Theory of Distorted Magneto-Inductive Ring Resonators

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Abstract

An analytic theory of periodically perturbed magneto-inductive waveguides is developed and used to determine the mode spectrum of distorted MI ring resonators. A set of coupled equations is first established from the recurrence equations for an infinite set of coupled elements, and used to determine the dispersion equation. Solutions for ring resonators are then found using periodic boundary conditions, which give simple predictions for resonance splitting.

1. Introduction

Most metamaterial research is concerned with perfect, infinite lattices, which are almost never obtained in practice. There is therefore an interest in developing simple analytic approaches to more realistic situations involving finite numbers of elements without excessive reliance on numerical solutions. Here we show how the method of periodic boundary conditions may be applied to the particular case of a distorted octagonal magneto-inductive ring resonator, which may have applications in magnetic resonance imaging [1]. Fig. 1a shows an undistorted ring, which consists of eight coupled L-C resonators. Each coil is typically rectangular to obtain high coupling. When used in transmit mode (for example) the ring can produce an extremely uniform distribution of magnetic field (Fig. 1b). For particular distortions, the properties of the ring alter in a simple way, which allows the eigenmodes to be found by applying periodic boundary conditions to the dispersion characteristics of an infinite line. The method can be applied to rings with different numbers of elements and to other distortions.

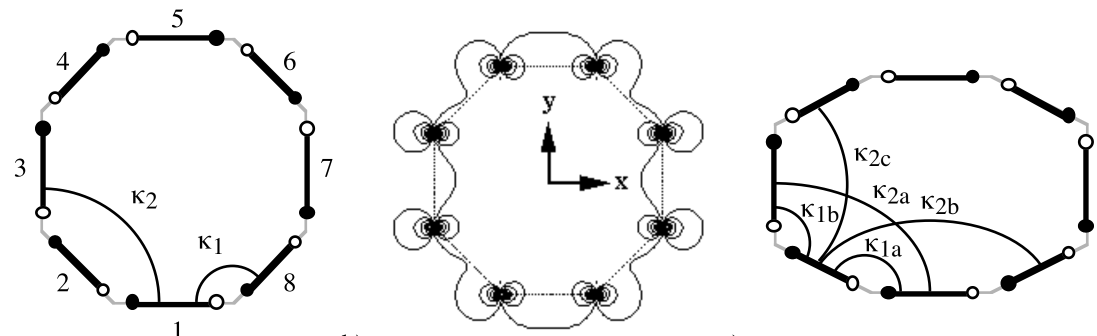

Fig. 1 Octagonal ring resonator: a) arrangement, b) distribution ofmagnetic field H_y, and c) symmetrically distorted.

2. Analytic theory

The undistorted ring has coupling coefficients κ_m between m^{th} neighbours. Assuming the travelling wave solution $I_n = I_0 \exp(-jnka)$ for the current in the n^{th} resonator, we obtain in the loss-less case the usual dispersion equation [2]:

$$\{1 - \omega_0^2/\omega^2\} + {}_m\Sigma\, \kappa_m \cos(mka) = 0 \tag{1}$$

Here $\omega_0^2 = 1/LC$ and $\kappa_m = 2M_m/L$, where M_m is the corresponding mutual inductance, and ka is the phase shift per element. The ring will resonate when the round trip phase is a whole number of multiples of 2π, so that $kNa = 2\mu\pi$ where N is the number of elements and μ is the mode number. For a ring with even N, there are N/2 + 1 resonances, with propagation constants:

$$k_\mu a = 2\mu\pi/N \qquad (\mu = 0, 1 \ldots N/2) \tag{2}$$

Once the values of $k_\mu a$ are known, the corresponding angular frequencies ω_μ may be obtained from the dispersion equation as [1]:

$$\omega_\mu/\omega_0 = 1/\sqrt{\{1 + {}_m\Sigma\, \kappa_m \cos(2\mu m\pi/N)\}} \tag{3}$$

For MRI applications, the mode $\mu = 1$ is the most important, since it may couple to the field of a rotating magnetic dipole.

When the ring is distorted, the mutual inductances and hence the coupling coefficients between different elements will alter. Fig. 1c shows the most important terms for an octagonal ring undergoing a horizontal stretching (non-shear) distortion. There are now two possible nearest neighbour coupling coefficients, κ_{1a} and κ_{1b}, and three second-nearest-neighbour terms, κ_{2a}, κ_{2b} and κ_{2c}. Horizontal compression produces a similar but rotated pattern of coefficients. Clearly, the eigenmodes could be found by substituting these coefficients into an 8 x 8 coupling matrix and using the standard numerical solution. The mathematical effort involved is equivalent to solving an 8^{th} order polynomial. Here, we use an analytic approach aimed at developing insight. For example, the resonant frequencies for stretching and compression distortions must be the same, and this feature should emerge naturally. For simplicity we consider the case when the second neighbour coefficients are negligible, but the nearest-neighbour coefficients are unequal. For Fig. 1c, this corresponds to a periodic pattern of coupling, repeating twice per cycle, so the recurrence relation between the currents can be written down immediately for a similar infinite line as follows:

$$\{1 - \omega_0^2/\omega^2\}I_n + \{\kappa/2 + \gamma \cos(n\pi/2 - 3\pi/4)\}I_{n-1} + \{\kappa/2 + \gamma \cos(n\pi/2 - \pi/4)\}I_{n+1} = 0 \tag{4}$$

Here, $\kappa_{1a} = \kappa + 2\gamma \cos(\pi/4) = \kappa + \gamma\sqrt{2}$, and $\kappa_{1b} = \kappa - \gamma\sqrt{2}$. For horizontal compression a similar expression may be obtained by writing $-\gamma$ for γ. Symmetry conditions then imply that any result for resonant frequencies should be independent of the sign of γ.

If Equation 4 may be converted into a dispersion relation, periodic boundary conditions may be used to find the resonant frequencies of the eight-element ring. Because periodic perturbations tend to couple together harmonic components [3], we assume the solution as a sum of travelling waves:

$$I_n = I_{00} \exp(-jnka) + I_{01} \exp[-jn(ka+\pi/2)] + I_{02} \exp[-jn(ka+\pi)] + I_{03} \exp[-jn(ka+3\pi/2)] \tag{5}$$

Here, I_{00}, I_{01}, I_{02} and I_{03} are constants. Substituting and equating each of the coefficients of exp(-jnka), exp[-jn(ka+π/2)], exp[-jn(ka+π/2)] and exp[-jn(ka+3π/2)] separately with zero we obtain:

$$\begin{aligned}
&\{(1 - \omega_0^2/\omega^2) + \kappa \cos(ka)\}\, I_{00} + j\gamma\, \{I_{03} \cos(ka - \pi/4) - I_{01} \cos(ka + \pi/4)\} = 0\\
&\{(1 - \omega_0^2/\omega^2) + \kappa \cos(ka)\}\, I_{01} + j\gamma\, \{I_{00} \cos(ka + \pi/4) + I_{02} \cos(ka - \pi/4)\} = 0\\
&\{(1 - \omega_0^2/\omega^2) + \kappa \cos(ka)\}\, I_{02} + j\gamma\, \{I_{03} \cos(ka + \pi/4) - I_{01} \cos(ka - \pi/4)\} = 0\\
&\{(1 - \omega_0^2/\omega^2) + \kappa \cos(ka)\}\, I_{03} - j\gamma\, \{I_{02} \cos(ka + \pi/4) + I_{00} \cos(ka - \pi/4)\} = 0
\end{aligned} \tag{6}$$

Equations 6 are a set of four coupled equations that can be uncoupled to yield a single dispersion equation. After some manipulation, we get:

$$\begin{aligned}
&\{(1 - \omega_0^2/\omega^2) + \kappa \cos(ka)\}^2 = 2\gamma^2 \cos^2(ka) \qquad \text{or}\\
&\{(1 - \omega_0^2/\omega^2) + \kappa \cos(ka)\}^2 = 2\gamma^2 \sin^2(ka)
\end{aligned} \tag{7}$$

So that:

$$\begin{aligned}
&\omega/\omega_0 = 1/\sqrt{\{1 + \kappa \cos(ka) \pm \sqrt{2}\gamma \cos(ka)\}} \qquad \text{or}\\
&\omega/\omega_0 = 1/\sqrt{\{1 + \kappa \cos(ka) \pm \sqrt{2}\gamma \sin(ka)\}}
\end{aligned} \tag{8}$$

This result implies that there are in general four solutions, although some are spurious because they follow from special cases of the coefficients in Equations 6. For the primary resonance of an eight element ring ($ka = \pi/4$) we obtain:

$$\omega_1/\omega_0 = 1/\sqrt{\{1 + \kappa/\sqrt{2} \pm \gamma\}} \tag{9}$$

This result is independent of the sign of γ. It may also be written as $\omega_1/\omega_0 = 1/\sqrt{\{1 + \kappa_{1a}/\sqrt{2}\}}$ or $\omega_1/\omega_0 = 1/\sqrt{\{1 + \kappa_{1b}/\sqrt{2}\}}$. Since there are two solutions, the original primary resonance has split into two. Fig. 2a shows the four general solutions (Equation 8), for the parameters $\kappa_a = -0.35$, $\kappa_a = -0.25$. For a ring, the discrete resonances must be obtained by applying Equation 2 once again. Clearly there can be no more than 8 resonances in an 8-element ring, so some of the solutions are redundant. The resonance pairs at $ka/\pi = 0.25$ and at $ka/\pi = 0.75$ are unambiguous. The remaining resonances are illustrated by comparison with the discrete eigenvalues obtained from the full numerical solution of the matrix problem. Fig. 2b shows the variation of the two primary mode frequencies (Equation 9) with the normalised distortion parameter γ/κ, for $\kappa = -0.3$. The degree of mode splitting clearly varies quasi-linearly with the amount of distortion.

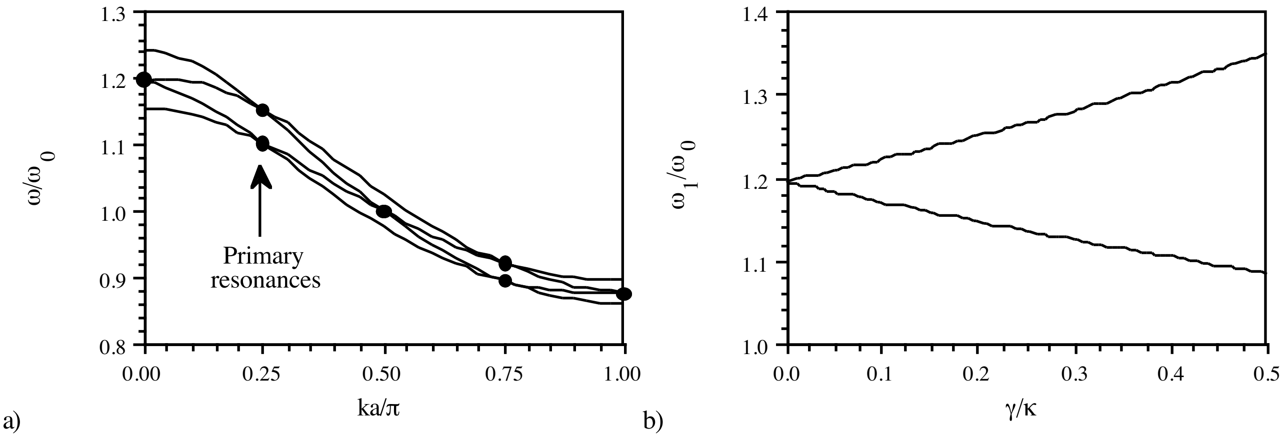

Fig. 2 a) Dispersion diagram; b) variation of primary mode resonant frequencies with γ/κ for distorted octagonal ring.

Substitution into Equations 6 then allows the relation between the current amplitudes to be found. For the primary resonances of the 8-element ring we get $I_{03} = \pm jI_{00}$. A similar relation may be found between I_{01} and I_{02}. Using Equations 6, we then get:

$$I_n = 2I_{00} \cos[(n \pm 1)\pi/4] + 2I_{01} \cos[(3n \pm 1)\pi/4] \quad (10)$$

This result implies that the geometric perturbation will distort the uniform current distribution into a periodically varying one, which will result in poor transmission uniformity or reception sensitivity in MRI. Ignoring the contributions of I_{01}, the variations are cosines, shifted in phase by $\pi/2$ as expected from the symmetry of Fig. 1c. These analytic mode shapes also agree with the numerical solution of the matrix eigenvector problem.

3. Conclusions

The method of periodic boundary conditions has been applied to an octagonal magneto-inductive ring resonator with symmetric geometrical distortions, and the solution has been shown to imply splitting of otherwise degenerate resonant modes. The splitting must be controlled for proper functioning in MRI applications, for example by arranging for the primary coupling coefficient to remain constant as the ring is distorted.

References

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