IMPERIAL

Imperial College London

Department of Mathematics

MSc Mathematics and Finance

2026-2027

September Preparation

London, September 2026

Dear all,

We are delighted to welcome you to the MSc in Mathematics and Finance at Imperial College London for the 2026-2027 academic year. This will be an exciting year, preparing you to the next stage of your professional life.

This year will be dense, full of lectures, career events, courseworks, exams and projects. It is also a stressful year because of job interviews and we strongly advise you to start preparing now, before the lectures actually start. Quantitative Finance is an exciting area, evolving fast and encompassing — and requiring — a large array of tools, from Mathematics (Statistics, Probability, Algebra, Analysis) to Computing (Python, C++), to Machine Learning and Artificial Intelligence, without obviously forgetting the human factor: you are and will be working with people all day, so communication is key!

The following notes gather useful information about the programme itself as well as some help about how to best prepare for interviews in Quantitative Finance.

Looking forward to having you here,

The MSc Mathematics and Finance team

Contents

Chapter 1

The MSc Programme at a glance

- Imperial College of Science, Technology and Medicine
- The Mathematical Finance group: research, teaching, consulting.
- MSc Mathematics and Finance:
  - Core and Elective Module guide
  - Careers in Quantitative Finance: every Tuesday 6-7pm
  - Practitioners' Lectures: every Wednesday 6-8pm
  - Summer Thesis: as part of an internship (how to find one.....?, internal.
  - Imperial MSc Math Finance LinkedIn Alumni group
  - Imperial MSc Math Finance Alumni reception: Tuesday 29 September

Start connecting together. Start practicing together....

Chapter 2

Tips and Resources

2.1 Your CV

Your CV is screened by recruiters and will often serve as support for the interviewers during an interview. It should be easy to read and should highlight what makes you an interesting person to work with. A few important things when writing it:

- No typo (use a spellchecker)
- No photo
- Instead of listing all the courses you took, list what distinguishes you from the rest of the crowd: project you worked on, specific exchange programme,...
- Always send it as PDF, not as a Word document (as formatting may be altered)
- If you cannot talk for 5 minutes about an activity/project/hobby that you list, you should remove it.

2.2 General Tips

Know what you are heading to

When you apply for a job, know what the job is about and show that you are interested in it. The world of Finance is broad, with many types of roles (Front Office, Back Office, Middle Office, Risk, trading, sales, quants, Equites, FX, Commodities, Structuring,.....) and many actors (banks, hedge funds, financial software firms, regulators, clearing houses,...). It is never too early to learn about the environment and to find out (or at least get some ideas about) where your interests lie. If your sole motivation for this role is *"Because it pays more"*, some priorities may need to be re-evaluated.....

Practice is key

Mock interviews (with your classmates for example), online quizzes. Interviews can take many forms, depending on the companies, on the roles and on the interviewers. The questions will usually center around (a combination of) the following:

- Finance
- Mathematics
- Coding
- Personality

The interview is not an easy exercise and the only way to be successful at it is to practice, by yourself and with others. We will list below many websites as well as examples for you to start practicing.

Personality question tips
The easiest – yet somehow most tricky one — is the personality type of questions. These questions are there to assess whether you will be able to work in a team, with other people, as opposed to working alone. Companies are successful because of team work, and we all benefit from other people's inputs. Depending on your background and your upbringing, you may be more shy or more extrovert. This is not necessarily an issue, but you need to show that you are an interesting person, able to express their ideas clearly. You should also always ask yourself the following: *Why is the interviewer asking me this question?*

2.3 Foundational Background and Resources

2.3.1 Mathematics

The MSc in Mathematics and Finance combines core Mathematics with applications to Finance and therefore requires many different (technical) skills. We would like to draw your attention to the following textbooks, which serve as a foundational basis for the course:

- W. Rudin - *Principles of Mathematical Analysis* (McGraw-Hill, 1976)
- G. Folland - *Real Analysis, Modern Techniques and their Applications* (Wiley, 1984)

Rudin's book should be part of your Undergraduate background. Folland's monograph goes deeper in Analysis, covering Functional Analysis and some elements of Measure Theory. For background on Probability and Statistics, you should look at

- G. Grimmett, D. Stirzaker -*Probability and Random Processes* (OUP, 2001)

- J. Jacod, P. Protter - *Probability Essentials* (Springer, 2004)
- D. Hand - *Statistics: A Very Short Introduction* (OUP, 2008)

Partial Differential Equations are also fundamental in Mathematical Finance, and we highly recommend the following book:

- S.J. Farlow - *PDEs for Scientists and Engineers* (Dover, 1993)

We highly recommend you familiarise yourself (or refresh your memories) on these topics. Grimmett and Stirzaker's book contains both standard Probability theory (random variables, generating functions, convergence), as well as some essential results—which will be covered in the MSc—on stochastic processes. Some familiarity with standard probability theory concepts would definitely be an advantage.

Note: The Imperial College library has many of them available online and some are freely (and legally) available on the Internet.

2.3.2 Finance

Even though the underlying tools of quantitative analysis in banks, hedge funds and FinTech are highly mathematical, one should not lose track of the surrounding contexts and objectives. Standard (non-mathematical) books about options derivatives are the following:

- J.C. Hull - *Options, Futures, and Other Derivatives* (Pearson, 2021)
- S. Neftci - *An Introduction to the Mathematics of Financial Derivatives* (Academic Press, 2000)

If you wish to learn about the history and the making of quantitative finance, we recommend the following easy-to-read novels, albeit to take with a pinch of critical mind:

- A. Admati, M. Hellwig - *The banker's new clothes* (PUP, 2014)
- S. Patterson - *The Quants, the Maths geniuses who brought down Wall Street* (Random House Business, 2011)
- M. Lewis - *Liar's Poker* (Hodder Paperbacks, 2006)
- M. Lewis - *Flash Boys* (Penguin, 2015)

Internet also contains a lot of information, and the following videos will get you familiar with quantitative finance:

- Interview: Marco Avellaneda: 'The era of the pure quant is over'.
- An interview with Jim Simons, the founder of Renaissance Technologies.

- Backlight - Money and Speed: Inside The Black Box
- A.E. Khandani and A.W. Lo - What Happened To The Quants In August 2007?

The following websites should also be checked regularly:

- Bloomberg is a financial software company providing analytics, Equity trading platform, data services, and news to financial companies.
- The Financial Times is a leading newspaper regarding business and economics.

2.3.3 Coding

Coding is an essential part of the daily task of quantitative analysts and data scientists, and C++ has historically been the main language in the financial industry. While we will teach you C++, it is highly recommended that you acquire preliminary notions. A good reference to start is

- B. Stroustrup (designer of C++) - *Programming: Principles and Practice Using C++* (Addison-Wesley Professional, 2024)

Aside from C++, Python has become an essential language in the (financial) industry; it is open source, interpreted, high-level, multipurpose and cross-platform. It also allows easy manipulation of data (with direct imports from Yahoo Finance or Google for example), an essential feature in the current Big Data context. Several modules in the MSc programme use Python, and we strongly recommend you have a first look at it. Full details about the language itself are available at www.python.org, and we recommend to install everything through www.anaconda.com. A good reference in the context of Finance is

- Y. Hilpisch - *Python for Finance: Analyze Big Financial Data* (O'Reilly, 2014)

There are of course many other useful programming languages and computing environment (R, C#, Java, MATLAB, S+, Q, KDB), but a large part of the financial industry (banks, hedge funds, regulators) seem to be now shifting towards a combination of C++ for speed and Python for ease of use and compatibility and for its wide-ranging libraries.

At the interface of Computing, Mathematics and Statistics, Machine Learning has become an essential tool in the financial industry, and a good overview is available at

OECD (2021) - *Artificial Intelligence, Machine Learning and Big Data in Finance: Opportunities, Challenges, and Implications for Policy Makers*

2.3.4 Interview Preparation

To start with, we strongly recommend the eFinancialCareers Banking Guide, where you will get an overview of the financial industry as well as general interview tips. The Michael Page's Interview tips are also extremely useful and should be read (and re-read regularly) carefully. There are zillions of websites, books, notes on how to prepare for this and that. We list below a few of them that should help you practice.

- X. Zhou - *A Practical Guide To Quantitative Finance Interview* (2008)
- M. Joshi - *Quant Job Interview Questions and Answers* (2013)
- D. Bester - *An Interview Primer for Quantitative Finance*
- T. Falcon Crack - *Heard on the Street, Quantitative Questions from Wall Street Job Interviews* (2007)
- D. Stefanica, R. Radoicic, T.-H. Wang - *150 Most Frequently Asked Questions on Quant Interviews* (FE Press, 2024)

Maths puzzles and Brainteasers

- TraderMaths
- Jane Street puzzles
- brainstellar
- puzzles.nigelcoldwell
- ZetaMac (basic questions, good starting point)

Coding

- CodeChef
- Sphere Online Judge
- PKU ONline Judge
- CodeForces Problem sets
- leetcode

Chapter 3

Interview Questions Examples

You will find below a few typical interview questions, from Mathematics, Coding, Finance and Personality. Most of them only assume Undergraduate level in Mathematics, but some can clearly be more demanding than others.

3.1 Finance

- What does the acronym CBOE stand for?
- What does the acronym FX stand for?
- What is the difference between buy-Side and Sell-Side?
- Can you tell me about some recent important financial news?
- What does an investment banking division do?
- Explain what you observe in the figure below.

3.2 Mathematics

Question 3.2.1. *Describe the difference between Bayesian vs Frequentist Statistics.*

Question 3.2.2. *Let* $\mathbf{Y} = \boldsymbol{\beta}\mathbf{X} + \boldsymbol{\varepsilon}$ *where* $\boldsymbol{\varepsilon}$ *is a vector of Normally distributed error terms with mean zero and variance* 1*. What is the value of optimal least-square estimator of* $\boldsymbol{\beta}$*?*

Question 3.2.3. *Let* $\{X_n\}_{n\in\mathbb{N}}$ *be an independent and identically distributed (iid) sequence such that for each* $n \in \mathbb{N}$*,* X_n *is a random variable taking values in* $\{-1, 1\}$ *with equal probability. For* $x \in \mathbb{R}$*, define now the parametric sequence* $(Z_n(x))_{n\in\mathbb{N}}$ *by*

$$Z_n(x) := \left(\exp\left\{\sum_{i=0}^{n} X_i - xn\right\}\right)_{n\in\mathbb{N}}, \qquad \text{for each } n \in \mathbb{N},$$

What value of x *makes the process* $(Z_n(x))_{n\in\mathbb{N}}$ *a discrete-time martingale, namely such that the conditional expectation* $\mathbb{E}[Z_n(x)|Z_{0:p}(x)]$ *is equal to* $Z_p(x)$ *for any* $0 \le p \le n$*? We denote here* $Z_{0:p}(x) := (Z_p(x), \dots, Z_0(x))$*.*

Question 3.2.4. *Throw a dice up to three times and you can repeat or take the money that you get, what is the expected value of the lottery?*

Question 3.2.5. *Solve the following equation:*

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{\cdots}}}} = x.$$

Question 3.2.6. *Which is larger,* e^{π} *or* π^{e}*?*

Question 3.2.7. *Probability of winning a game of tennis? Current score is 30-30 and you have a 0.6% chance of winning each point.*

Question 3.2.8. *What is* 2^{40}*?*

Question 3.2.9. *Given two data sets* $\mathcal{X}$ *and* $\mathcal{Y}$*, we run two linear regressions to obtain* $y \approx ax + b$ *and* $x \approx cy + d$*. What are the bounds on the product ac?*

Question 3.2.10. *How many piano tuners are there in Oxford?*

Question 3.2.11. *You have two ropes coated in an oil. Each rope takes exactly* 1 *hour to burn. However, the ropes do not burn at constant rates (there are spots where they burn a little faster and spots where they burn a little slower). With a lighter to ignite the ropes, how can you measure exactly* 45 *minutes?*

Question 3.2.12. *What is the angle between the hour hand and the minute hand when the clock shows* 3.15*pm?*

Question 3.2.13. *Suppose the two of us go into a pub and order a pint of beer. I drink half the pint. You drink half of the remainder. I drink half of what is left after this, and we continue. How much of the pint do I drink, and how much do you?*

Question 3.2.14. *For a fair coin, what is the expected number of tosses to get three heads in a row?*

Question 3.2.15. *The office hour session is assigned to be between 9am and 10am. The professor and the student then randomly pick a time within this interval, show up and wait for the other for fifteen minutes. What is the probability of them meeting?*

Question 3.2.16.

a) *Approximate* $\sqrt{145}$ *to four decimal places.*

b) *Approximate* $\sqrt{1.1}$ *to four decimal places.*

c) *Approximate* $\sqrt{0.1}$ *to four decimal places.*

Question 3.2.17. *We are in a queue ready to get over a plane, we are 100. The first guy in the queue looses his ticket and seats randomly, then each person, following the order, takes his seat unless it is already occupied, in that case he chooses randomly. What's the probability I seat in the correct seat if I'm boarding last?*

Question 3.2.18. *There are* n *ropes in a bag. Each time, you pick two rope ends at random, tie them together and put it back in the bag. The process terminates when there is no free ends left in the bag. What is the expectation of the number of loops in the bag at the end of the process?*

Question 3.2.19.

a) *What is the amount of following zeros at the end of* $100! + 200!$ *?*

b) *You roll a fair dice till you get 3. What is the expectation of number of rolls?*

c) *What is the probability of at least 2 people having the same day birthday within a group of* n *people?*

Question 3.2.20. *A population of amoebas starts with there being just one amoeba. After 1 period that amoeba can divide into 1, 2, 3, or 0 (it can die) with equal probability. What is the probability that the entire population dies out eventually?*

Question 3.2.21. *For a continuous random variable* X *on* $[0, 1]$*, give an upper bound of* $\mathbb{V}[X]$.

Question 3.2.22. *Take an ordinary deck of* 52 *playing cards. Turn up the cards from the top until the first ace appears. How many cards on average are required to be turned before seeing the first ace?*

Question 3.2.23. *You have* 12 *almost identical balls except for one with an odd weight (lighter or heavier). You are given a set of balance scales with three possible readings:*

- *Left is heavier;*

- *Right is heavier;*
- *Both sides are equal.*

Given that you can only use the scale three times, how do you find the odd ball?

Question 3.2.24. *What is the probability you draw two cards of the same color from a standard 52-card deck? You are drawing without replacement.*

Question 3.2.25. *A bag contains N socks, some of which are black, and some of which are red. If two random socks are picked, the probability that they are both red is $1/2$. What is the smallest possible value of N for which this is possible?*

Question 3.2.26. *Given three random variables X, Y and Z such that $\mathrm{corr}(X, Y) = \mathrm{corr}(X, Z) = \mathrm{corr}(Z, Y) = \rho$, what is the admissible range of values for ρ?*

Question 3.2.27. *Let $\mathfrak{X} := (X_1, \ldots, X_n)$ denote an iid sample from a Uniform distribution on $[0, 1]$. Compute $\mathbb{E}\left[\max(\mathfrak{X}) - \min(\mathfrak{X})\right]$.*

Question 3.2.28. *Compute the determinant of the matrix*

$$A = \begin{pmatrix} n & 1 & 1 & \cdots & 1 \\ 1 & n & 1 & \cdots & 1 \\ 1 & 1 & n & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n \end{pmatrix}.$$

Question 3.2.29. *There were 2 boys and 3 girls in a room. A new baby entered the room, equally likely to be a boy or a girl. A nurse then came into the room and picked one baby randomly. It turned out to be a boy. What's the probability that the new baby was a boy?*

Question 3.2.30. *You enter a game where you roll a dice up to 3 times. If you roll number x, you get this amount in £. The first time you roll the dice, you can decide to stop the game and pocket the amount obtained on the dice or to continue to the next roll. You can do so until the third and final roll, where you eventually receive the amount rolled and the game stops. Using an optimal strategy, how much do you expect to receive on average when playing this game ?*

Question 3.2.31.

- *What is the difference between* Machine Learning *and* Artificial Intelligence*?*
- *What is a* feedforward neural network*?*

3.3 General / Personality

- Why do you want to work in this industry?
- Tell us about yourself
- Where do you see yourself in five years?
- What's your greatest weakness?
- Why should I hire you?
- What is your greatest failure, and what did you learn from it?
- What motivates you?
- What is your programming experience?
- Can you explain to me what you did in XXX project?
- Why this company?
- Why this role?
- Tell us about a recent event in markets that has interested you
- Tell us about a recent development in tech/ML that has interested you
- Project related questions (from your studies/internships if applicable)
- Why should we pick you over the other candidates?
- What can you bring to the company?

3.4 Coding

Question 3.4.1. *What is inheritance in C++? What is encapsulation?*

Question 3.4.2. *What are the main characteristics of Python? What are its advantages and disadvantages?*

Question 3.4.3. *You are given a list of n integers. Find the biggest difference between any two integers in the list, such that larger element appears after the smaller one. Make sure that the time complexity is $\mathcal{O}(n)$.*

Question 3.4.4. *Given a total amount X (in British Pounds), suppose that we wish to obtain some change as coins, and we have infinite supply of each of $C = \{C_1, C_2, \ldots, C_m\}$ valued coins, what is the minimum number of coins to make the change? How many of each coin do we need? The time complexity should be $\mathcal{O}(mX)$.*

Question 3.4.5. *We are given numbers in form of triangle, by starting at the top of the triangle and moving to adjacent numbers on the row below, find the maximum total from top to bottom.*

```
Input :
   3
  7 4
 2 4 6
8 5 9 3
Output : 23
Explanation : 3 + 7 + 4 + 9 = 23
```

Question 3.4.6. *The following* `python` *code generates the figure below. Discuss any issues.*

```python
N = 10000
radius = np.random.rand(N)
angle = 2.*np.pi*np.random.rand(N)
x = radius*np.cos(angle)
y = radius*np.sin(angle)
plt.figure(figsize=(10, 10))
plt.scatter(x, y, s=2)
plt.show()
```

Listing 3.1: Generating random number on a disk

Question 3.4.7.

1. *Explain Python tuples and lists. What are immutable and mutable objects?*
2. *Explain Python dictionary, hashmap and hash functions.*
3. *Why adding a two* `numpy` *arrays is faster than adding two regular python arrays?*
4. *How are arrays stored in memory? How do you sort an array? Chose one algorithm and explain it.*

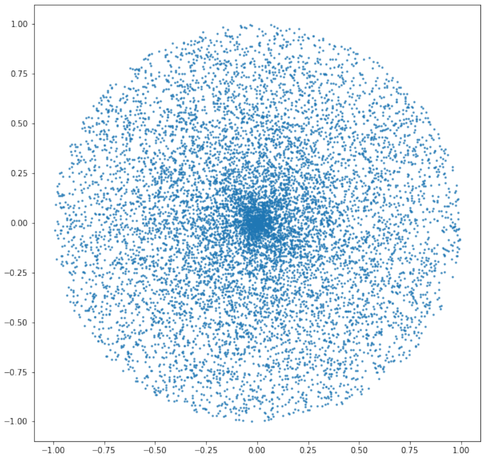

Figure 3.1: Figure output of Question 3.4.6

5. *Dictate me a hello world program in* **`python`** *from beginning to end.*

6. *How does a computer generate random numbers?*

Question 3.4.8. *Let $\mathcal{T} = \{t_i\}_{i=0,\dots,n}$ (for some strictly positive integer n) denote a discrete time grid of the time interval $[0, T]$, for example taking $t_i = \frac{iT}{n}$. Consider the discrete-time process $X = (X_t)_{t\in\mathcal{T}}$ defined as*

$$\begin{cases} X_{t_0} &= 0, \\ X_{t_{i+1}} - X_{t_i} &= \left(r - \dfrac{\sigma^2}{2}\right)(t_{i+1} - t_i) + \sigma\mathcal{N}_i, \qquad \text{for } i=0,\dots,\ n\text{-}1, \end{cases}$$

for some $r \in \mathbb{R}$ and $\sigma > 0$, and where $(\mathcal{N}_i)_{i=0,\dots,n-1}$ denotes an iid sequence Normal random variables with mean 0 and variance 1. We are interested in simulating m paths of X. The **`python`** *function below does so.*
Question: How could you modify this function to (still in `python`) to make it a lot faster?

```python
def simulateX(T, r, sigma, n, m):
    tt = np.linspace(0., T, n+1)
    dt = T / n
    x_paths = []
    for i in range(m):
        xpath = [0]
        for t in tt[1:]:
            N = np.random.normal(0.,1.)
            temp = xpath[-1] - sigma**2*dt / 2.+ sigma*np.sqrt(dt)*N
            X.append(temp)
        x_paths.append(xpath)
    return tt, x_paths
```

Question 3.4.9. *Given three points A, B, C that form a triangle and an additional point P, write a simple condition depending only on triangle areas to determine if P is inside the triangle (including the boundary).*

Question 3.4.10. *Given three positive integers i, j, k where $i, k < j$. Write a* `python` *routine to determine*

$$i + (i + 1) + \cdots + j + (j - 1) + \cdots + k.$$

Question 3.4.11. *Start with a given array of integers and an arbitrary initial value x. Write a* `python` *routine to calculate the running sum of x plus each array element, from left to right. The running sum must never get below 1. Write a* `python` *routine to determine the minimum value of x.*

Question 3.4.12. *Given a* `python` *list, how do you reverse it? What if it is a string?*

Question 3.4.13. *What is the difference between inheritance and polymorphism?*

Chapter 4

Solutions

4.1 Mathematics Solutions

Solution (Question 3.2.2)**.** *The optimal OLS estimator is:* $\widehat{\boldsymbol{\beta}} = \left(\mathbf{X}^\top \mathbf{X}\right)^{1} \mathbf{X}^\top \mathbf{Y}$.

Solution (Question 3.2.3)**.** *For any $n \geq p$, we have*

$$
\begin{aligned}
\mathbb{E}[Z_n(x)|Z_{0:p}(x)] &= \mathbb{E}\left[\exp\left\{\sum_{i=0}^{n} X_i - xn\right\}\middle| Z_{0:p}(x)\right] \\
&= \mathbb{E}\left[\exp\left\{\sum_{i=0}^{p} X_i + \sum_{i=p+1}^{n} X_i - xn\right\}\middle| Z_{0:p}(x)\right] \\
&= \mathbb{E}\left[\exp\left\{\left(\sum_{i=0}^{p} X_i - xp\right) + \left(\sum_{i=p+1}^{n} X_i - x(n-p)\right)\right\}\middle| Z_{0:p}(x)\right] \\
&= \mathbb{E}\left[Z_p(x)\exp\left\{\sum_{i=p+1}^{n} X_i - x(n-p)\right\}\middle| Z_{0:p}(x)\right] \\
&= Z_p(x)\mathbb{E}\left[\exp\left\{\sum_{i=p+1}^{n} X_i - x(n-p)\right\}\middle| Z_{0:p}(x)\right],
\end{aligned}
$$

where we used the fact that, conditional on $Z_{0:p(x)}$ then $Z_p(x)$ is known. To satisfy the martingale condition, we thus require the last conditional expectation to be equal to 1, *for any* $0 \leq p \leq n$. *Since the sequence* $(X_n)_{n\in\mathbb{N}}$ *is iid, then*

$$\begin{aligned}
\mathbb{E}\left[\exp\left\{\sum_{i=p+1}^{n} X_i - x(n-p\right\}\middle| Z_{0:p}(x)\right] &= \mathbb{E}\left[\exp\left\{\sum_{i=p+1}^{n} X_i\right\}\middle| Z_{0:p}(x)\right] \mathrm{e}^{-x(n-p)} \\
&= \left(\mathbb{E}\left[\mathrm{e}^{X_1}\middle| Z_{0:p}(x)\right]\right)^{n-p} \mathrm{e}^{-x(n-p)} \\
&= \left(\mathrm{e}^{1}\mathbb{P}(X_1 = 1) + \mathrm{e}^{-1}\mathbb{P}(X_1 = -1)\right)^{n-p} \mathrm{e}^{-x(n-p)} \\
&= \left(\frac{\mathrm{e} + \mathrm{e}^{-1}}{2}\right)^{n-p} \mathrm{e}^{-x(n-p)} \\
&= \left(\frac{\mathrm{e} + \mathrm{e}^{-1}}{2\mathrm{e}^{x}}\right)^{n-p}.
\end{aligned}$$

The only solution to this being equal to 1 *is then*

$$x = \ln\left(\frac{\mathrm{e} + \mathrm{e}^{-1}}{2}\right).$$

Solution (Question 3.2.4). *Let us approach this problem using optimal stopping for Markov chains with state space* Ω. *Our goal is to maximise the value function*

$$V(x) := \sup_{n\in\mathcal{T}} \mathbb{E}_x[f(X_n)],$$

where $\mathcal{T} = \{0, 1, 2, \dots\}$ *and* $f : \Omega \to \mathbb{R}_+$ *is the payoff function. The optimal strategy can be described as* $\tau^* := \inf\{n \in \mathcal{T} : X_n \in \mathcal{S}\}$, *where* $\mathcal{S} = \{x \in \Omega : f(x) = V(x)\}$ *is the stopping region. In our case the state space is*

$$\Omega = \Omega_0 \cup \{2, 1, 0\} \times \{1, 2, 3, 4, 5, 6\},$$

where Ω_0 *is the starting state,* $n \in \{2, 1, 0\}$ *is number of rolls we have left (*$n = 0$ *is an absorbing state) and* $d \in \{1, 2, 3, 4, 5, 6\}$ *is the number of dots. The payoff is simply the number of dots on the dice so* $f(x) = x$.

	n					
	2		1		0	
	v	x	**v**	x	**v**	x
	4.25	1	**3.5**	1	**1**	1
	4.25	2	**3.5**	2	**2**	2
d	**4.25**	3	**3.5**	3	**3**	3
	4.25	4	**4**	4	**4**	4
	5	5	**5**	5	**5**	5
	6	6	**6**	6	**6**	6
Pv	**4.67**		**4.25**		**3.5**	

We start at $n = 0$ *and work our way backward using* $V(x) = \max\{x, (Pv)(x)\}$, *where* $(Pv)(x) = \sum_{y\in\Omega} p(x, y)v(y)$ *with* $p(x, y) = \frac{1}{6}$. *So the expected value of the lottery is* 4.67 *and the optimal strategy is to keep rolling the dice until* $v(x) = x$.

Solution (Question 3.2.9). *For simplicity one can assume that both data sets have mean 0, as they can always be scaled to mean 0. Then we simply use the covariance definition of slope and finish with the Cauchy-Schwarz inequality on the sums. The bounds will be between 0 and 1.*

Solution (Question 3.2.10). *Initially this might seem like an impossible question. But the point of this question is to test how the answer is structured and which assumptions were used to get to there. Nobody expects you to give an exact answer. First we have to remember the population of Oxford, say* 150000. *We can assume that individuals do not usually own a piano, but that households do. The average household size is around* 5, *so there are about* 30000 *households in Oxford. Since people who live in Oxford seem to like playing the piano, we assume one piano for every* 5 *households, that is* 6000 *pianos. Next we assume that each household is somewhat diligent and tunes the piano once a year. An average piano tuner can tune, say* 4 *pianos a day, so if there are* 200 *working days in a year each piano tuner can tune* 800 *pianos per year. We can then approximate there are* $6000/800 \approx 8$ *piano tuners in Oxford.*

Solution (Question 3.2.11). *Light one of the ropes on fire on both ends and light the second rope on one end at the same time. When the first rope burns out,* 30 *minutes have passed. At that exact moment, you light the unlit end of the second rope. Because the second rope burned for half an hour, half an hour more remains. Lighting the other end at the moment the first rope burns up will cause the remaining part of the second rope to burn up in* 15 *minutes. Once the second rope has been consumed by the flames, exactly* 45 *minutes have passed.*

Solution (Question 3.2.12). *At* $3:15$, *the minute hand is perfectly horizontal pointing towards 3, whereas the hour hand is already moving towards* 4. *More precisely, the hour hand must have covered* $\frac{1}{4}$ *of angle between* 3 *and* 4. *The angle between two different digits is* $\frac{360°}{12} = 30°$. *Thus,* $\frac{1}{4}$ *of it is* $7.5°$.

Solution (Question 3.2.13). *I first drink half of the glass, leaving the other glass for you. You drink half of the remainder or forth of the entire glass. I drink half of the rest and so on... The amounts drank are, therefore*

$$\text{Me: } \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}}$$

$$\text{You: } \sum_{k=0}^{\infty} \frac{1}{2^{2k+2}} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}}.$$

Hence, I drank twice as much as you did, given that the entire pint was drunk. For this we need to check that $\sum_{k=1}^{\infty} \frac{1}{2^k}$ *converges to* 1.

Solution (Question 3.2.14). *There are many possible solutions to this problem. We show an interesting approach using a gambler strategy.*

We gamble in a way so that we make many on heads (H), but if tails (T) drop on the n-th toss, our position is $-n$. Meaning we gamble 1 unit on the first toss and on each toss after T. After one H we gamble 3, after two H we gamble 7 and so on, so that in case of T, we get back to $-n$ again. Our gambling winnings is thus a martingale, since we are making finite trades in a martingale.

After three H, our position is $11 - (n-3) = 14 - n$. Time taken to get three heads is a stopping time with a finite expectation, so if we stop at it, we still have a martingale (using the Optional Sampling Theorem). Thus

$$\mathbb{E}[14 - n] = 0$$

and $n = 14$, which is the solution, given that the stopping time really has a finite expectation. To show this, note that the probability the coins at tosses $3k, 3k+1, 3k+2$ are all heads is $\frac{1}{8}$. So the probability that we have not obtained three H in a row after $3k$ draws is less then $\left(\frac{1}{8}\right)^k$. The expected time to achieve three heads is therefore less than $\sum_k (3k)\left(\frac{1}{8}\right)^k$, which is clearly finite.

Solution (Question 3.2.15)**.** *The approach can be solved using integration, we take a route of expressing the problem as a two-dimensional event. Let us define two independent uniforms x, y on $[0,1]$. Then we ask what is the probability of $|x-y| \leq 0.25$. Put differently, what is the probability that a pair (x,y) picked from the uniform distribution on $[0,1] \times [0,1]$ lies in the set*

$$E = \{(x,y) \mid |x-y| \leq 0.25\}.$$

The set E will correspond to a strip inside $[0,1] \times [0,1]$ with sides $y = x + 0.25$ and $y = x - 0.25$. Now that the complement of this set is two congruent right-angled triangles. Both short sides of these triangles are of length 0.75. The two triangles together form a square with side $3/4$, giving an area of $9/16$. The final answer is thus $1 - \frac{9}{16} = \frac{7}{16}$.

Solution (Question 3.2.16)**.** *Here are two different ways to approach this problem:*

a) Using the continued fractions:

$$\sqrt{n^2+1} = n + \cfrac{1}{2n + \cfrac{1}{2n + \cfrac{1}{2n + \cfrac{1}{2n + \ldots}}}}$$

and

$$\sqrt{12^2+1} \approx 12 + \frac{1}{24} \approx 12.0416$$

b) Using the Taylor expansion around one:

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 (4^n)} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$

so

$$\sqrt{1+0.1} \approx 1 + \frac{1}{2}\frac{1}{10} - \frac{1}{8}\frac{1}{100} \approx 1.0468$$

Solution (Question 3.2.17). *Consider the complement of the event i.e. not getting our sit. We now analyse the probability of our sit being taken, while the passengers take their sits one by one.*

Passenger 1. $p_1 = \frac{1}{100}$

Passenger 2. $p_2 = \frac{1}{100} + \frac{1}{100}\frac{1}{99} = p_1\left(1 + \frac{1}{99}\right)$

Passenger 3. $p_3 = \frac{1}{100} + \frac{1}{100}\frac{1}{99} + \frac{1}{100}\frac{1}{98} + \frac{1}{100}\frac{1}{98}\frac{1}{97} = p_2\left(1 + \frac{1}{98}\right)$
$\dots$

Passenger k. $p_k = p_{k-1}\left(1 + \frac{1}{101-k}\right)$.

Solving this recursively we can spot a pattern

Passenger 1. $p_1 = \frac{1}{100}$

Passenger 2. $p_2 = \frac{1}{99}$

Passenger 3. $p_3 = \frac{1}{98}$
$\dots$

Passenger k. $p_k = \frac{1}{101-k}$.

So for the passenger just before us

Passenger 99. $p_{99} = \frac{1}{101-99} = \frac{1}{2}$.

And the probability of the complement event is

$$p = 1 - \frac{1}{2} = \frac{1}{2}.$$

Solution (Question 3.2.18). *Each time one picks two rope ends either one of two things happen can happen:*

1. *the two ends are of the same rope, therefore one additional loop was added and there are $n-1$ ropes left (loop does not have ends to pick any more)*

2. *the two ends are of different ropes, no loop was added, and effectively the two ropes were replaced with a longer rope and again only $n-1$ ropes are left*

There are $\binom{2n}{2}$ ways to pick two rope ends from the pile in the firs step and only n of them result in Case (1). The probability of Case (1) happening is thus

$$\frac{n}{\binom{2n}{n}} = \frac{1}{2n-1}.$$

The expected number of loops added in the first step is therefore simply

$$\left(\frac{1}{2n-1}\right)1+\left(1-\frac{1}{2n-1}\right)0=\frac{1}{2n-1}$$

Then one start over, but now with only $n-1$ ropes. Relying on the linearity of the expectation we obtain the expected number of loops

$$\frac{1}{2n-1}+\frac{1}{2n-3}+\cdots+\frac{1}{3}+1.$$

Solution (Question 3.2.19)**.**

a) *A number $a\in\mathbb{R}$ is divisible by 10 if it is divisible by both 5 and 2. The largest power of 10 that divides a is thus the minimum of the largest power of 5 that divides it or the largest power of 2 that divides a. It is also clear that the power of 2 dividing $n!$ is larger than the power of 5 dividing $n!$. We, therefore, need to find the largest power of 5 that divides $n!$. One cannot simply divide n by 5, since there exist cases like the number 25 for example, which are divisible by 5^2 and have to be re-counted. This gives us the following formula for the number of trailing zeros in a factorial $n!$*

$$\text{TrailingZeros}(n)=\sum_{i=1}^{\lfloor\log_5 n\rfloor}\left\lfloor\frac{n}{5^i}\right\rfloor.$$

Now notice, that the number of tailing zeros is equal to the number of tailing zero's in the smallest factorial. Therefore

$$\text{TrailingZeros}(100)=\sum_{i=1}^{\lfloor\log_5 100\rfloor}\left\lfloor\frac{100}{5^i}\right\rfloor=\sum_{i=1}^{2}\frac{100}{5^i}=\frac{100}{5}+\frac{100}{25}=24\,.$$

b) *The geometric distribution gives the probability that the first occurrence of success requires k independent trials, each with success probability p. The expected value of a geometrically distributed random variable $X\sim\text{Geo}(p)$ is $\mathbb{E}[X]=\frac{1}{p}$. So in the case of a 6-sided dice, we have $p=\frac{1}{6}$ and the expected number of trials is 6.*

c) *Let us simplify the problem by disregarding the leap years (can you think of a solution that considers leap years as well?). It is easier to consider the compliment of this situation i.e. the one in which at least two people out of n have the same birthday. The probability of this happening is*

$$\frac{365!}{(365-n)!365^n}$$

and the solution is hence

$$1-\left(\frac{365!}{(365-n)!365^n}\right).$$

Solution (Question 3.2.20)**.** *Let us assume that there exists such a probability of extinction p. Probability for a population of amoebas to become extinct is equal to a sum of probabilities of the offspring to become extinct, therefore:*

$$p_{\text{parent}} \; = \frac{1}{4}p^3_{\text{offspring}} + \frac{1}{4}p^2_{\text{offspring}} + \frac{1}{4}p_{\text{offspring}} + \frac{1}{4}$$

Setting equal both probabilities, we have

$$p = \frac{1}{4}p^3 + \frac{1}{4}p^2 + \frac{1}{4}p + \frac{1}{4}.$$

The roots of the polynomial are $p_1 = 1, p_2 = \sqrt{2} - 1, p_3 = -\sqrt{2} - 1$. We can immediately exclude the root p_3, since probabilities cannot be negative. We can also exclude p_1, because the root represents the following statement

> ***If*** *the probability for the offspring to become extinct is equal to* 1 ***then*** *the probability for the parent to become extinct is equal to* 1

which is a vacuous truth, thus the correct answer is $p = \sqrt{2} - 1$.

Solution (Question 3.2.21)**.** *The easiest way to tackle this problem is by the use of Popoviciu's inequality on variances, which gives us an upper bound on the variance σ^2 of any bounded probability distribution. It states*

$$\sigma^2 \leq \frac{1}{4}\left(\sup\{x \in \operatorname{supp}(X)\} - \inf\{x \in \operatorname{supp}(X)\}\right)^2.$$

So the upper bound on the variance of a variable over $[0, 1]$ is simply

$$\mathbb{V}[X] \leq \frac{(1-0)^2}{4} = \frac{1}{4}.$$

Solution (Question 3.2.22)**.** *There are four Aces and 48 other cards; label those 48 from 1 to 48 and introduce*

$$X_i := \begin{cases} 1, & \text{if card } i \text{ is flipped before all the aces,} \\ 0, & \text{otherwise.} \end{cases} \qquad \text{and} \qquad X := \sum_{i=1}^{48} X_i.$$

It is easy to see that X represents the number of non-Ace cards flipped before seeing the first ace, so that the answer to the question is $1 + \mathbb{E}[X]$. Since the sequence $(X_i)_{i=1,\dots,48}$ is iid Bernoulli, then X is a Binomial random variable with $\mathbb{E}[X] = 48\pi$, where $\pi := \mathbb{P}[X_1 = 1]$. Now, the cards from top to bottom can be viewed as the following sequence:

$$\cdots\cdots A \cdots\cdots A \cdots\cdots A \cdots\cdots A \cdots\cdots$$

Since the deck is ordinary and shuffled randomly, then the non-Ace card i can be in any of the dotted gaps, in particular in the first one with probability $\mathbb{P}[X_i = 1] = \frac{1}{5}$. Thus

$$1 + \mathbb{E}[X] = 1 + \sum_{i=1}^{48} \mathbb{E}[X_i] = 1 + \sum_{i=1}^{48} \frac{1}{5} = 10.6.$$

Solution (Question 3.2.23). *We proceed by dichotomy. We name the balls* 1 *to* 12 *for clarity, and split them into three groups* $G_1 := [1, 2, 3, 4]$, $G_2 := [5, 6, 7, 8]$, $G_3 := [9, 10, 11, 12]$, *and write* $W(G)$ *the weight of group* G. *We denote by* X *the odd ball.*

- *If* $W(G_1) = W(G_2)$, *then* $X \in G_3$. *Split then* $F_1 := [6, 7, 8]$ *and* $F_2 := [9, 10, 11]$*:*
  - *If* $W(F_1) = W(F_2)$, *then* $X = 12$, *and check against any other ball to see whether it is heavier or lighter.*
  - *If* $W(F_1) > W(F_2)$, *then* F_1 *contains a heavy ball, so if* $W([9]) = W([10])$, *then* $X = 11$ *is heavy, else* X *is the heavier between* 9 *and* 10*;*
  - *If* $W(F_1) < W(F_2)$, *then* F_1 *contains a light ball, so if* $W([9]) = W([10])$, *then* $X = 11$ *is light, else* X *is the lighter between* 9 *and* 10*;*
- *If* $W(G_1) > W(G_2)$, *then consider* $F_1 = [5, 6, 1]$ *and* $F_2 = [7, 2, 12]$.
  - *If* $W(F_1) = W(F_2)$, *then* $X = 8$ *is the light ball or* $X \in \{3, 4\}$ *is the heavy ball. Check* $W([3])$ *vs* $W([4])$.
  - *and so on...*
- *If* $W(G_2) > W(G_1)$, *take* $F_1 = [1, 2, 5]$ *and* $F_2 = [3, 6, 12]$, *and proceed as above.*

Solution (Question 3.2.24). *You either draw a black or a red card first. Then, there are* 51 *cards left in the deck and* 25 *of these cards have the same color. Thus, the probability is* 25/51.

Solution (Question 3.2.25). *Let* R_i *indicate the event where the* i*-th sock is red and* r *be the number of red socks in a bag of* N *socks. We have, by assumption,*

$$\frac{1}{2} = \mathbb{P}[R_1 \cap R_2] = \mathbb{P}[R_1]\mathbb{P}[R_1|R_2] = \frac{r}{N}\frac{r-1}{N-1}.$$

When $N = 2$, *this probability should be equal to one, so it does not work. For* $N = 3$,

$$\frac{r}{3}\frac{r-1}{2} = \frac{r(r-1)}{6} = \frac{1}{2},$$

but this has no integer solution. When $N = 4$,

$$\frac{r}{4}\frac{r-1}{3} = \frac{r(r-1)}{12} = \frac{1}{2},$$

has an integer solution $r = 3$, *so the smallest possible value for* N *is* 4.

Solution (Question 3.2.26). *The correlation matrix of the vector* (X, Y, Z) *is equal to*

$$\mathrm{R} := \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix},$$

and we can compute its determinant, using the matrix principal minors, as

$$\begin{aligned}\det(\mathrm{R}) &= \det\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix} - \rho\det\begin{pmatrix}\rho & \rho\\ \rho & 1\end{pmatrix} + \rho\det\begin{pmatrix}\rho & 1\\ \rho & \rho\end{pmatrix}\\ &= 1-\rho^2-\rho(\rho-\rho^2)+\rho(\rho^2-\rho)\\ &= 1-3\rho^2+2\rho^3.\end{aligned}$$

The plot below shows the value of the determinant as a function of the correlation parameter ρ. Now, we know that a correlation matrix has to be positive semi-definite,

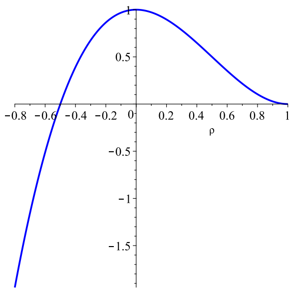

Figure 4.1: Plot of the determinant of R as a function of the correlation ρ.

which in particular holds as soon as the determinant is greater or equal to zero. This clearly happens as soon as $\rho \geq -\frac{1}{2}$, and therefore the range of admissible values is $[-\frac{1}{2}, 1]$.

Note: *You could also note that $3+6\rho = \mathbb{V}(X+Y+Z) \geq 0$ and conclude from there.*

Solution (Question 3.2.27)**.** *We proceed by integration. Since for any $z \in [0,1]$, $\max(\mathfrak{X}) \leq z$ is equivalent to $X_i \leq z$, for all $i = 1, \dots, n$, by independence we can write*

$$\begin{aligned}\mathbb{E}\left[\max(\mathfrak{X})\right] &= \int_0^1 \cdots \int_0^1 \max(x_1,\dots,x_n) f_{X_1}(x_1)\cdots f_{X_n}(x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n\\ &= \int_0^1 \cdots \int_0^1 \max(x_1,\dots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n\\ &= n\int_0^1 x_1\left(\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_1}\mathrm{d}x_n\right)\mathrm{d}x_1\\ &= n\int_0^1 x_1^n\mathrm{d}x_1 = \frac{n}{n+1}.\end{aligned}$$

where in the second line, we argue by symmetry. Therefore

$$\mathbb{E}\left[\max(\mathfrak{X})-\min(\mathfrak{X})\right] = 2\mathbb{E}\left[\max(\mathfrak{X})\right]-1 = \frac{n-1}{n+1}.$$

Solution (Question 3.2.28). *Summing all the rows into the last one gives*

$$B = \begin{pmatrix} n & 1 & 1 & \cdots & 1 \\ 1 & n & 1 & \cdots & 1 \\ 1 & 1 & n & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n-1 & 2n-1 & 2n-1 & \cdots & 2n-1 \end{pmatrix}.$$

Consider the matrix

$$\widetilde{B} = \begin{pmatrix} n & 1 & 1 & \cdots & 1 & 1 \\ 1 & n & 1 & \cdots & 1 & 1 \\ 1 & 1 & n & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & n & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

i.e. replacing the last row of B by 1, then $\det(B) = (2n-1)\det(\widetilde{B})$. Subtract this row from all the other rows, so that $\widetilde{B}$ becomes

$$\overline{B} = \begin{pmatrix} n-1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

This is a lower triangular matrix, so that its determinant is precisely equal to the product of the diagonal entries, namely $(n-1)^{n-1}$, and therefore $\det(A) = (2n-1)(n-1)^{n-1}$.

Solution (Question 3.2.29). *3/5*

Solution (Question 3.2.30). *4.66*

4.2 Coding Solutions

Solution (Question 3.4.9). *check if area(ABC)=area(A,B,P)+area(A,P,C)+area(P,B,C).*

Solution (Question 3.4.10).

```
x=j-i
y=j-k
return (x+1)i + .5*(x+1)x + yj - .5*(y+1)y
```

Solution (Question 3.4.11).

```
sum=1
reverse iteration
if sum<1:
    sum=1
sum-=v[i]
```

Solution (Question 3.4.6)**.** *Consider the two-dimensional random variable (X, Y) distributed as uniform over the disc $\Gamma := \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$. Using polar coordinates where R denotes the radius in $[0, 1]$ and Θ the angle in $[0, 2\pi]$, we can write*

$$\begin{cases} X = R\cos(\Theta), \\ Y = R\sin(\Theta). \end{cases} \tag{4.2.1}$$

The joint density of (X, Y) reads $f_{X,Y}(x, y) = \frac{1}{\pi}\mathbf{1}_{\{(x,y)\in\Gamma\}}(x, y)$ (since the area of Γ is equal to π) and thus the joint density of (R, Θ) reads (by change of variables)

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(X(r, \theta), Y(r, \theta)) \det(J(r, \theta)), \tag{4.2.2}$$

where J denote the Jacobian matrix

$$J(r, \theta) := \begin{pmatrix} \partial_r X(r, \theta) & \partial_r Y(r, \theta) \\ \partial_\theta X(r, \theta) & \partial_\theta Y(r, \theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{pmatrix}.$$

Its determinant is $\det(J(r, \theta)) = r\left(\cos(\theta)^2 + \sin(\theta^2)\right) = r$ *so that* (4.2.2) *becomes*

$$f_{R,\Theta}(r, \theta) = \frac{r}{\pi}\mathbf{1}_{\{(X(r,\theta),Y(r,\theta))\in\Gamma\}}(X(r, \theta), Y(r, \theta)) = \frac{r}{\pi}\mathbf{1}_{\{(r,\theta)\in[0,1]\times[0,2\pi]\}}(r, \theta)$$

and in particular the marginal densities of R and Θ read

$$\begin{aligned} f_\Theta(\theta) &= \int_0^1 f_{R,\Theta}(r, \theta)\mathrm{d}r = \frac{1}{2\pi}, \qquad \textit{for } \theta \in [0, 2\pi], \\ f_R(r) &= \int_0^{2\pi} f_{R,\Theta}(r, \theta)\mathrm{d}\theta = 2r, \qquad \textit{for } r \in [0, 1]. \end{aligned}$$

This implies that, while the angle Θ is uniformly distributed over $[0, 2\pi]$, the radius R is not and its cumulative distribution function reads $\mathbb{P}[R \leq r] = \int_0^r 2z\mathrm{d}z = r^2$.

Suppose that instead the polar coordinates (4.2.1) *read*

$$\begin{cases} X = \psi(R)\cos(\Theta), \\ Y = \psi(R)\sin(\Theta). \end{cases} \tag{4.2.3}$$

An analogous analysis yields that the density of the couple (R, Θ) reads

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(X(r, \theta), Y(r, \theta)) \det(J_\psi(r, \theta)), \tag{4.2.4}$$

where J denote the Jacobian matrix

$$J_\psi(r, \theta) := \begin{pmatrix} \partial_r X(r, \theta) & \partial_r Y(r, \theta) \\ \partial_\theta X(r, \theta) & \partial_\theta Y(r, \theta) \end{pmatrix} = \begin{pmatrix} \psi'(r)\cos(\theta) & \psi'(r)\sin(\theta) \\ -\psi(r)\sin(\theta) & \psi(r)\cos(\theta) \end{pmatrix}.$$

Its determinant now reads

$$\det(J_\psi(r,\theta)) = \psi'(r)\psi(r)\left(\cos(\theta)^2 + \sin(\theta^2)\right) = \psi'(r)\psi(r)$$

and therefore (4.2.4) *becomes*

$$f_{R,\Theta}(r,\theta) = \frac{r}{\pi}\mathbf{1}_{\{(X(r,\theta),Y(r,\theta))\in\Gamma\}}(X(r,\theta),Y(r,\theta)) = \frac{\psi'(r)\psi(r)}{\pi}\mathbf{1}_{\{(r,\theta)\in[0,1]\times[0,2\pi]\}}(r,\theta)$$

with marginal densities

$$f_\Theta(\theta) = \int_0^1 f_{R,\Theta}(r,\theta)\mathrm{d}r = \int_0^1 \frac{\psi'(r)\psi(r)}{\pi}\mathrm{d}r = \frac{\psi(1)^2-\psi(0)^2}{2\pi}, \qquad \textit{for } \theta\in[0,2\pi],$$
$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta)\mathrm{d}\theta = \int_0^{2\pi} \frac{\psi'(r)\psi(r)}{\pi}\mathrm{d}\theta = 2\psi'(r)\psi(r), \qquad \textit{for } r\in[0,1].$$

and $\mathbb{P}[R\le r] = \int_0^r 2\psi'(z)\psi(z)\mathrm{d}z = \psi(r)^2-\psi(0)^2$*. In order to obtain a uniform distribution for* (R,Θ) *over* $[0,1]\times[0,2\pi]$*, the only possibility is therefore* $\psi(r)=\sqrt{r}$*.*

Note [extra]: *Consider now the case of a function* $\psi(r)=r^\gamma$ *for* $\gamma>0$*. Clearly* Θ *remains uniformly distributed over* $[0,2\pi]$ *since* $\psi(1)^2-\psi(0)^2=1$*. But now*

$$\mathbb{P}[R\le r] = \int_0^r 2\psi'(z)\psi(z)\mathrm{d}z = \psi(r)^2 = r^{2\gamma}.$$

The smaller the value of γ *the more mass accumulates around the origin of the disk, and vice versa. A graphical help can be seen in the following pictures:*