Noise in metamaterials

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Metamaterials made up of resonant elements containing lossy metallic conductors will invariably give rise to Johnson noise. A model based on nearest neighbor interaction of magnetically coupled elements is shown to predict the propagation of noise waves and the excitation of resonances in regular arrays. The power spectral density (PSD) of the noise is calculated for rectangular arrays of different dimension. It is shown that the effect of coupling is to alter the PSD, leaving the noise bandwidth unaltered. The implications for passive and active devices are examined using the simple model of a lossy one-dimensional interconnect with distributed parametric amplification, and it is shown that the improvements to the noise factor offered by amplification are limited.
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I. INTRODUCTION

Following some seminal investigations,[1,2] artificial media with novel electromagnetic properties such as negative permittivity or permeability have attracted considerable interest. The media (collectively known as metamaterials) are based on arrays of electrically resonant structures. Examples of resonant elements include split-ring resonators (SRRs) and 'Swiss rolls'.[2–4] Periodic circuits provide models for such media,[5] and it is simple to show that coupling allows the propagation of lattice waves; for example magnetoinductive[4,6] and electroinductive[7] waves. Applications include interconnects,[8] near-field imaging devices[9] and cloaks of invisibility.[10] However, RF metamaterials are inherently lossy. Conductors and dielectric materials are dissipative sources of propagation loss, while scattering from disorder in periodic lattices can also give rise to effective attenuation. As a result, methods of overcoming loss such as parametric amplification are being actively considered.[11,12]

Here, we focus on the effect of conductors. In addition to attenuation, such elements must give rise to electrical noise, and this aspect has so far escaped significant attention. However, because noise may have profound implications for practical applications it deserves consideration. Since the observation of thermal noise in conductors by Johnson[13] and its theoretical explanation by Nyquist,[14] electrical engineers have attempted to minimize its effect on lumped-element circuits.[15,16] The propagation of 'noise waves' in distributed circuits has also been noted.[17,18] Noise must clearly propagate as a wave in any coupled metamaterial. Using the specific example of multi-dimensional magnetoinductive (MI) media,[19] we show in this paper how the power spectral density (PSD) of the noise can be found by adding together the contributions of noise waves or noise resonances. We then show how magnetic coupling alters the PSD in arrays of different dimension. In each case, we provide exact analytical solutions, rather than representative averages. Finally, we examine the implications for the noise factor of simple passive and active metamaterial links, and show that the improvements offered by amplification are limited.

A. Infinite one-dimensional arrays and noise waves

We first follow Pierce[15] and consider a single lossy *L-C* resonator containing a voltage source $V(\omega) = V_0 \exp(j\omega t)$ at angular frequency $\omega = 2\pi f$ as shown in the inset to Fig. 1. If the voltage arises from Johnson noise in the resistor R, the average value of $V_0V_0^*$ in a small frequency interval df is 4KTRdf, where K is Boltzmann's constant and T is absolute temperature (assumed constant throughout this paper). Assuming the current also varies as $I = I_0 \exp(j\omega t)$, its modulus square is $I_0I_0^* = P_0V_0V_0^*/R^2$, where P_0 (the normalized power spectral density of the noise for the 0-dimensional case) is:

$$P_0(\omega) = 1/\{1 + Q^2(1 - \omega_0^2/\omega^2)^2\} \quad (1)$$

Here, $\omega_0 = 1/\sqrt{(LC)}$ is the angular resonant frequency, $Q_0 = \omega_0 L/R$ is the quality factor and $Q = Q_0\omega/\omega_0$. This analysis shows that the reactive elements add no noise, but instead form a filter that alters the noise PSD from its original flat distribution. Figure 1 shows the frequency variation of P_0, for $Q_0 = 100$. There is a single peak at $\omega = \omega_0$, with a monotonic reduction on either side. Substituting for $V_0V_0^*$ we get $I_0I_0^* = P_0 4KTGdf$, where $G = 1/R$ is the conductance. The noise bandwidth may be found by comparing this result with the standard expression for a flat noise distribution ($I_0I_0^* = 4KTGB$) to get $B = {}_0\int^\infty P_0(f)\, df$. This integral was performed analytically by Pierce,[15] who obtained a bandwidth $B = \omega_0/4Q_0$. These results represent well-known benchmarks, and our interest is whether they also hold in coupled arrays or not.

Sets of identical elements may be arranged in 1D, 2D and 3D arrays, as shown in Figs. 2(a)–(c).[19] Obviously there are many possible arrangements but here we restrict ourselves to rectangular arrays with rectangular unit cells. We assume magnetic coupling between nearest neighbors, and

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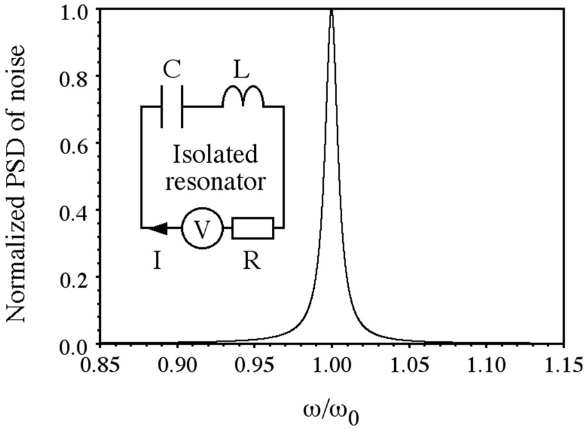

FIG. 1. Lossy L-C resonator with internal noise voltage source due to the resistor R (inset) and its normalized power spectral density of noise (main figure).

begin by considering 1D arrays consisting of an infinite set of identical lossy L-C resonators with lattice spacing a. With no voltage sources, the circuit equation for the n^{th} element is $(R + j\omega L + 1/j\omega C)I_n + j\omega M(I_{n-1} + I_{n+1}) = 0$, where M is the mutual inductance between elements. Assumption of the wave solution $I_n = I_0 \exp(-jnka)$ leads to the dispersion equation $1 - {\omega_0}^2/\omega^2 - j/Q + \kappa\cos(ka) = 0$ (Ref. 6). Here, $\kappa = 2M/L$ is the coupling coefficient, and k is the propagation constant. Positive and negative values of M lead to forward and backward waves, over the range $1/\sqrt{(1 - |\kappa|)} \le \omega/\omega_0 \le 1/\sqrt{(1 - |\kappa|)}$. Finite Q-factors render the propagation constant complex (so that $k = k' - jk''$) and allow out-of-band propagation.

To find the effect of noise, we assume an array with n extending from minus to plus infinity, containing to start with a single noise source of amplitude V_0 in element zero. For forward-waves, the resulting equations may be solved by assuming the solution $I_n = I_0 \exp(-j\,|\,n\,|\,ka)$, i.e., noise waves propagating away from the source on either side. It is simple to show that $I_0 = V_0/\{R\kappa Q \sin(ka)\}$, and hence that the modulus-square of the current in the n^{th} element is:

$$I_n I_n^* = (V_0 V_0^*/R^2)\exp(-2|n|k''a)/\{\kappa^2 Q^2|\sin(ka)|^2\} \quad (2)$$

Similarly, the response in element n for a noise source in element n′ is:

$$I_{n,n'}\, I^*_{n,n'} = (V_0 V_0^*/R^2)\exp(-2|n - n'|k''a)/\{\kappa^2 Q^2|\sin(ka)|^2\} \quad (3)$$

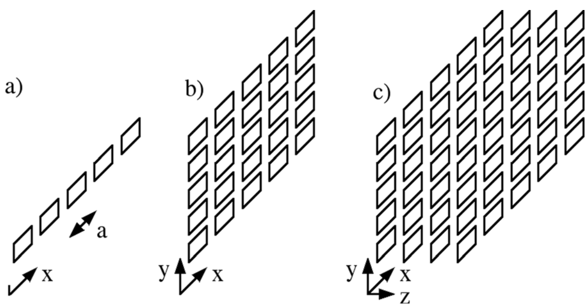

FIG. 2. (a)–(c) Regular arrays of magnetically coupled resonators of different dimension.

Equation 3 implies that elements n and n′ are in equilibrium, since their effects on each other balance. The overall response $_{n'}\Sigma\, I_{0,n'} I_{0,n'}{}^*$ in (say) element zero can be found by summing independent contributions from all noise sources to get $I_0 I_0{}^* = P_1 V_0 V_0{}^*/R^2$, where P_1 (the PSD of the noise for a 1D lattice) is:

$$P_1(\omega) = 1/\{\kappa^2 Q^2 \tanh(k''a)|\sin(ka)|^2\} \quad (4)$$

P_1 may be evaluated following numerical solution of the lossy dispersion equation. Figure 3 shows its frequency variation, for $Q_0 = 100$ and different values of κ. For small κ, we recover the PSD of an isolated element (P_0). However, as κ increases, the results are entirely different. The peak value of P_1 decreases and the frequency range of significant noise increases to include the whole MI band. Within the band, the PSD is approximately flat, but there are small peaks at the band edges. The same results are obtained for negative κ. Clearly, coupling does alter the spectral distribution of noise. However, numerical integration of P_1 still yields the previous result ($B = \omega_0/4Q_0$), implying that the equivalent noise bandwidth is unchanged, despite the introduction of coupling.

B. Finite one-dimensional arrays and noise resonances

We now consider the excitation of resonances in finite arrays by noise waves. In the loss-less case, the spatial resonances supported by a finite 1D array may be found by imposing periodic boundary conditions. Assuming the array extends from $n = 1$ to N, we obtain $k_\nu a = \nu\pi/(N+1)$, with $\nu = 1 \ldots N$, and eigenfrequencies ω_ν given by $\omega_0^2/\omega_\nu^2 = 1 + \kappa\cos\{\nu\pi/(N+1)\}$. The associated current distributions can be written in normalized form as $j_{\nu n} = \sqrt{\{2/(N+1)\}}\sin\{\nu\pi/(N+1)\}$ and represented as N-element eigenvectors j_ν. In the lossy case, it can be shown that the result of excitation by a set of voltages (written as an N-element vector V) is the current distribution given by the vector I:[20]

$$I = (1/R)\ _\nu\Sigma < V, j_\nu > j_\nu/\{1 + jQ(\omega_0^2/\omega_\nu^2 - \omega_0^2/\omega^2)\} \quad (5)$$

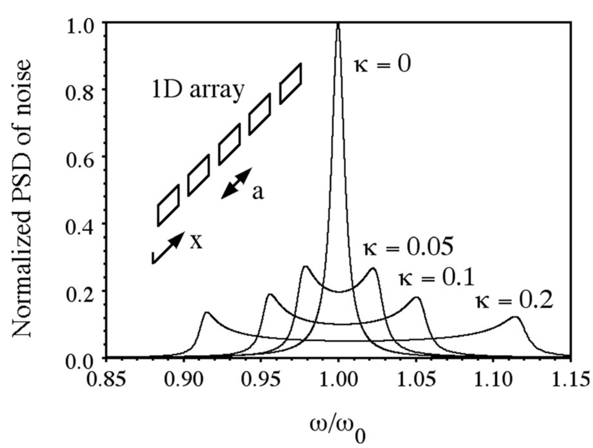

FIG. 3. 1D magnetically coupled resonator array (inset), and its normalized power spectral density of noise (main figure), for $Q_0 = 100$ and different values of κ.

Here, $< V, j_\nu >$ is the inner product of V and j_ν. This result implies that each mode will be excited by an amount that depends on the correlation between its shape and the excitation pattern and also on the closeness of its resonance to the operating frequency. It is valid for arbitrary MI systems with identical elements, but is especially valuable in regular arrays when j_ν and ω_ν are known. Using Eq. (5), the response at element n to a noise source at element n′ is:

$$I_{n,n'} I^*_{n,n'} = (V_0 V^*_0/R^2)\{_\nu \Sigma j_{\nu n'}\, j_{\nu n}/\lambda_\nu\}\{_\mu \Sigma j_{\mu n'}\, j_{\mu n}/\lambda^*_\mu\} \quad (6)$$

Here $\lambda_\nu = 1 + jQ({\omega_0}^2/{\omega_\nu}^2 - {\omega_0}^2/\omega^2)$. Summing the effect of all noise sources, we then obtain:

$$I_n I^*_n = (V_0 V^*_0/R^2)_{n'}\Sigma\{_\nu \Sigma_\mu \Sigma j_{\nu n'}\, j_{\nu n}\, j_{\mu n'}\, j_{\mu n}/\lambda_\nu \lambda^*_\mu\} \quad (7)$$

Changing the order of the sums, and using the orthonormality of the modes we then obtain:

$$I_n I^*_n = (V_0 V^*_0/R^2)\{_\nu \Sigma {j_{\nu n}}^2/\lambda_\nu \lambda^*_\nu\} \quad (8)$$

Comparison with previous results shows that the noise PSD must be $P_1 = {}_\nu \Sigma {j_{\nu n}}^2/\lambda_\nu \lambda_\nu *$, or:

$$P_1(n, \omega) = \{2/(N+1)\}_\nu \Sigma \sin^2(n\nu\pi/(N+1)\}/ \{1 + Q^2(\omega_0^2/\omega_\nu^2 - \omega_0^2/\omega^2)^2\} \quad (9)$$

Before proceeding, we note that this result could have been obtained somewhat differently. In previous analysis we have assumed that the noise sources are all applied separately. However, for excitation by all the sources simultaneously, we obtain:

$$I_n I^*_n = (1/R^2)\{_\nu \Sigma_\mu \Sigma < j_\nu, V > < V^*, j_\mu > j_{\nu n} j_{\mu n}/\lambda_\nu \lambda^*_\mu\} \quad (10)$$

Here V is an n-element vector containing elements V_0, and we have reversed the order of the first inner product. Since the noise voltages must be un-correlated, the outer product of V and V^* is $V_0 V_0^*\, i$, where i is the identity matrix. In this case, it can be seen that Eq. (8) is obtained directly.

Equation (9) is an alternative to Eq. (4), and predicts that the noise PSD must actually depend on position in a finite array. Figure 4 shows the frequency variation of P_1 for infinite arrays (smooth curves) and finite arrays with $N = 15$ (oscillatory curves), calculated both at the center and at the edge of the array. In each case, $Q_0 = 100$ and $\kappa = 0.1$. For small N, individual noise resonances may now be distinguished. As N rises, the size and period of the oscillations decreases, and the variation at the array center tends to the result for an infinite array. Peaks in the limiting PSD may then be ascribed to the crowding of resonances near the bandedge. The variation at the array edge is entirely different, with a peak rather than a trough at the band center. This result is due to the local environment of elements at the edge, which sample noise from a different set of sources. Similar calculations may be carried out for 2D and 3D arrays, to obtain double and triple sums, respectively.

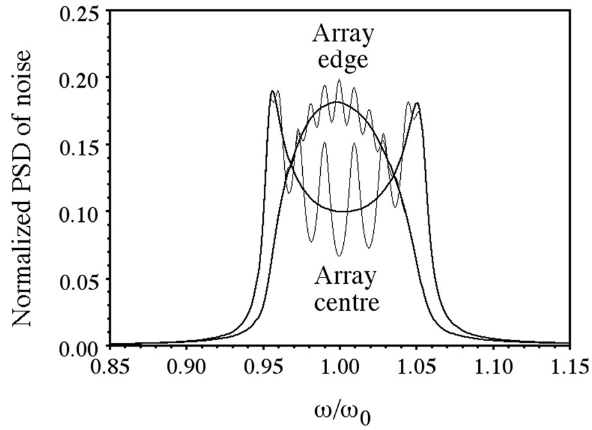

FIG. 4. Power spectral density of noise PSD in 1D magnetically coupled resonator arrays, for $Q_0 = 100$ and $\kappa = 0.1$. Smooth lines — infinite array; oscillatory lines — finite array with $N = 15$.

C. Infinite multi-dimensional arrays

When the array is large, the PSD near the center must be approximately uniform, and the summation must tend to an integral. Making use of the fact that the resonances are at regular intervals in *k*-space, it is simple to show that the relevant integral is:

$$P_1(\omega) = (1/\pi)\int_0^\pi dk_\nu a/\{1 + Q^2(\omega_0^2/\omega_\nu^2 - \omega_0^2/\omega^2)^2\} \quad (11)$$

Comparison with Eq. (1) suggests the noise spectrum in the array is the average in *k*-space of the PSDs of a set of equally weighted noise resonances. The integral may be evaluated by eliminating the resonant frequency term using the lossless dispersion relation to get:

$$P_1(\omega) = (1/\pi)\int_0^\pi dk_\nu\, a/\ \{1 + Q^2(1 + \kappa\cos(k_\nu a) - \omega_0^2/\omega^2)^2\} \quad (12)$$

Despite their dissimilar appearances, numerical evaluation shows that Eq. (12) predicts exactly the result of Eq. (4). The approach may be extended to higher dimensions very simply. For example, for a 2D array with coupling coefficients κ_x and κ_y in the *x*- and *y*-directions and equal lattice spacing *a*, we obtain the double integral:

$$P_2(\omega) = (1/\pi^2)\int_0^\pi\int_0^\pi dk_\nu a\, dk_\mu a/\{1 + Q^2(1 + \kappa_x\cos(k_\nu a) + \kappa_y\cos(k_\mu a) - \omega_0^2/\omega^2)^2\} \quad (13)$$

Similarly, addition of a coupling term κ_z in the *z*-direction allows the 3D PSD to be found as:

$$P_3(\omega) = (1/\pi^3)\int_0^\pi\int_0^\pi\int_0^\pi dk_\nu a\, dk_\mu a\, dk_o a \,/\, \{1 + Q^2[1 + \kappa_x\cos(k_\nu a) + \kappa_y\cos(k_\mu a) + \kappa_z\cos(k_o a) - \omega_0^2/\omega^2]^2\} \quad (14)$$

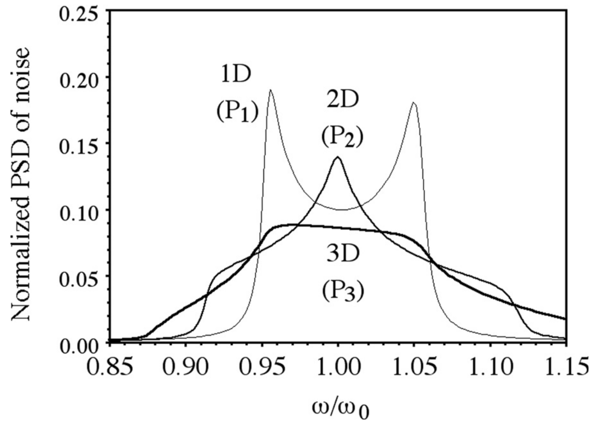

FIG. 5. Power spectral density of noise PSD in infinite magnetically coupled resonator arrays with different dimension, for $Q_0 = 100$ and $\kappa_x = -0.1$, $\kappa_y = -0.1$, $\kappa_z = 0.1$.

Figure 5 compares the PSDs obtained for 1D, 2D and 3D arrays, assuming $\kappa_x = -0.1$, $\kappa_y = -0.1$ and $\kappa_z = 0.1$ and $Q_0 = 100$. The results clearly depend strongly on the dimension of the array.

D. Noise factor of passive interconnects

Internal noise must affect the performance of metamaterial interconnects. We illustrate the likely effects with the example of a passive linear MI array, used to link a voltage source V_S with output impedance R_S to a load R_L as shown in Fig. 6(a). We assume that source and load are matched, and that the array is operated at mid-band. The available signal power at the input is $S_I = V_{S0}{}^2/4R_S$ (Ref. 21). Since R_S will provide a source of noise V_{Sn0} such that $V_{Sn0}V_{Sn0}{}^* = 4KTBR_S$, the available noise power is $N_I = V_{Sn0}V_{Sn0}{}^*/4R_S = KTB$. The input signal-to-noise ratio (SNR) is therefore:

$$S_I N_I = V_{S0}{}^2/4KTBR_S \tag{15}$$

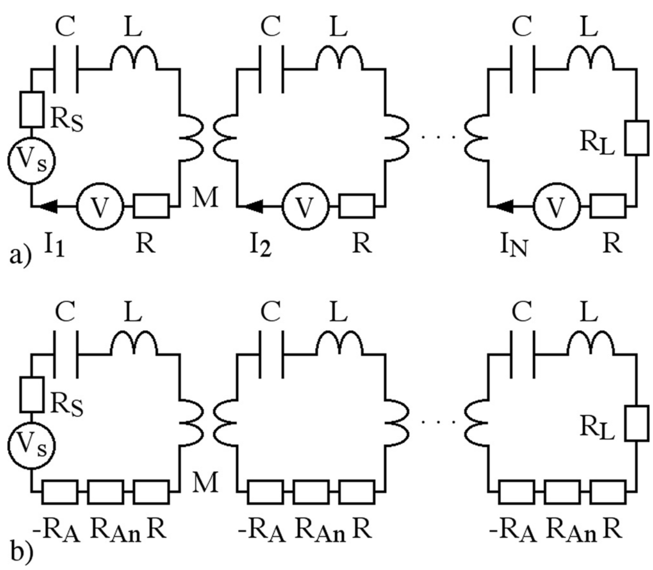

FIG. 6. (a) Passive and (b) active magneto-inductive arrays used to connect a source and a load.

Ignoring losses, the MI interconnect has a real impedance at mid-band, given by $Z_0 = \omega_0 M$. Clearly, Z_0 may be matched to R_S by choosing $\omega_0 M = R_S$ and the signal will then propagate from source to load without reflection. Including propagation losses, but ignoring any small reflections, the output signal power is $S_O = (V_{S0}{}^2/4R_S)\exp\{-2(N-1)k''a\}$. For small losses, $k''a \approx 1/\{\kappa Q \sin(k'a)\}$, so $k''a \approx 1/\kappa Q$ at mid-band. If the line is impedance matched, $\kappa Q = 2R_S/R$, so $S_O = (V_{S0}{}^2/4R_S)\exp\{-(N-1)R/R_S\}$. Similarly, the output noise power N_{OS} due to R_S is $N_{OS} = KTB \exp\{-(N-1)R/R_S\}$.

With no other sources of noise, the SNR is unchanged. However, each resistor R in the resonant elements will generate additional noise, which lower the SNR. To find these contributions, we turn again the solution for central excitation of an infinite line, namely $I_n = I_0 \exp(-j|n|ka)$ where $I_0 = V_0/\{R\kappa Q \sin(ka)\}$. This result implies that each noise source will generate a wave propagating away on either side. At mid-band, $I_0 = V_0/2R_S$, where $V_0 V_0{}^* = 4KTBR$. Ignoring propagation loss, the power delivered to the load from element *n* is therefore $I_n I_n{}^* R_S = KTB(R/R_S)$. Including propagation loss, and assuming the earlier value of $k''a$, this result modifies to $N_{On} = KTB(R/R_S)\exp\{-(N-n)R/R_S\}$. Clearly, each element produces a similar contribution, and these must be summed to obtain the total noise N_{OE} as $N_{OE} = KTB(R/R_S)\ {}_n\Sigma \exp\{-(N-n)R/R_S\}$. The total noise delivered to the load is then $N_O = N_{OS} + N_{OE}$. Evaluating the sum above, the output SNR can be found as:

$$S_O/N_O = (V_{S0}^2/4KTBR_S)/[1 + (R/R_S)\exp\{(N-1)R/R_S\} \times \{1-\exp(-NR/R_S)\}/\{1-\exp(-R/R_S)\}] \tag{16}$$

The noise factor $F = (S_I/N_I)/(S_O/N_O)$ is then:

$$F = 1 + (R/R_S)\exp\{(N-1)R/R_S\} \{1-\exp(-NR/R_S)/\{1-\exp(-R/R_S)\} \tag{17}$$

If $N = 1$, this result reduces to $F = 1 + R/R_S$, the noise factor of a single resistor of value R. If $N \neq 1$, it can be written as $F = 1 + R_N/R_S$, where R_N is an equivalent resistance given by:

$$R_N = R\exp\{(N-1)R/R_S\}\{1-\exp(-NR/R_S)\}/ \{1-\exp(-R/R_S)\} \tag{18}$$

These results contain only N, R and R_S, suggesting that the noise factor cannot be improved by alteration to the reactive parameters of the magneto-inductive link. If NR/R_S is small, $R_N \approx NR$, suggesting that F will increase linearly for small N. The upper curve in Fig. 7 shows the variation of F with N, for $R/R_S = 0.02$ (corresponding to $R_S = 50\Omega$ and $R = 1\Omega$, typical of experimental lines). The noise factor initially increases linearly with N, but the rate of increase soon starts to rise and the noise factor rapidly becomes poor in this example.

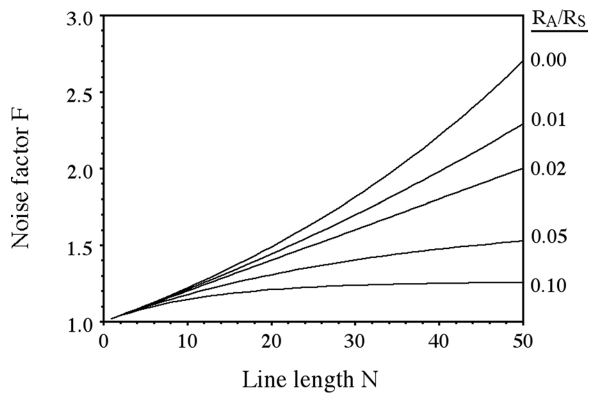

FIG. 7. Variation of noise factor F with line length N, for magneto-inductive interconnects with loss defined by $R/R_S = 0.02$ and parametric amplification defined by different values of R_A/R_S.

E. Noise factor of active interconnects

The relatively poor performance of a passive interconnect suggests that it would be useful to know the effect on the noise factor of amplification. Given the interest in the use of distributed parametric processes to overcome loss in metamaterials, we focus on this example. Specifically, we assume amplification in each loop via a separate varactor-based three-frequency amplifier.[12,22] If the loop capacitance is adjusted to compensate for insertion of the varactor, such an amplifier can be represented as a negative resistance $-R_A$. Its noise arises from two sources, the series resistance of the varactor and down-converted idler noise. Because both are Johnson-type, we shall assume that they can be represented as a single noisy resistor R_{An}. A simple model for a MI link with distributed amplification is therefore as shown in Fig. 6(b). In this case the previous analysis can be re-used, replacing R by $R' = R + R_{An}$ to calculate noise and R by $R'' = R - R_A$ to calculate attenuation. The modified noise factor is then:

$$F = 1 + (R'/R_S)\exp\{(N-1)R''/R_S\} \times \{1 - \exp(-NR''/R_S)\}/\{1 - \exp(-R''/R_S)\} \quad (19)$$

The lower curves in Fig. 7 show the variation of F with line length N for an amplified MI waveguide with background losses defined by $R/R_S = 0.02$. We assume that amplification is noiseless, so that $R_{An}/R_S = 0$, and show results for different values of R_A/R_S up to 0.1. Bearing in mind that the oscillation threshold for a single loop is reached when $R_A/R_S = R/R_S = 0.02$, the level of amplification needed to improve F is significant, even when the amplification process itself is assumed to be noise-free. The explanation of this poor result is that distributed amplification must amplify all the noise, even the cable noise. This result implies that distributed parametric processes may indeed overcome distributed loss, but make little impact against any noise associated with the loss. The only solution is the standard one of a high-gain front-end amplifier,[21] so that the source noise remains dominant.

II. CONCLUSIONS

The arrangement of magnetically coupled resonant elements in an array has been shown to have a fundamental effect on the power spectral density of Johnson noise arising from lossy conductive elements. The PSD is modified by coupling and by the dimension of the array, and is significantly different at the edge of an array and at the center. The effects can be explained in terms of noise waves and the resonances excited by such waves in a highly consistent manner. This simple analysis has considered only rectangular unit cells with magnetic coupling between nearest neighbors, and it is to be expected that further variations will be found when different unit cells and coupling mechanisms, different array shapes and non-nearest neighbor effects are considered.

Clearly the generation of internal noise must have an impact on performance in any application in which a metamaterial is imposed between a source and a load (for example, RF interconnects or near-field imaging devices). While losses may clearly be improved by internal amplification, the possible improvements to signal-to-noise ratio are limited (and often extremely so) and this aspect may be crucial in measurement systems. Noise may also be significant in applications that merely involve a source and a metamaterial. One such example is the use of a metamaterial invisibility cloak, where the emission of noise with a characteristic spectral signature might represent an advertisement of the presence of the metamaterial.

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