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White polarization sandwiches: optical elements with non-orthogonal eigenpolarizations

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Abstract
Optical elements with non-orthogonal eigenpolarizations have complex anisotropic properties, which are not yet well understood. As an example of such elements, we studied theoretically the class of white polarization sandwiches. Light passing successively through two identical white sandwiches preserves its polarization state, so that, following the terminology of Berry and Dennis, white polarization sandwiches can be regarded as non-trivial square roots of unity.

This paper presents a general Jones matrix of white polarization sandwiches and discusses its spectral properties. It further discusses a way to synthesize white sandwiches. Our analysis shows that a general white polarization sandwich has complex anisotropic properties comprising all four basic anisotropy mechanisms: linear and circular dichroism and linear and circular birefringence. Several simple examples of synthesized white sandwiches show, however, that the form, orientation and rotation direction of their eigenpolarization ellipses can be easily controlled by changing a single parameter of the constituent elements. The results could contribute to understanding the properties of optical elements with non-orthogonal eigenpolarizations and, more generally, elements with anisotropic absorption.

Keywords: polarization, optical element, Jones matrix, eigenpolarization, eigenvalue, anisotropy

1. Introduction

Any non-depolarizing optical element belongs to one of two broad classes [1]: homogeneous elements, which have orthogonal eigenpolarizations, and inhomogeneous elements, which have non-orthogonal eigenpolarizations. Homogeneous elements are traditionally regarded as simpler, and more basic, than inhomogeneous ones. The basic anisotropy mechanisms [2] (linear and circular dichroism and linear and circular birefringence) are homogeneous and so are the basic polarization elements (polarizers, retarders, and rotators) [3, 4]. In addition, any combination of non-absorbing elements—and, more generally, any non-absorbing optical crystal—is also homogeneous [5].

Inhomogeneous elements, on the other hand, often have more complex anisotropic properties than homogeneous ones. Non-orthogonal eigenpolarizations of inhomogeneous elements require a combination of several anisotropic mechanisms, including anisotropic absorption. Degenerate polarization elements [3], whose eigenpolarizations collapse into one, are the classical example of inhomogeneous elements. An upsurge of interest in the properties of inhomogeneous elements in the mid-90s was triggered in a paper by Lu and Chipman [6] who applied the polar decomposition [7, 8] for

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their analysis. Degenerate elements were further studied in a number of recent publications (see e.g. [9–11]). Another class of inhomogeneous elements, non-orthogonal polarizers with one zero eigenvalue, was discussed by Tudor [10, 12]. Despite the recent developments, much remains to be done in understanding the properties of inhomogeneous elements.

In [13], Berry and Dennis considered yet another class of inhomogeneous elements: they showed that optical elements consisting of any specimen placed between two crossed ideal polarizers have equal eigenpolarizations. They called such elements black polarization sandwiches. Because the 2×2 Jones matrices [3, 4] of black sandwiches obey the relation

$$\mathbf{T}_{\text{black}}\mathbf{T}_{\text{black}} = \mathbf{0}, \tag{1}$$

these elements can be regarded as non-trivial square roots of zero [13].

This paper discusses the class of elements complementary to black sandwiches, with the Jones matrices obeying the relation

$$\mathbf{T}_{\text{white}}\mathbf{T}_{\text{white}} = A\exp(\mathrm{i}\psi)\mathbf{I}, \tag{2}$$

where $\mathbf{I}$ is the 2×2 identity matrix; A and ψ are arbitrary constants. Following Berry and Dennis [13], we call these elements white polarization sandwiches. Light interacting with the combination of two identical white sandwiches preserves its polarization state. White sandwiches can be thus seen as non-trivial square roots of unity; they are a non-trivial model of the free space. The constants A and ψ characterize an arbitrary isotropic absorption and phase shift; the isotropic absorption assures that white sandwiches remain passive [14].

In section 2, we derive and analyze expressions for the Jones matrices of white sandwiches and show that white sandwiches are inhomogeneous elements. In section 3, we discuss the synthesis of white polarization sandwiches by a combination of four basic polarization elements, and we further show how the eigenpolarizations of white sandwiches can be controlled. Finally, we draw conclusions in section 4.

2. Analysis of white polarization sandwiches

Presenting the Jones matrix of a white sandwich in the form

$$\mathbf{T}_{\text{white}} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{3}$$

and substituting it into (2), we obtain

$$T_{11} = -T_{22}. \tag{4}$$

Equation (4) is necessary and sufficient for a 2×2 Jones matrix to describe a white polarization sandwich.

Two eigenpolarizations and two eigenvalues constitute the spectrum [15] of a Jones matrix. A convenient way to describe eigenpolarizations mathematically is by the complex variable, χ, [3, 4] given by

$$\chi_{1,2} = \frac{T_{22} - T_{11} \pm \sqrt{(T_{22} - T_{11})^2 + 4T_{21}T_{12}}}{2T_{12}}. \tag{5}$$

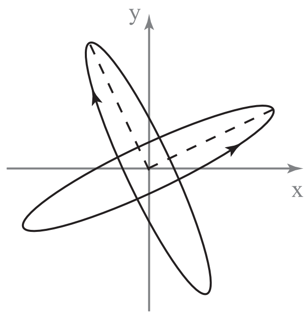

Figure 1. Two orthogonal polarization states have polarization ellipses of equal form, with opposite rotation direction, and with orthogonal axes.

With the help of the complex variable, the orthogonality condition for two polarization states reads

$$\chi_1\chi_2^* = -1, \tag{6}$$

where the asterisk denotes complex conjugation.

Polarization states can be graphically presented by the polarization ellipse [3, 4]. The polarization ellipses of two orthogonal polarization states have equal form, opposite rotation directions, and their axes are perpendicular to each other, as shown in figure 1.

Substituting (4) into (5), we obtain for the eigenpolarizations of white sandwiches

$$\chi_{1,2} = \frac{-T_{11} \pm \sqrt{-\det\mathbf{T}_{\text{white}}}}{T_{12}}. \tag{7}$$

Because (6) does not, in general, hold for (7), these eigenpolarizations are not orthogonal. The eigenvalues, characterizing the absorption and phase shift of the eigenpolarizations, have for the white sandwiches the form

$$V_{1,2} = \pm\sqrt{|\det\mathbf{T}_{\text{white}}|}. \tag{8}$$

Because $|V_1/V_2| = 1$, both eigenpolarizations are equally absorbed.

3. Synthesis of white polarization sandwiches

The analysis presented in section 2 has been quite straightforward. Equation (4) determines the Jones matrix model for white polarization sandwiches, and it provides the basis for the analysis of the spectral properties of these elements, (7) and (8). Written in symbolic form, (4) does not determine, however, which anisotropic mechanisms are able to constitute a white sandwich. It also does not answer a closely related question of how a white sandwich can be synthesized by conventional optical elements: polarizers, retarders and rotators.

The problems of the analysis and synthesis of white polarization sandwiches in terms of realistic, measurable, anisotropy parameters can be solved by applying the general decomposition theorem formulated in [16]. This theorem states that any non-depolarizing element can be presented by

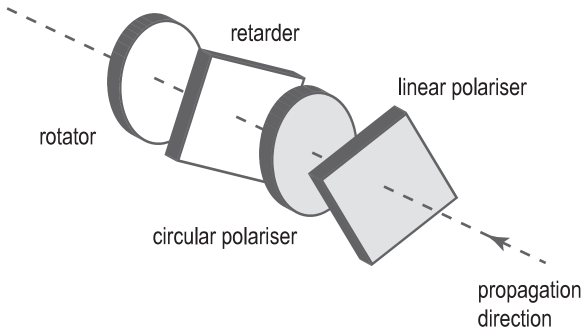

Figure 2. An arbitrary polarization element can be presented by a sandwich consisting of a partial linear polarizer, a partial circular polarizer, a retarder and a rotator.

a sandwich of four basic elements: a partial linear polarizer, a partial circular polarizer, a retarder and a rotator (figure 2). The Jones matrix, $\mathbf{T}$, of an arbitrary polarization element can then be written in the form [16]

$$\mathbf{T} = \mathbf{T}^{\mathrm{Rot}}(\varphi)\,\mathbf{T}^{\mathrm{Ret}}(\Delta,\alpha)\,\mathbf{T}^{\mathrm{CirPol}}(R)\,\mathbf{T}^{\mathrm{LinPol}}(P,\gamma), \qquad (9)$$

where

- $\mathbf{T}^{\mathrm{LinPol}}(P,\gamma)$ is the matrix of the linear polarizer

$$\mathbf{T}^{\mathrm{LinPol}}(P,\gamma) = \begin{pmatrix} \cos^2\gamma + P\sin^2\gamma & (1-P)\cos\gamma\sin\gamma \\ (1-P)\cos\gamma\sin\gamma & \sin^2\gamma + P\cos^2\gamma \end{pmatrix} \qquad (10)$$

  with the relative absorption of two orthogonal linear polarizations P and the azimuth γ;
- $\mathbf{T}^{\mathrm{CirPol}}(R)$ is the matrix of the circular polarizer

$$\mathbf{T}^{\mathrm{CirPol}}(R) = \begin{pmatrix} 1 & -\mathrm{i}R \\ \mathrm{i}R & 1 \end{pmatrix} \qquad (11)$$

  with the relative absorption of the left- and right-circular polarization states R;
- $\mathbf{T}^{\mathrm{Ret}}(\Delta,\alpha)$ is the matrix of the retarder

$$\mathbf{T}^{\mathrm{Ret}}(\Delta,\alpha) = \begin{pmatrix} \cos^2\alpha + \exp(-\mathrm{i}\Delta)\sin^2\alpha & [1-\exp(-\mathrm{i}\Delta)]\cos\alpha\sin\alpha \\ [1-\exp(-\mathrm{i}\Delta)]\cos\alpha\sin\alpha & \sin^2\alpha + \exp(-\mathrm{i}\Delta)\cos^2\alpha \end{pmatrix} \qquad (12)$$

  with the phase shift between two orthogonal linear polarizations Δ and the azimuth α;
- $\mathbf{T}^{\mathrm{Rot}}(\varphi)$ is the matrix of the rotator

$$\mathbf{T}^{\mathrm{Rot}}(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \qquad (13)$$

  with the rotation angle φ.

Explicit expressions for the elements of the generalized matrix (9) are unwieldy; they are given in the appendices of [16] and [11], and we do not reproduce them here. Substituting equations (A1)–(A3) from the appendix of [11] into (4), we obtain, after some algebraic calculations, the following expressions for the anisotropy parameters of the generalized matrix (9)

$$\begin{aligned} &(1+P)\cos\left(\frac{\Delta}{2}\right)\cos\varphi + (1-P)R\sin\left(\frac{\Delta}{2}\right) \\ &\quad\times \sin[\varphi - 2(\alpha-\gamma)] = 0 \\ &(1+P)R\cos\left(\frac{\Delta}{2}\right)\sin\varphi + (1-P)\sin\left(\frac{\Delta}{2}\right) \\ &\quad\times \cos[\varphi - 2(\alpha-\gamma)] = 0. \end{aligned} \qquad (14)$$

We wish to emphasize here that (14) defines an arbitrary white sandwich and does so in terms of measurable anisotropy parameters. Together with (9), this expression presents the generalized matrix model of white sandwiches. Solution of (14) allows one to determine which anisotropic mechanisms can constitute a white sandwich and also to synthesize an arbitrary white polarization sandwich by a combination of basic optical elements. Below, we consider three examples of synthesized white sandwiches that consist only of two elements.

3.1. Linear polarizer followed by a retarder

Taking $R = 0$ (no circular polarizer), $\varphi = 0$ (no rotator), $\Delta = \pi$ (half-wavelength retarder), $\gamma = 0$, and $\alpha = \pi/4$ satisfies (14). The corresponding element consists of a linear polarizer with zero azimuth followed by a half-wavelength retarder with the azimuth $\pi/4$; its Jones matrix (ignoring the isotropic multiplier) is

$$T = \begin{pmatrix} 0 & P \\ 1 & 0 \end{pmatrix}. \qquad (15)$$

As easily seen, $\mathbf{TT} = P\mathbf{I}$, hence the element considered is indeed a white sandwich. Its eigenpolarizations are

$$\chi_{1,2} = \pm\frac{1}{\sqrt{P}}, \qquad (16)$$

and the corresponding eigenvalues are

$$V_{1,2} = \pm\sqrt{P}. \qquad (17)$$

The eigenpolarizations are two linear polarizations oriented symmetrically against the x-axis, figure 3(a), at the angle given by

$$\tan 2\theta = \frac{2\sqrt{P}}{1-P}. \qquad (18)$$

By changing the relative absorption of the polarizer, P, it is possible to control the angle between the eigenpolarizations.

For the ideal polarizer, $P = 0$, the eigenpolarizations collapse into one linear polarization oriented along the x-axis. The corresponding eigenvalues become zero, indicating total absorption of the eigenpolarizations: the white sandwich turns into a black one.

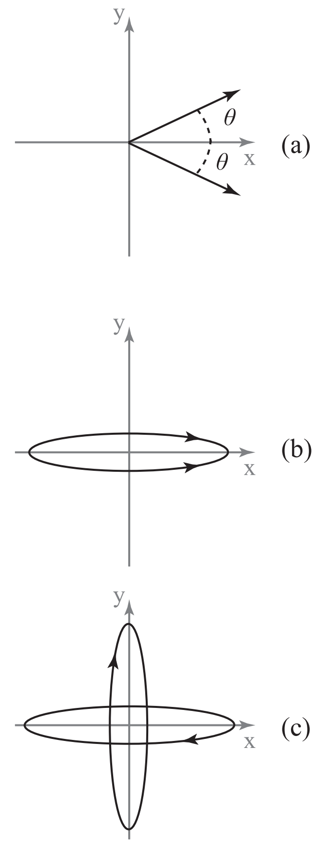

Figure 3. The eigenpolarizations of white sandwiches are non-orthogonal; they are linear for the sandwich consisting of a linear polarizer and a half-wavelength retarder (a), and elliptical for the sandwich consisting of a linear polarizer and a 90° rotator (b) and for the sandwich consisting of a circular polarizer and a half-wavelength retarder (c). The angle between the eigenpolarizations (a) and the form of the polarization ellipses (b) and (c) can be controlled by changing the relative absorption of the polarizers.

3.2. Linear polarizer followed by a rotator

A similar polarization element is obtained by taking, in (14), $R = 0$ (no circular polarizer), $\Delta = 0$, $\alpha = 0$ (no retarder), $\gamma = 0$, $\varphi = \pi/2$. The element consists of a linear polarizer followed by a 90° rotator; its Jones matrix is

$$T = \begin{pmatrix} 0 & P \\ -1 & 0 \end{pmatrix}. \tag{19}$$

The eigenpolarizations and eigenvalues of this white sandwich are

$$\chi_{1,2} = \pm\frac{\mathrm{i}}{\sqrt{P}}, \qquad V_{1,2} = \pm\mathrm{i}\sqrt{P}. \tag{20}$$

The eigenpolarizations have two equal polarization ellipses oriented along the x-axis but with two different rotation directions, figure 3(b). For the ellipticities, we get

$$\sin 2|\varepsilon| = \frac{2\sqrt{P}}{1-P}, \tag{21}$$

and, therefore, by changing the relative absorption of the polarizer, it is possible to control the form of the eigenpolarization ellipses.

This polarization sandwich is also interesting because it belongs to the class of elements described by the second Jones equivalence theorem [5]. Whereas the fist Jones equivalence theorem, capable of describing the optical properties of crystals outside their absorption bands, is widely used, the second theorem has not received much attention. The above example shows that the Jones matrix described by the second equivalence theorem is the model of a two-element white sandwich.

3.3. Circular polarizer followed by a retarder

For the next example, we choose $P = 1$, $\gamma = 0$ (no linear polarizer), $\varphi = 0$ (no rotator), $\Delta = \pi$, $\alpha = 0$. The white sandwich consists of a circular polarizer followed by a half-wavelength retarder; the corresponding Jones matrix is

$$T = \begin{pmatrix} 1 & -\mathrm{i}R \\ \mathrm{i}R & -1 \end{pmatrix}. \tag{22}$$

The eigenpolarizations and eigenvalues are

$$\chi_{1,2} = \mathrm{i}\frac{-1 \pm \sqrt{1-R^2}}{R}, \qquad V_{1,2} = \pm\sqrt{1-R^2}. \tag{23}$$

The ellipses of the eigenpolarizations are oriented along the x-axis and the y-axes, figure 3(c). Note, however, that because the rotation direction is the same for both ellipses, the eigenpolarizations are not orthogonal (compare with figure 1). For the ellipticity of both ellipses, we get

$$\sin 2\varepsilon = R, \tag{24}$$

so that the form of the ellipses can be controlled by changing the partial absorption of the circular polarizer.

4. Conclusions

The class of white polarization sandwiches considered in this paper is a counterpart of the class of black polarization sandwiches of Berry and Dennis. Whereas the eigenpolarizations of black sandwiches always coincide, the eigenpolarizations of white sandwiches can differ from each other, being non-orthogonal in general. As a result, the anisotropic properties of white sandwiches are more complex than those of black sandwiches. The derived matrix model shows that an arbitrary white sandwich can comprise polarization elements with all four basic types of anisotropy. By considering several examples of synthesized white sandwiches, we have shown that the orientation of the eigenpolarization ellipses, their form, and rotation direction can be controlled by changing the properties of the constituent elements. These results could contribute to further understanding the anisotropic properties of inhomogeneous optical elements, and more generally, systems demonstrating anisotropic absorption, such as absorbing optical crystals [17], liquid crystals [18], complex solutions [19] and metamaterials [20, 21], a new and rapidly developing area of optics.

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