IMPERIAL

Modelling Accumulation of Resources: Queueing Theory and First-Passage Processes

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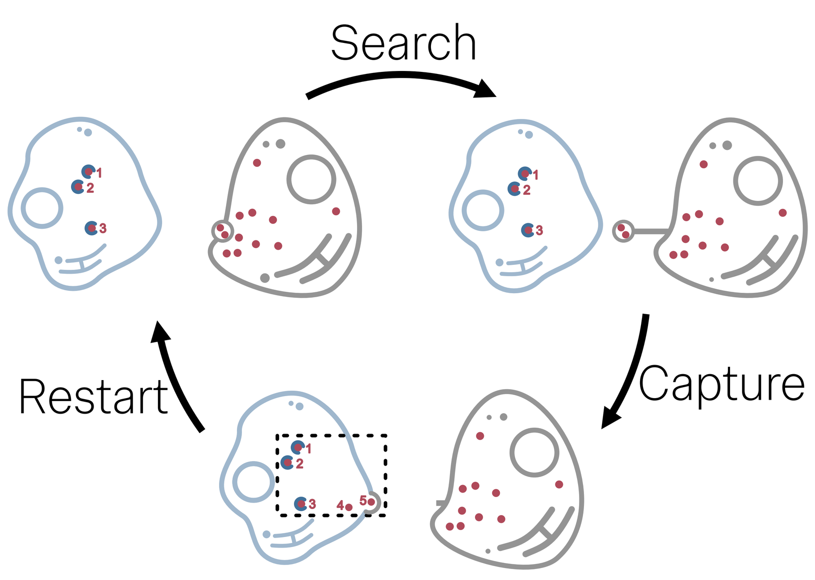
Goal

We study the **accumulation of resources** within a target due to the interplay between continuous delivery, driven by **1D stochastic search processes**, and the sequential consumption of resources by a **finite number of servers**.

The Model

Delivery

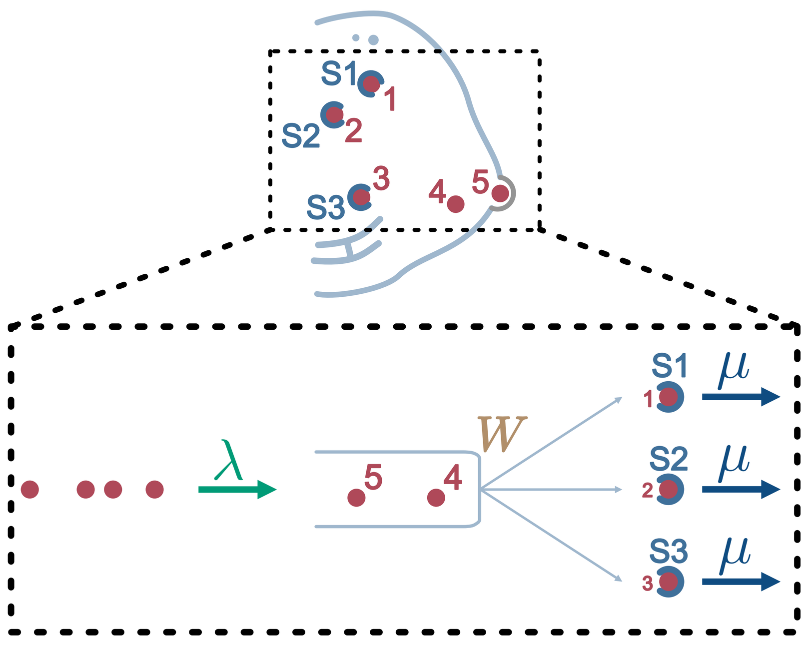
1. A searcher cell carries a resource on the tip of a filament, which is **randomly searching** for a target cell.
2. Once the searcher finds the target it delivers the resources, returns to its initial position and the process restarts.



This **search-and-capture** process repeats, making the searcher deliver resources with rate λ.

Consumption

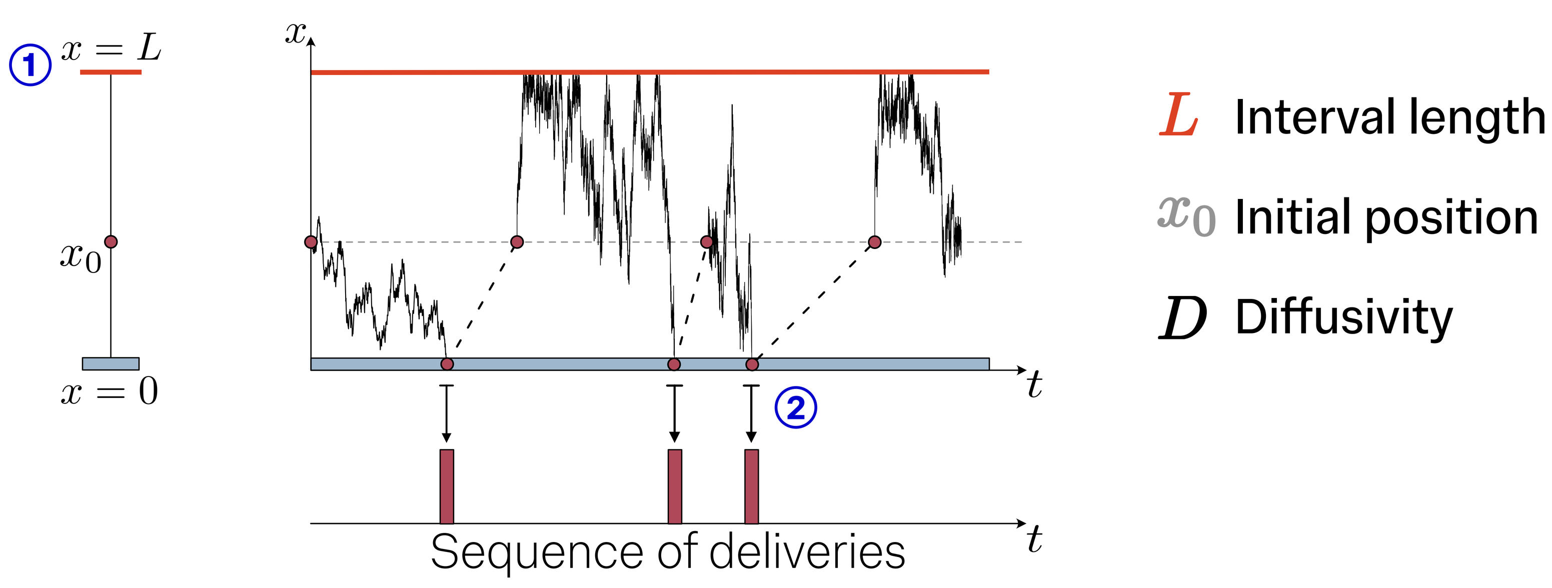
- The resources within the target are consumed by a finite number of servers.
- Consumption in each server takes an exponentially distributed random time with rate μ.
- The resources are consumed in the order they arrive (first in, first out).

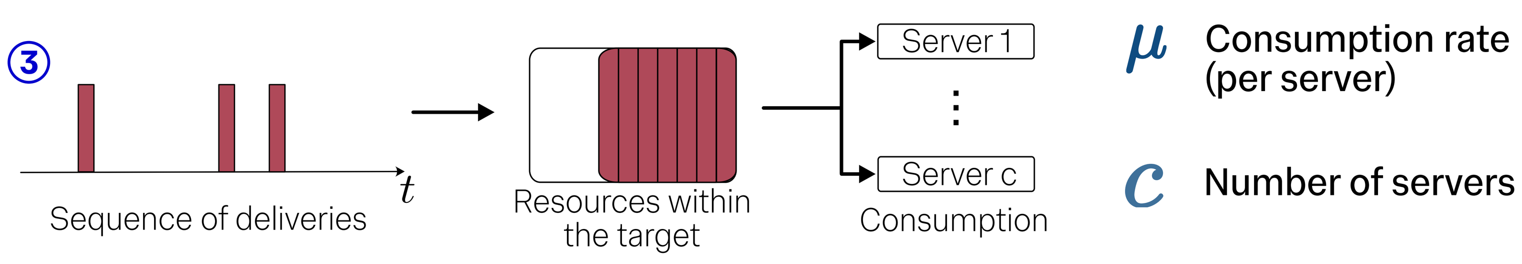


Each one of the resources may **wait an additional random time before being consumed, W,** if all the servers are occupied.

Mathematical Description

1. **Searcher:** a particle following a 1D Brownian motion inside the interval $[0, L]$ with a reflecting boundary at L and an absorbing boundary at 0.
2. **Delivery:** Every time the particle finds the absorbing target it delivers a resource and returns to its initial position, x_0.





3. **Accumulation:** we use a $G/M/c$ queueing system.
   - Resources arrive to the system with a **General** distribution, determined by the search-and-capture process.
   - Resources are consumed in exponential (**Markovian**) times.
   - The system has $c < \infty$ **servers**.

Blow-up Regions

Queueing Theory

If the arrival rate (λ) is greater than the removal rate ($c\mu$), then the number of resources in the system grows to infinity and it blows up [1].

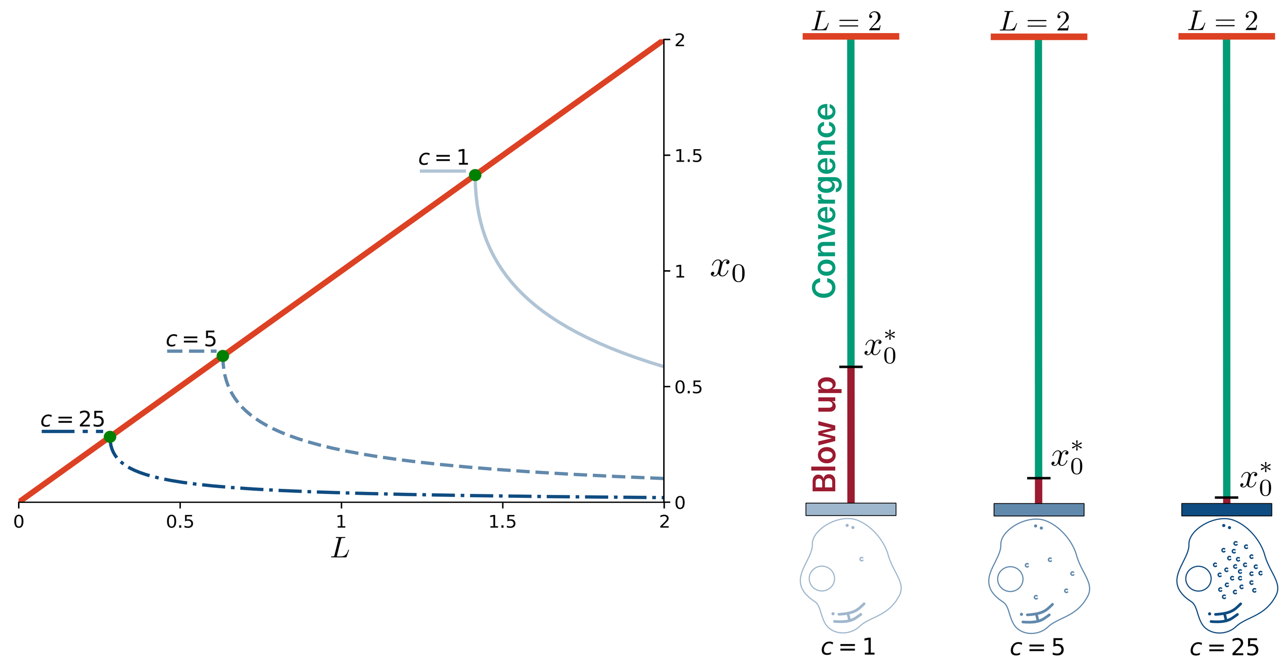
First-Passage Time Theory

We can determine the arrival rate for a **diffusing particle in an interval** in terms of the starting position, the interval length and the diffusivity [2].

$$\lambda = \frac{2D}{(2L - x_0)x_0}.$$

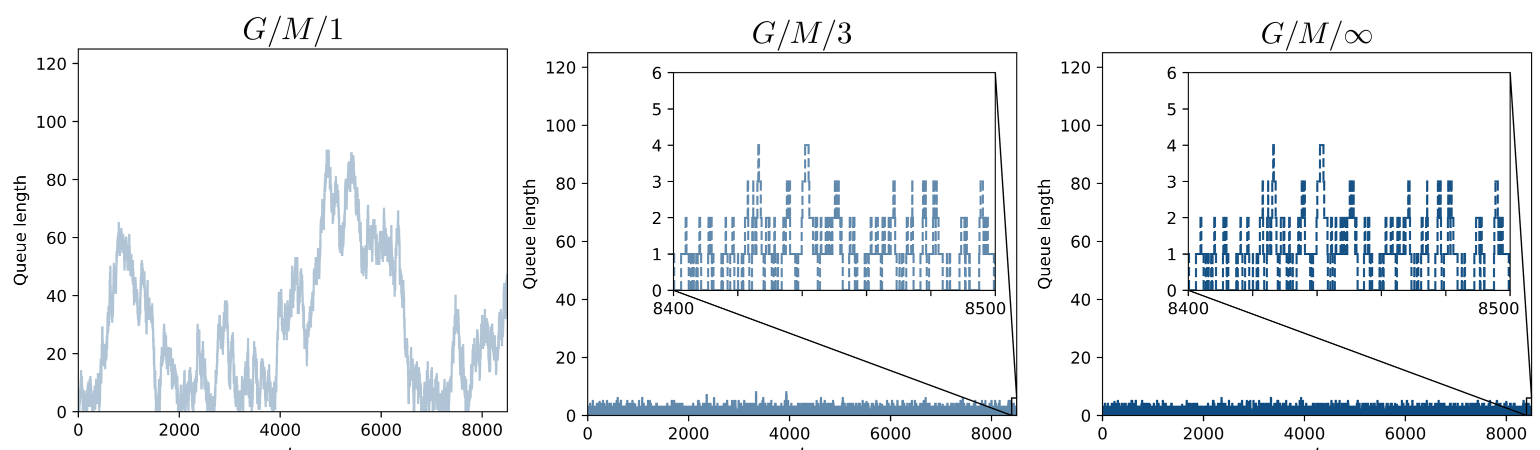
Joining these two results we obtain a **spatial condition for convergence to steady state**:

$$\lambda > c\mu \iff x_0 < x_0^* = L - \sqrt{L^2 - \frac{2D}{c\mu}}.$$



Finite vs. Infinite number of servers

This type of model has been studied before in the special case of a $G/M/\infty$ **queueing system** [3]. The $G/M/\infty$ queue **always converges to a steady state** and has **null waiting times before service** (W).

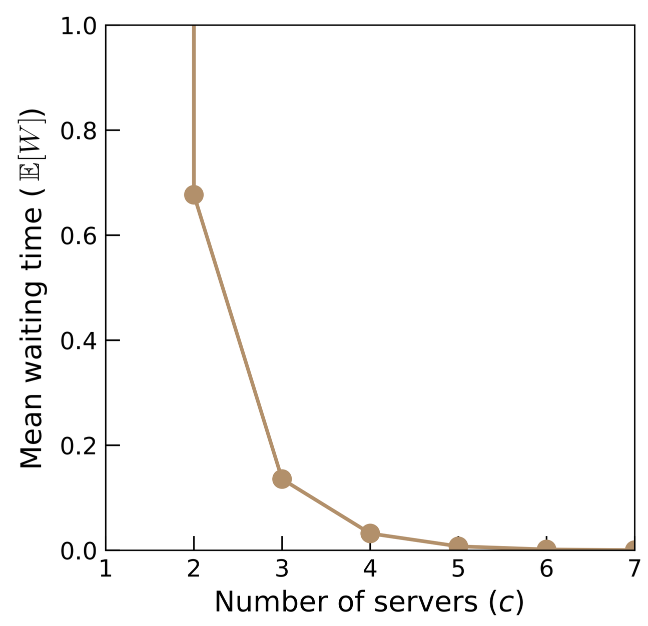


$\mathbb{E}[W] \to \infty$ $\mathbb{E}[W] = 0.0266$ $\mathbb{E}[W] \equiv 0$

A Measure of Performance

As W is a non-negative random variable, we know that if $\mathbb{E}[W] = 0$ then $W \equiv 0$.

Moreover, the **mean waiting time** corresponds to the **Wasserstein distance** between the distributions of the waiting times in the $G/M/\infty$ and the waiting times in the $G/M/c$.



Adding a few servers takes $\mathbb{E}[W]$ from ∞ to almost 0!

Future work

- Extend to **more complex domains**: higher dimensions and unbounded domains.
- Consider **multiple targets** and **multiple searchers**.
- Applications to **cellular transport**.

References

[1] Cooper R B 1981 Introduction to queueing theory 2nd ed (New York: North Holland) ISBN 978-0-444-00379-9

[2] Redner S 2001 A Guide to First-Passage Processes (Cambridge: Cambridge University Press) ISBN 978-0-521-65248-3 Holland) ISBN 978-0-444-00379-9

[3] Bressloff P C 2020 Queueing theory of search processes with stochastic resetting Phys. Rev. E **102**

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