B. $-1, 1$
C. $1, \sqrt{2}$
D. $-2, 2$

Solution: Squaring Equation 2 and substituting into Equation 1 results in,

$$y^2 = x^2 + 2xc + c^2$$
$$2x^2 + 2xc + (c^2 - 1) = 0$$

To solve for a single root (one intersection), the discriminant $(b^2 - 4ac)$ of the quadratic equation must be zero. This yields,

$$4c^2 - 8c^2 + 8 = 0$$
$$8 - 4c^2 = 0$$
$$8 = 4c^2$$
$$c^2 = 2$$
$$\therefore c = \pm\sqrt{2}$$

26. The polynomial [2 marks]

$$2x^4 + 16x^3 + 51x^2 + 70x + 35$$

can be expressed in the form

$$A(x+B)^4 + C(x+1)^2$$

Find A, B and C.

A. $A = 3, B = 2, C = 1$
B. $A = 2, B = -3, C = 1$
C. $A = 3, B = 1, C = -2$
D. $A = 2, B = 2, C = 3$

Solution: By inspection, $A = 2$. Comparing coefficients of x^0 yields,

$$35 = C + 2B^4$$
$$70 = 2C + 4B^4$$

Comparing coefficients of x^1 yields,

$$70 = 2C + 8B^3$$

Therefore,

$$B = 2 \qquad C = 3$$

27. Two sequences a_n and b_n are defined by: [1 mark]

$$a_1 = 1 \qquad a_{n+1} = a_n + 3 \tag{1}$$
$$b_1 = 20 \qquad b_{n+1} = b_n + 1 \tag{2}$$

for $n \geq 1$. Find the smallest n for which $a_n > b_n$.

A. 7
B. 9
C. 11
D. 13

Solution: From the question, it can be deduced that,

$$a_n = 3n - 2$$
$$b_n = n + 19$$

For the condition, $a_n > b_n$,

$$\begin{aligned} 3n - 2 &> n + 19 \\ 2n &> 21 \\ n &> 10.5 \end{aligned}$$

Therefore,

$$n = 11$$

28. Consider the polynomial: [1 mark]

$$p(x) = x^3 + x^2 - 14x - 24$$

Given that $p(-2) = 0$, find the other two solutions to $p(x) = 0$.

A. $x = -3, 2$
B. $x = -3, -4$
C. $x = -2, 1$
D. $x = -3, 4$

Solution: Given that $p(-2) = 0$, $(x + 2)$ is a factor of $p(x)$,

$$\begin{aligned} p(x) &= x^3 + x^2 - 14x - 24 \\ &= (x + 2)(x^2 - x - 12) \\ &= (x + 2)(x - 4)(x + 3) \end{aligned}$$

Therefore, the other two solutions are,

$$x = -3 \qquad x = 4$$

29. The equation [1 mark]

$$x^2 - 6x + y^2 - 4y - 3 = 0$$

describes a circle with radius r and centre c. Find r and c.

A. $r = 3, c = (1, 3)$
B. $r = 3, c = (3, 1)$
C. $r = 4, c = (3, 2)$
D. $r = 4, c = (2, 3)$

Solution: Completing the square of the circle equation yields,

$$(x-3)^2 + (y-2)^2 - 3 - 9 - 4 = 0$$
$$(x-3)^2 + (y-2)^2 = 16$$

Therefore,

$$c = (3, 2) \qquad r = 4$$

30. A sequence a_n is defined by, [1 mark]

$$a_1 = 1 \qquad a_{n+1} = \frac{1}{3}a_n$$

Evaluate

$$\sum_{n=1}^{\infty} a_n$$

A. $\frac{2}{3}$
B. 1
C. $\frac{3}{2}$
D. 2

Solution: The sequence is a geometric series with $a_1 = 1$ and $r = \frac{1}{3}$. For a geometric series where $|r| < 1$, the closed form solution of the sum to infinity is,

$$S_\infty = \frac{a_1}{1-r}$$

Therefore,

$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{1}{\frac{2}{3}}$$
$$= \frac{3}{2}$$

Question	Points	Score
1	1	
2	1	
3	1	
4	1	
5	1	
6	1	
7	1	
8	2	
9	1	
10	1	
11	2	

Question	Points	Score
12	2	
13	2	
14	2	
15	2	
16	2	
17	1	
18	1	
19	1	
20	1	
21	1	
22	2	

Question	Points	Score
23	2	
24	1	
25	1	
26	2	
27	1	
28	1	
29	1	
30	1	
Total:	40	

End of AMAT