IMPERIAL

Phase transitions for interacting particle systems on random graphs

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Motivation

- Interacting particle systems (IPS) model collective behaviour in systems ranging from gas particles, to neurons, to opinions.
- These systems can evolve into ordered or disordered states, depending on key parameters.
- In real systems, interactions occur on networks, not uniformly.
- We study IPS on random graphs, and show that critical points depend on the network structure.

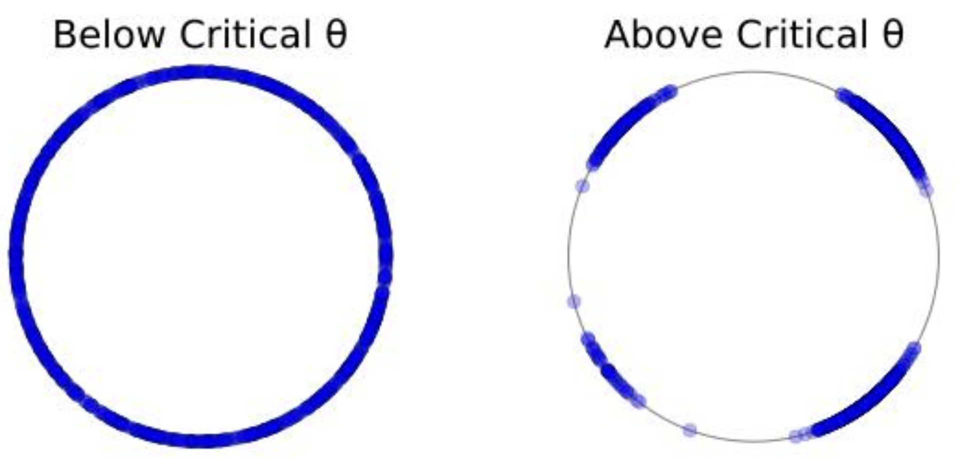

Figure 1: Distribution of particle phases below (left) and above (right) the critical coupling angle, showing the onset of synchronization.

The model

We study N particles on $[0,2\pi]$, each following a stochastic differential equation (SDE):

$$dX_t^i = -\frac{\theta}{N}\sum_{j=1}^{N} W_{N,ij} D'(X_t^i - X_t^j)dt + \sqrt{2\beta^{-1}}dB_t^i$$

θ and β are physical parameters, D is the interaction potential, and $W_{N,ij}$ encodes the network.

We use multichromatic potentials (sums of cosine modes) to model diverse behaviour.

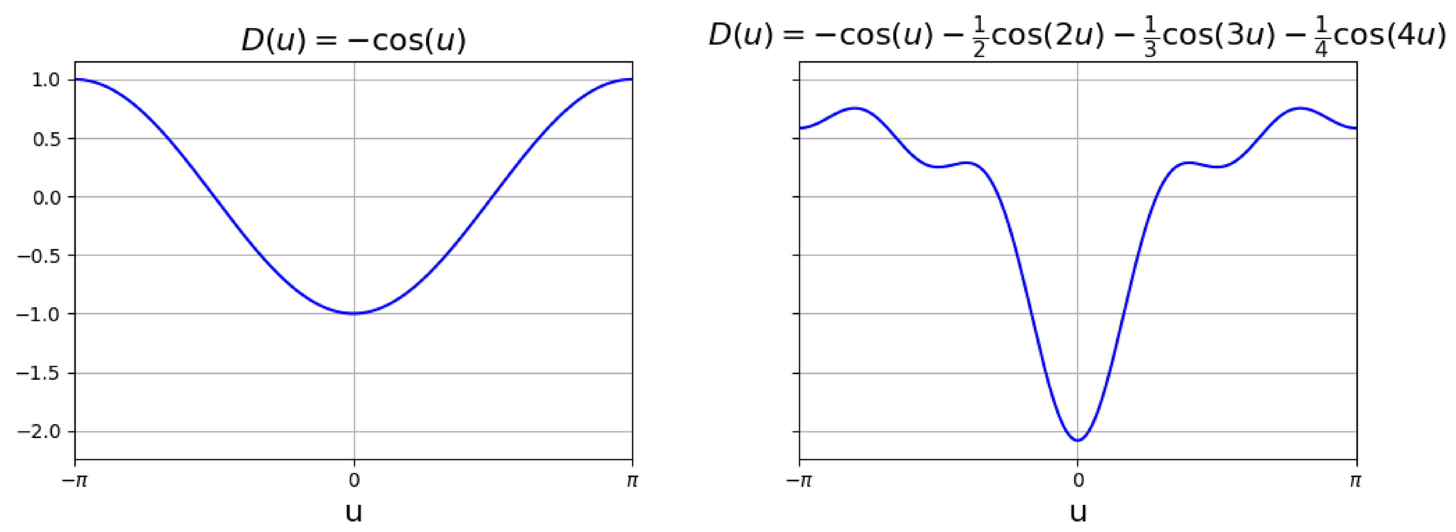

Figure 2: Examples of two interaction potentials D. Increasing the number of cosine modes adds local minima, allowing for multiple steady states.

As $N \to \infty$, the system is described by the McKean-Vlasov PDE for the probability density ρ of an average particle:

$$\partial_t \rho = \theta \partial_u \left(\rho \int W(x,y) D'(u-v)\rho(v,y)\,dv\,dy\right) + \beta^{-1}\partial_u^2 \rho.$$

$W(x,y)$, called a graphon, describes how likely two particles at positions x and y are to interact. Examples include:

- Erdős-Rényi: $W(x,y) = p$, a constant probability
- Small-World: favours nearby nodes
- Power-Law: favours a few highly linked nodes.

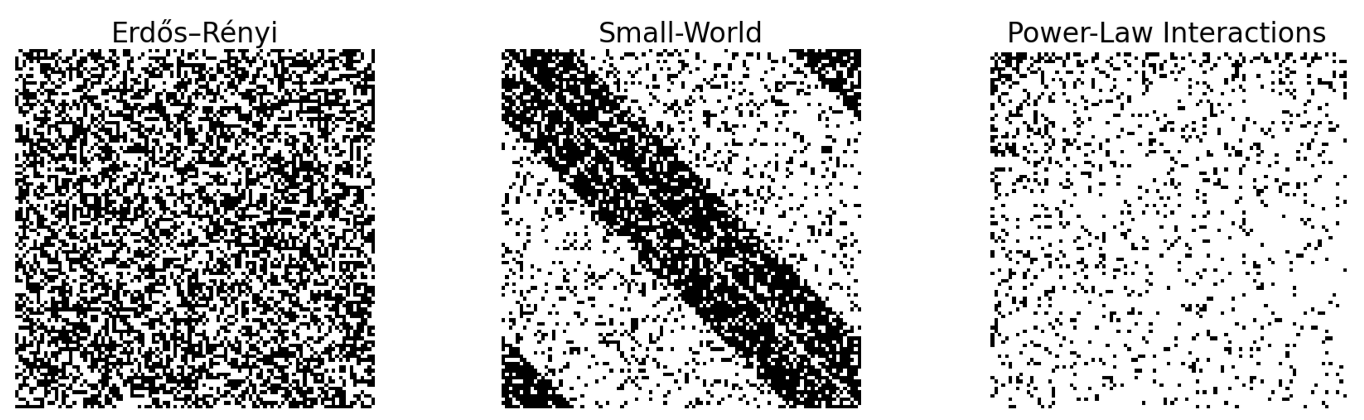

Figure 3: Network structures: adjacency matrices show distinct patterns of connectivity.

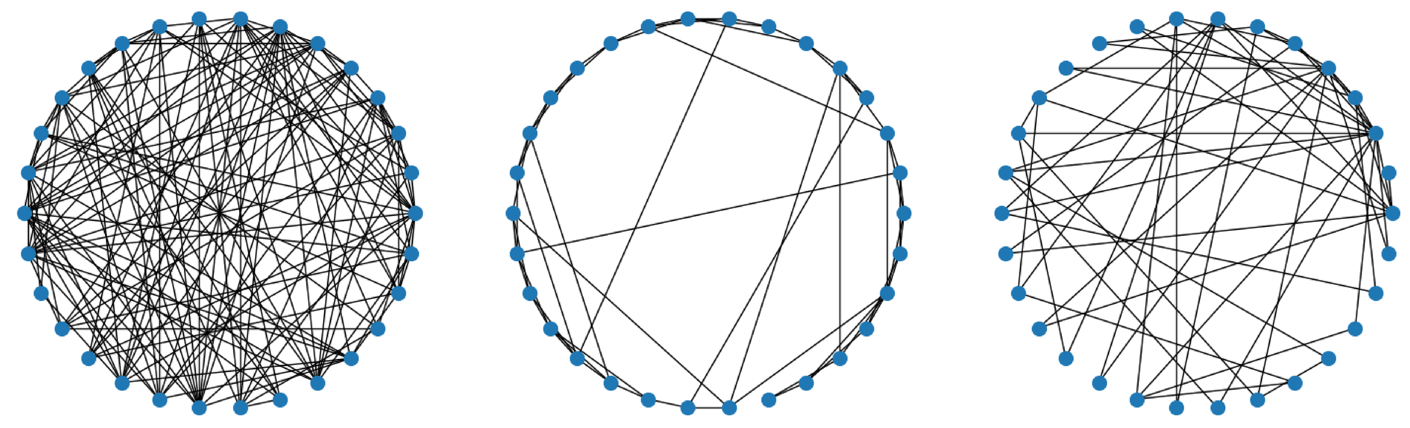

Figure 4: Erdős–Rényi (left), Small-World (centre), and Power-Law (right) graphs. Particles (nodes) are placed on a circle; edges show interactions.

Methods

- Stability analysis: identifies when the uniform state becomes unstable
- Bifurcation theory: detects when new solution branches appear
- Numerical simulations: compare SDE behaviour across networks

Main results

The system undergoes a phase transition at a critical θ determined by the interaction potential and the graph structure. In particular, ([1])

$$\theta_c = \min\{\lambda_{m,l}\},$$

where $\lambda_{m,l}$ are the eigenvalues of:

$$T\rho = -\frac{\beta}{2\pi}\int W(x,y)D(u-v)\rho(v,y)\,dv\,dy.$$

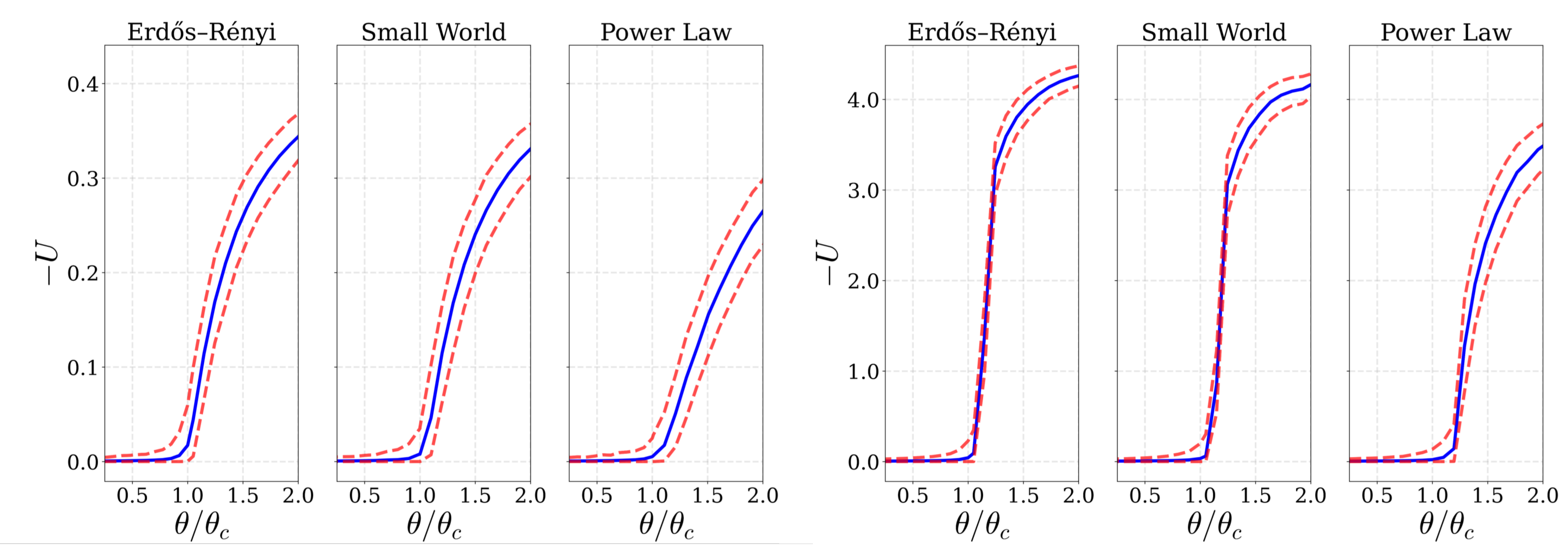

Figure 5: Mean interaction energy versus normalized coupling for Kuramoto (left) and multichromatic (right) potentials.

For multichromatic D, the system can admit multiple steady states ([2]), leading to metastability, where it temporarily settles into unstable configurations before reaching equilibrium.

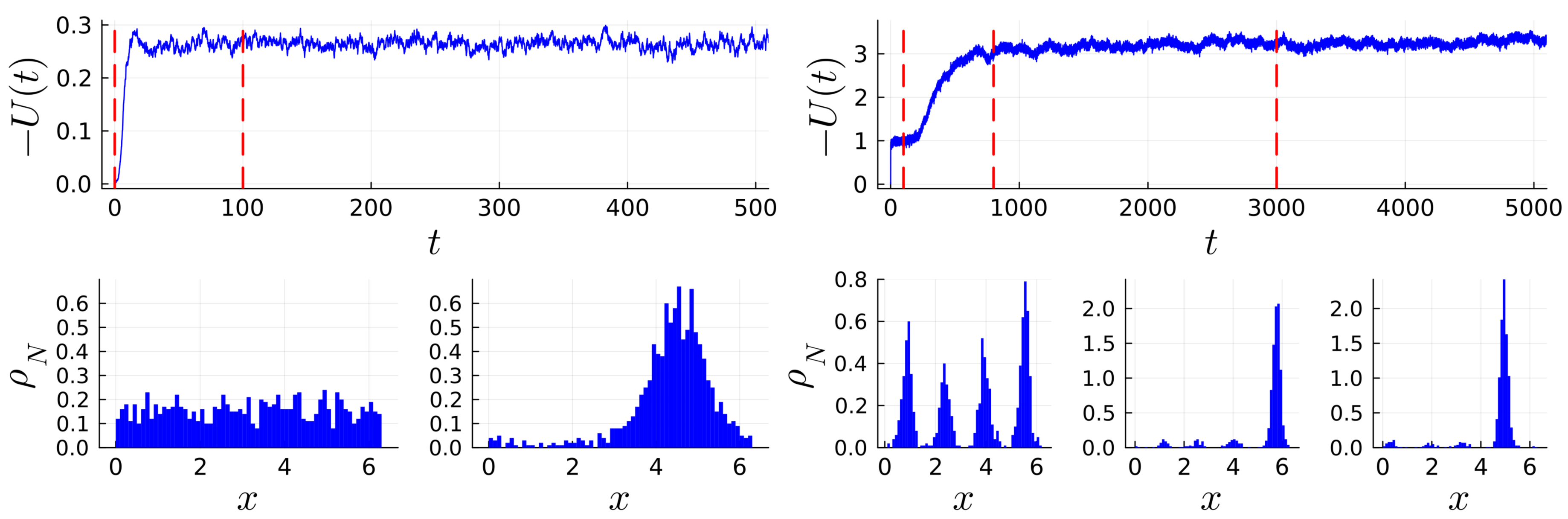

Figure 6: Metastable dynamics: Time evolution of the energy (top) and empirical particle distributions (bottom) for the Kuramoto model (left) and a multichromatic potential (right).

Conclusions

- Long-term behaviour depends both on the network and on the interaction potential. Multichromatic potentials lead to metastability and multiple steady states.
- This framework connects models of synchronization, clustering, and pattern formation across fields.

References:
[1] Bertoli, B., Pavliotis, G. A., & Zagli, N. (2025). *arXiv:2504.02721*.
[2] Bertoli, B., Goddard, B. D., & Pavliotis, G. A. (2025). *IMA Journal of Applied Mathematics*, hxaf001.