A new approach to liquidity risk

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• What is *liquidity risk*?

  – **Treasurer**’s answer: the risk of running short of cash

  – **Trader**’s answer: the risk of trading in *illiquid* markets, i.e. markets where exchanging assets for cash may be difficult or uncertain

  – **Central Bank**’s answer: the risk of concentration of cash among few economic agents

  ⇒ Setting a precise mathematical framework is not easy
● The theoretical framework
  – Portfolios and Marginal Supply-Demand Curves
  – Liquidation value vs. usual mark-to-market value
  – Liquidity policies and general mark-to-market values

● Coherent/convex risk measures and liquidity risk

● Some numerical examples
It is possible to trade in

- \( N \geq 1 \) illiquid assets
- cash, which is by definition the only liquidity risk-free asset

We define

- A **portfolio** is a vector \( \mathbf{p} \in \mathbb{R}^{N+1} \)
- \( p_0 \) is the amount of cash
- \( \mathbf{p} = (p_1, \ldots, p_N) \) is the assets’ position
- \( p_n \) is the number of assets of type \( n \)
- Perfectly liquid market ($S_0(t) \equiv 1$)
  
  - $S_n(t)$ is the unique price, at time $t$, for selling/buying a unit of asset $n$; this price does **not** depend on the size of the trade
  
  - $V(p, t) = p_0 + \sum_{n=1}^{N} p_n S_n(t)$ is linear

- Illiquid markets ($S_0(t) \equiv 1$)
  
  - $S_n(t) = S_n(t, x)$ will depend on the size $x \in \mathbb{R}$ ($x > 0$ is a sale) of the trade
  
  - $V(p, t)$ need not be linear anymore. A first idea is:
    \[
    V(p, t) = p_0 + \sum_{n=1}^{N} p_n S_n(t, p_n)
    \]
  
  - But this is not the only sensible notion of value
a typical MSDC

Bids

Asks
Some basic definitions

- A Marginal Supply-Demand Curve (msdc) is a decreasing function
  \[ m : \mathbb{R} \to (0, +\infty) \]

- \( m^+ = m(0+) \) and \( m^- = m(0-) \) are the best bid (sell) and ask (buy) prices. Of course \( m^+ \leq m^- \).

- Let \( x \) be the size of the transaction (\( x > 0 \) sale, \( x < 0 \) purchase). The unit price is
  \[ S(x) = \frac{1}{x} \int_0^x m(y) \, dy > 0 \]
  and the proceeds are
  \[ P(x) = xS(x) = \int_0^x m(y) \, dy \geq 0 \]

"Same" setting as in Cetin-Jarrow-Protter 2005, but our focus is on \( m \).
We can also allow for (care is needed with the details):

- Assets which are not securities (e.g. swaps) and can display negative (marginal) prices: \( m : \mathbb{R} \to \mathbb{R} \)

- Securities with finite depth market: \( m(x) = +\infty \) for \( x << 0 \) and/or \( m(x) = 0 \) for \( x >> 0 \)

- Swaps with finite depth market: \( m(x) = +\infty \) for \( x << 0 \) and/or \( m(x) = -\infty \) for \( x >> 0 \)
Given \( m = (m_1, \ldots, m_N) \) a vector of msdc. Let \( p \in \mathbb{R}^{N+1} \) be a portfolio.

- The **Liquidation Value** of \( p \) is
  \[
  L(p) = p_0 + \sum_{n=1}^{N} p_n S_n(p_n) = p_0 + \sum_{n=1}^{N} \int_{0}^{p_n} m_n(x) \, dx
  \]

- The **Usual Mark-to-Market Value** of \( p \) is
  \[
  U(p) = p_0 + \sum_{p_n > 0} p_n m_n^+ + \sum_{p_n < 0} p_n m_n^-
  \]
  as if only the best bid and ask would matter.

Note that \( U(p) \geq L(p) \) for any \( p \).
Some properties of $L$ and $U$:

**concavity.** both $L$ and $U$ are *concave* (but not linear)

**additivity.** $L$ is *subadditive* ($L(p + q) \leq L(p) + L(q)$) whenever $p$ and $q$ are concordant ($p_n q_n \geq 0$ for $n \geq 1$); it is *superadditive* for discordant portfolios $U$ is always *superadditive*, and it is additive for concordant portfolios

**scaling** if $\lambda \geq 1$

\[
L(\lambda p) \leq \lambda L(p) \quad U(\lambda p) = \lambda U(p)
\]
• Liquidation Mark-to-Market Value ($L$):
  measure of the portfolio value as if we are forced to entirely liquidate it (so, liquidity risk is a big concern)

• Usual MtM Value ($U$):
  measure of the portfolio value as if we don’t have to liquidate even a small part of it (so, liquidity risk is not a concern)

Our aim is to introduce notions of value between the two extreme cases. Whether and what to liquidate is a need that may vary.
First we give a notion of acceptability for a portfolio:

**A liquidity policy** is a convex and closed subset $\mathcal{L} \subseteq \mathbb{R}^{N+1}$ such that

1. $p \in \mathcal{L}$ implies $p + a \in \mathcal{L}$ for any $a \geq 0$ (adding cash cannot worsen the liquidity properties of a portfolio)

2. $(p_0, \overrightarrow{0}) \in \mathcal{L}$ implies $(p_0, \overrightarrow{0}) \in \mathcal{L}$ (if a portfolio is acceptable, its cash component is acceptable as well)

$\mathcal{L}$ collects the portfolios whose liquidity risk is not a concern and thus may be valued through $U$
Examples of liquidity policies:

1. \( \mathcal{L} = \mathbb{R}^{N+1} \) \hspace{1cm} Every portfolio is acceptable: no need to liquidate (this will lead to \( U \))

2. \( \mathcal{L} = \{ \mathbf{p} : \overrightarrow{p} = \overrightarrow{0} \} \) \hspace{1cm} Only pure-cash portfolios are acceptable: need to entirely liquidate \( \mathbf{p} \) (this will lead to \( L \))

3. \( \mathcal{L} = \{ \mathbf{p} : p_0 \geq a \} \) (\( a \geq 0 \) fixed) \hspace{1cm} This is a typical requirement imposed by the ALM of an institution

4. other examples may be based on bounds on concentration...
1. Start with a portfolio $p$, which need not be acceptable

2. Make it acceptable by liquidating the assets’ (sub)position $q \in \mathbb{R}^N$
   
   \[ r = p - q + L(0, q) = (p_0 + L(0, q), p - q) \in \mathcal{L} \]

3. Find the best way to do this, maximizing the Usual MtM value $U(r)$

4. Note that $L$ is used in 2. and $U$ in 3.:
   in 2. we care about liquidity risk, in 3. we don’t as $r \in \mathcal{L}$
— Having fixed a liquidity policy \( \mathcal{L} \) we can define the **associated MtM Value** (\( \sup \emptyset = -\infty \))

\[
V_L(p) = \sup \{ U(r) : r = p - \overrightarrow{q} + L(0, \overrightarrow{q}) \in \mathcal{L}, \overrightarrow{q} \in \mathbb{R}^N \}
\]

— \( r^* \in \mathbb{R}^{N+1} \) is optimal if \( V_L(p) = U(r^*) \).

It is immediate to see that \( V_L(r^*) = U(r^*) = V_L(p) \) (there is no change in value passing from \( p \) to \( r^* \))

— The set over which \( U \) (concave) is maximized is convex. Thus

The optimization program defining \( V_L \) is always **convex**.
1. If $\mathcal{L} = \mathbb{R}^{N+1}$, then

$$V_{\mathcal{L}}(p) = U(p)$$

2. If $\mathcal{L} = \{p : \vec{p} = \vec{0}\}$, then

$$V_{\mathcal{L}}(p) = L(p)$$

3. If $\mathcal{L} = \{p : p_0 \geq a\}$, then

$$V_{\mathcal{L}}(p) = \sup\{U(p - \vec{q}) + L(0, \vec{q}) : L(0, \vec{q}) \geq a - p_0, \vec{q} \in \mathbb{R}^{N+1}\}$$

which is not trivial (and non-linear)
• If $\mathcal{L} \subset \mathcal{L}'$, then $V_{\mathcal{L}} \leq V_{\mathcal{L}'}$.

• Thus,

$$V_{\mathcal{L}}(p) \leq U(p) \quad \forall \mathcal{L}$$

• For any $\mathcal{L}$, $V_{\mathcal{L}}$ is concave and translational supervariant

$$V_{\mathcal{L}}(p + a) \geq V_{\mathcal{L}}(p) + a \quad \forall a \geq 0$$
• As the problem defining $V_\mathcal{L}$ is convex, many fast algorithms are available

• An analytical solution is sometime easy. Assume:

  – $\{p : p_0 \geq a\}$

  – $m_i$ continuous and strictly decreasing $\forall i$

Then

  – if $p_0 \geq a$ ($p \in \mathcal{L}$) then $r^* = p$ and $V_\mathcal{L}(p) = U(p)$

  – if $p_0 < 0$ then

    $$r^*_i = m_i^{-1} \left( \frac{m_i(0)}{1 + \lambda} \right)$$

    where $\lambda$ is determined by $L(r^*) = p_0 - a$. 

Coherent risk measures (CRM) $\rho : L \rightarrow \mathbb{R}$ ($L$ space of r.v.) are characterized by (Artzner-Delbaen-Eber-Heath-98)

1. **Translation invariance**: $\rho(X + c) = \rho(X) - c \quad \forall c \in \mathbb{R}$;
2. **Monotonicity**: $\rho(X) \leq \rho(Y)$ whenever $X \geq Y$
3. **Positive homogeneity**: $\rho(\lambda X) = \lambda \rho(X) \quad \forall \lambda \geq 0$
4. **Subadditivity**: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Axioms 3 and 4 do not seem to take into account liquidity risk: *if I double my portfolio, its risk should more than double in many cases.*

They were replaced (Follmer-Schied02, Frittelli-Rosazza02) by the weaker axiom of convexity.
In our opinion, CRM are appropriate to deal with liquidity risk. The key point is that:

If I double my portfolio... means $p \rightarrow 2p$, not $X \rightarrow 2X$

The relation between $p$ and its value $X$ is **not linear**.

We define risk measures defined directly on portfolios $R = R(p)$ that are not necessarily positively homogeneous or subadditive.
Given

- a liquidity policy $\mathcal{L}$
- a probability space $(\Omega, \mathcal{F}, P)$ describing randomness up to $T > 0$
- a coherent risk measure defined on some $L \subset L^0(\Omega, \mathcal{F}, P)$
- the random future msdc: $(m_i(x, T))$ for any $i$, where
  - for any $x$, $m_i(x, T)$ is a r.v.
  - for any $\omega$, $x \mapsto m_i(x, T)(\omega)$ is decreasing

We compute $V_\mathcal{L}(p) = V_\mathcal{L}(p, T)(\omega)$ for any $\omega$ (it is a r.v.) and set

$$R_\mathcal{L}(p) = \rho(V_\mathcal{L}(p))$$
Some properties (for general $\mathcal{L}$ and $\rho$)

1. $R_\mathcal{L}$ is convex

2. $R_\mathcal{L}$ is translational subvariant: $R_\mathcal{L}(p + c) \leq R_\mathcal{L}(p) - c$

3. $R_\mathcal{L}$ is in general not homogeneous, nor subadditive

4. specific properties for $R_\mathcal{L}$ may be derived from properties of $V_\mathcal{L}$ (and coherency of $\rho$)

5. no monotonicity property can be introduced for $R$
Consider \((T \text{ is fixed})\)

\[ m_i(x) = \alpha_i \exp\{-\beta_i x\}, \]

where, \(A_i > 0\) and \(\beta_i \geq 0\) are r.v. There can be

- **Market risk only**:
  \(\alpha_i\) jointly lognormal, \(\beta_i = 0\)

- **Market and "non-random" liquidity risk**:
  \(\alpha_i\) jointly lognormal, \(\beta_i > 0\) non-random

- **Market and independent random liquidity risk**:
  \((\alpha_i, \beta_i)_i\) jointly lognormal, with \(\alpha_i \perp \beta_i\)

- **Market and correlated random liquidity risk**:
  \((\alpha_i, \beta_i)_i\) jointly lognormal, with \(\alpha_i\) and \(\beta_i\) negatively correlated

- **Market and correlated random liquidity risk with shocks**:
  \((\alpha_i, \tilde{\beta_i})_i\) jointly lognormal, with \(\alpha_i\) and \(\tilde{\beta}_i\) negatively correlated, \(\beta_i = \tilde{\beta}_i + \varepsilon_i\)
For a given portfolio $p$ and $\mathcal{L} = \{ q : q_0 \geq a \}$, in any of the 5 previous situations we:

- set $I = 10$, $\alpha_i$ and $\beta_i$ id. distr. for different $i$ 
- we perform 100k simulations of $(m_i(x))_i$ 
- for any outcome of the simulation we compute $V_\mathcal{L}(p)$ 
- we repeat for different inputs $(p, a, \text{mean, variances and correlations of } \alpha_i \text{ and } \beta_i)$

A typical outcome is:
Messages:

- Liquidity risk arises when msdc are ignored
- Liquidity risk can be captured by a redefinition of the concept of value, which depends on a liquidity policy
- Coherent risk measures are perfectly adequate to deal with liquidity risk

To do:

- study possible realistic (yet analytically tractable) stochastic models for a msdc
- portfolio optimization with liquidity risk