Markovian Integral Equations

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Abstract

We analyze multidimensional Markovian integral equations that are formulated with a time-inhomogeneous progressive Markov process that has Borel measurable transition probabilities. In the case of a path-dependent diffusion process, the solutions to these integral equations lead to the concept of mild solutions to semilinear parabolic path-dependent partial differential equations (PPDEs). Our goal is to establish uniqueness, stability, existence, and non-extendibility of solutions among a certain class of maps. By requiring the Feller property of the Markov process, we give weak conditions under which solutions become continuous. Moreover, we provide a multidimensional Feynman-Kac formula and a one-dimensional global existence- and uniqueness result.

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1 Introduction

Markovian integral equations arise when dealing with diffusion processes and mild solutions to semilinear parabolic partial differential equations (PDEs). This fact was utilized by Dynkin [4,5] to give probabilistic formulas for mild solutions via the log-Laplace functionals of superprocesses. In this context, Schied [18] used Markovian integral equations to solve problems of optimal stochastic control in mathematical finance. By introducing path-dependent diffusion processes, the connection of Markovian equations to PDEs can be extended to path-dependent partial differential equations (PPDEs)[1], as verified in the companion paper [13]. Inspired by the applications of one-dimensional Markovian equations, the aim of this paper is to construct solutions even in a multidimensional framework.

Let S be a separable metrizable topological space, $T > 0$, and $\mathscr{X} = (X, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a consistent progressive Markov process on some measurable space $(\Omega, \mathscr{F})$ with state space S that has Borel measurable transition probabilities. We consider the following *multidimensional Markovian integral equation* coupled with a terminal value condition:

$$\begin{aligned} E_{r,x}[u(t,X_t)] &= u(r,x) + E_{r,x}\bigg[\int_r^t f(s,X_s,u(s,X_s))\,\mu(ds)\bigg], \\ u(T,x) &= g(x) \end{aligned} \tag{M}$$

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[1]For a recent analysis of PPDEs in the context of classical and viscosity solutions, we refer the reader to Peng [15,16], Peng and Wang [17], Ji and Yang [11], Ekren, Keller, Touzi, and Zhang [7], and Henri-Labordere, Tan, and Touzi [9].

for all $r, t \in [0, T]$ with $r \leq t$ and each $x \in S$. Here, we assume implicitly that $k \in \mathbb{N}$, $D \in \mathscr{B}(\mathbb{R}^k)$ has non-empty interior, $f : [0, T] \times S \times D \to \mathbb{R}^k$ is product measurable, μ is an atomless Borel measure on $[0, T]$, and $g : S \to D$ is Borel measurable and bounded.

We first remark that for $D = \mathbb{R}^k$ a Picard iteration and Banach's fixed-point theorem produce existence of solutions to (M) locally in time. This can be found, for example, in Pazy [14, Theorem 6.1.4] when $\mathscr{X}$ is a diffusion process. Regarding existence, we will suppose more generally that D is convex. By modifying analytical methods from the classical theory of ordinary differential equations (ODEs), we will derive unique non-extendible solutions to (M) that are admissible in an appropriate topological sense. Moreover, weak conditions ensuring the continuity of the derived solutions will be provided. In the particular case when $D = \mathbb{R}^k$ and f is an affine map in the third variable $w \in \mathbb{R}^k$, we will prove a representation for solutions to (M). This gives a multidimensional generalization to the Feynman-Kac formula in Dynkin [6, Theorem 4.1.1].

Let us also emphasize that non-negative solutions to one-dimensional Markovian integral equations are well-studied. Namely, for $k = 1$ and $D = \mathbb{R}_+$, solutions to (M) have been deduced by a Picard iteration approach. For instance, the classical references are Watanabe [19, Proposition 2.2], Fitzsimmons [8, Proposition 2.3], and Iscoe [10, Theorem A]. In these works the existence of solutions to (M) is used for the construction of superprocesses. Dynkin [2, 3, 6] establishes superprocesses with probabilistic methods by means of branching particle systems, which in turn yields another existence result to our Markovian integral equations.

These treatments of (M) in one dimension require that the function f admits a representation that is related to measure-valued branching processes. To give one of the main examples, the following case is included in [2, 3, 6]:

$$f(t, x, w) = b_1(t, x)w^{\alpha_1} + \cdots + b_n(t, x)w^{\alpha_n} \tag{1.1}$$

for each $(t, x, w) \in [0, T] \times S \times \mathbb{R}_+$, where $n \in \mathbb{N}$, $b_1, \ldots, b_n : [0, T] \times S \to \mathbb{R}_+$ are Borel measurable and bounded, and $\alpha_1, \ldots, \alpha_n \in [1, 2]$. Here, the bound $\alpha_i \leq 2$ for all $i \in \{1, \ldots, n\}$ is strict. However, this paper intends to derive solutions without imposing a specific form of f. Rather, as in the multidimensional case, we will introduce regularity conditions for f with respect to the Borel measure μ like local Lipschitz μ-continuity. This will allow for a more general treatment of (M). In particular, our approach includes the case

$$f(t, x, w) = a(t, x) + b_1(t, x)\varphi_1(w) + \cdots + b_n(t, x)\varphi_n(w)$$

for all $(t, x, w) \in [0, T] \times S \times \mathbb{R}_+$, where $a : [0, T] \times S \to (-\infty, 0]$ is Borel measurable and bounded, and $\varphi_1, \ldots, \varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$ are locally Lipschitz continuous with $\varphi_i(0) = 0$ for each $i \in \{1, \ldots, n\}$. Hence, (1.1) is also feasible if $\alpha_i > 2$ for some $i \in \{1, \ldots, n\}$. Note that we will not restrict our attention to the case $D = \mathbb{R}_+$. In fact, the one-dimensional global existence and uniqueness result, we will establish, is applicable provided D is a non-degenerate interval. In this connection, the same weak conditions as before grant the continuity of solutions to (M).

The paper is structured as follows. In Section 2 we set up the framework. First, in Section 2.1 we consider product spaces endowed with a pseudometric and introduce several map spaces. Section 2.2 presents regularity conditions for multidimensional measurable maps relative to a Borel measure. In Section 2.3 we give an adjusted definition of a Markov process that is in line with the classical notion. In Section 2.4 we introduce the Markovian terminal value problem (M), by defining (approximate) solutions. In Section 2.5 the main results are presented. Section 3 shows our approach to the main results. In Section 3.1 we compare solutions, prove their stability, and also investigate their growth behavior, while in Section 3.2 we construct solutions locally in time. Finally, the main results are proven in Section 4.

2 Preliminaries and main results

Throughout the paper, let S be a separable metrizable topological space, $T > 0$, and μ be an atomless Borel measure on $[0,T]$. We fix $k \in \mathbb{N}$ and let $\mathbb{I}_k$ be the identity matrix in $\mathbb{R}^{k\times k}$. To keep notation simple, we use $|\cdot|$ for the absolute value function, the Euclidean norm on $\mathbb{R}^k$, and the Frobenius norm on $\mathbb{R}^{k\times k}$.

2.1 Time-space Cartesian products

We endow $[0,T]\times S$ with a pseudometric d_S generating a topology that is coarser than the product topology, which ensures that $\mathscr{B}([0,T]\times S) \subset \mathscr{B}([0,T])\otimes\mathscr{B}(S)$, since S separable. For instance, d_S could be any product metric on $[0,T]\times S$, in which case the Borel σ-field would coincide with the product σ-field. However, the presence of a pseudometric allows us to include path processes of path-dependent diffusions as specific strong Markov processes.

Let for the moment I be a non-degenerate interval in $[0,T]$ and $(E, \|\cdot\|)$ be a normed space, then we call a map $u : I\times S \to E$ *consistent* if $u(r,x) = u(s,y)$ for all $(r,x),(s,y) \in I\times S$ such that $d_S((r,x),(s,y)) = 0$. Moreover, u is said to be *right-continuous* if for each $(r,x) \in I\times S$ and every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|u(s,y) - u(r,x)\| < \varepsilon$$

for all $(s,y) \in I\times S$ with $s \geq r$ and $d_S((s,y),(r,x)) < \delta$. Clearly, if u is (right-)continuous, then it is consistent. In addition, (right-)continuity of u implies that $u(\cdot,x)$ is (right-)continuous for each $x \in S$ and $u(t,\cdot)$ is continuous for all $t \in [0,T]$, which entails that u is Borel measurable.

Example 2.1. Assume that $S = C([0,T],\mathbb{R}^d)$ for some $d \in \mathbb{N}$ and let ρ be a complete metric on S that is equivalent to the maximum metric, then S equipped with ρ is Polish. Denote each map $x \in S$ stopped at time $t \in [0,T]$ by $x^t \in S$, that is, $x^t(s) = x(s\wedge t)$ for all $s \in [0,T]$. Let

$$d_S((r,x),(s,y)) = |r-s| + \rho(x^r, y^s)$$

for every $(r,x),(s,y) \in [0,T]\times S$, then $[0,T]\times S$ endowed with d_S is a separable complete pseudometric space whose topology is indeed coarser than its product topology. Further, the map u is consistent if and only if it is *non-anticipative* in the sense that $u(t,x) = u(t,x^t)$ for all $(t,x) \in [0,T]\times S$. This framework is used in [7] and [13] to deal with PPDEs.

Finally, for every $D \in \mathscr{B}(E)$, we let $B(S,D)$ and $B(I\times S,D)$ denote the sets of all D-valued Borel measurable maps on S and $I\times S$, respectively. By $B_b(S,D)$ and $B_b(I\times S,D)$ we denote the set of all bounded $g \in B(S,D)$ and $u \in B(I\times S,D)$, respectively.

2.2 Regularity with respect to Borel measures

We recall that for each non-degenerate interval I in $[0,T]$, a function $a \in B(I,\mathbb{R})$ is locally μ-integrable if and only if $\int_r^t |a(s)|\,\mu(ds) < \infty$ for all $r,t \in I$ with $r \leq t$.

Definition 2.2. Suppose that $I \subset [0,T]$ is a non-degenerate interval, $(E,\|\cdot\|)$ is a normed space, and $a \in B(I\times S,E)$.

(i) The map a is called *(locally) μ-dominated* if there is a (locally) μ-integrable $\overline{a} \in B(I,\mathbb{R}_+)$ such that $\|a(\cdot,y)\| \leq \overline{a}$ for all $y \in S$ μ-a.s. on I.

(ii) We say that a is μ-suitably bounded if for each $r,t \in I$ with $r \leq t$ there is a μ-null set $N \in \mathscr{B}([0,T])$ such that $\sup_{(s,y)\in(N^c\cap[r,t])\times S} \|a(s,y)\| < \infty$.

By using the notation in above definition, we see immediately that the set of all E-valued product measurable locally μ-dominated maps on $I \times S$ is a linear space that contains every E-valued product measurable μ-suitably bounded map on $I \times S$.

Definition 2.3. Let $f : [0,T] \times S \times D \to \mathbb{R}^k$ be $\mathscr{B}([0,T] \times S) \otimes \mathscr{B}(D)$-measurable.

(i) We call f *affine μ-bounded* if there exist two μ-dominated $a, b \in B([0,T] \times S, \mathbb{R}_+)$ such that $|f(t,x,w)| \leq a(t,x) + b(t,x)|w|$ for all $(t,x,w) \in [0,T] \times S \times D$. If one can take $b = 0$, then f is called *μ-bounded.*

(ii) We say that f is *locally μ-bounded* at $\hat{w} \in \overline{D}$ if there is a neighborhood W of $\hat{w}$ in $\overline{D}$ for which $f|([0,T] \times S \times (W \cap D))$ is μ-bounded. The map f is called locally μ-bounded if it is locally μ-bounded at each $\hat{w} \in D$.

(iii) Let $k = 1$, then f is said to be *affine μ-bounded from below* if $f(t,x,w) \geq -a(t,x) - b(t,x)|w|$ for all $(t,x,w) \in [0,T] \times S \times D$ and some μ-dominated $a, b \in B([0,T] \times S, \mathbb{R}_+)$. If $b = 0$ is possible, then f is *μ-bounded from below.* Moreover, f is *(affine) μ-bounded from above* if $-f$ is (affine) μ-bounded from below.

For a $\mathscr{B}([0,T] \times S) \otimes \mathscr{B}(D)$-measurable map $f : [0,T] \times S \times D \to \mathbb{R}^k$ to be locally μ-bounded, it is sufficient that it is affine μ-bounded. If f is locally μ-bounded, then the Borel measurable map $f(\cdot,\cdot,\hat{w})$ is μ-dominated for each $\hat{w} \in D$. Of course, for $k = 1$ the function f is (affine) μ-bounded if and only if it is (affine) μ-bounded from below and from above.

Definition 2.4. Let $f : [0,T] \times S \times D \to \mathbb{R}^k$ be $\mathscr{B}([0,T] \times S) \otimes \mathscr{B}(D)$-measurable.

(i) We call f *Lipschitz μ-continuous* if there is a μ-dominated $\lambda \in B([0,T] \times S, \mathbb{R}_+)$ satisfying $|f(t,x,w) - f(t,x,w')| \leq \lambda(t,x)|w - w'|$ for all $(t,x) \in [0,T] \times S$ and each $w, w' \in D$.

(ii) We call f *locally Lipschitz μ-continuous* at $\hat{w} \in \overline{D}$ if there is a neighborhood W of $\hat{w}$ in $\overline{D}$ such that $f|([0,T] \times S \times (W \cap D))$ is Lipschitz μ-continuous. The map f is locally Lipschitz μ-continuous if it is locally Lipschitz μ-continuous at every $\hat{w} \in D$.

In what follows, the linear space of all $\mathbb{R}^k$-valued $\mathscr{B}([0,T] \times S) \otimes \mathscr{B}(D)$-measurable, locally μ-bounded, and locally Lipschitz μ-continuous maps on $[0,T] \times S \times D$ is denoted by

$$BC_\mu^{1-}([0,T] \times S \times D, \mathbb{R}^k). \tag{2.1}$$

Clearly, if $f : [0,T] \times S \times D \to \mathbb{R}^k$ is a $\mathscr{B}([0,T] \times S) \otimes \mathscr{B}(D)$-measurable map that is locally Lipschitz μ-continuous and $f(\cdot,\cdot,\hat{w})$ is μ-dominated for all $\hat{w} \in D$, then f is locally μ-bounded. If instead f is Lipschitz μ-continuous and $f(\cdot,\cdot,\hat{w})$ is μ-dominated for at least one $\hat{w} \in D$, then f is affine μ-bounded.

Examples 2.5. (i) Let $a \in B([0,T] \times S, \mathbb{R}^k)$ and $b \in B([0,T] \times S, \mathbb{R}^{k \times k})$ be μ-dominated. Assume that $\varphi : D \to \mathbb{R}^k$ is Borel measurable and fulfills

$$f(t,x,w) = a(t,x) + b(t,x)\varphi(w)$$

for all $(t,x,w) \in [0,T] \times S \times D$. Then the following two assertions hold:

(1) f is (affine) μ-bounded whenever φ is (affine) bounded. If instead φ is locally bounded, then f is locally μ-bounded. For $k = 1$ and $b \geq 0$, it follows that f is (affine) μ-bounded from below (resp. from above) if φ is (affine) bounded from below (resp. from above).

(2) From the (local) Lipschitz continuity of φ the (local) Lipschitz μ-continuity of f follows. Thus, if φ is locally Lipschitz continuous, then $f \in BC_\mu^{1-}([0,T] \times S \times D, \mathbb{R}^k)$.

(ii) Let $(U,\mathscr{U})$ be a measurable space and n be a kernel from $[0,T]\times S$ to $(U,\mathscr{U})$. Suppose that $\varphi: U\times D\to\mathbb{R}^k$ is $\mathscr{U}\otimes\mathscr{B}(D)$-measurable and $\varphi(\cdot,w)$ is $n(t,x,\cdot)$-integrable for every $(t,x,w)\in[0,T]\times S\times D$. Let f be of the form

$$f(t,x,w)=\int_U \varphi(u,w)\,n(t,x,du)$$

for each $(t,x,w)\in[0,T]\times S\times D$. Then the subsequent two assertions are valid:

(1) f is locally μ-bounded if for each $\hat{w}\in D$ there are a neighborhood W of $\hat{w}$ in D and an $\mathscr{U}$-measurable $a: U\to[0,\infty]$ with $|\varphi(u,w)|\leq a(u)$ for all $(u,w)\in U\times W$ such that $\int_U a(u)\,n(\cdot,\cdot,du)$ is finite and μ-dominated.

(2) f is locally Lipschitz μ-continuous if to all $\hat{w}\in D$ there are a neighborhood W of $\hat{w}$ in D and an $\mathscr{U}$-measurable $\lambda: U\to[0,\infty]$ with $|\varphi(u,w)-\varphi(u,w')|\leq\lambda(u)|w-w'|$ for all $u\in U$ and each $w,w'\in W$ such that $\int_U \lambda(u)\,n(\cdot,\cdot,du)$ is finite and μ-dominated.

2.3 Time-inhomogeneous Markov processes

In the sequel, let $\mathscr{X}$ be a *consistent Markov process* on some measurable space $(\Omega,\mathscr{F})$ with state space S and *Borel measurable transition probabilities*, which is a triple $(X,(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$ that is composed of a process $X:[0,T]\times\Omega\to S$, a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ to which X is adapted, and a set $\mathbb{P}=\{P_{r,x}\,|\,(r,x)\in[0,T]\times S\}$ of probability measures on $(\Omega,\mathscr{F})$ such that the following three conditions hold:

(i) $d_S((r,X_r),(r,y))=0$ for all $r\in[0,s]$ $P_{s,y}$-a.s. for each $(s,y)\in[0,T]\times S$.

(ii) The function $[0,t]\times S\to[0,1]$, $(s,x)\mapsto P_{s,x}(X_t\in B)$ is consistent and Borel measurable for all $t\in[0,T]$ and every $B\in\mathscr{B}(S)$.

(iii) $P_{r,x}(X_t\in B|\mathscr{F}_s)=P_{s,X_s}(X_t\in B)$ $P_{r,x}$-a.s. for all $r,s,t\in[0,T]$ with $r\leq s\leq t$, each $x\in S$, and every $B\in\mathscr{B}(S)$.

Hence, if d_S is a product metric, then (i) reduces to $X_r=y$ for all $r\in[0,s]$ $P_{s,y}$-a.s. for each $(s,y)\in[0,T]\times S$ and we recover the classical definition of a time-inhomogeneous Markov process with Borel measurable transition probabilities. Moreover, let $\mathscr{X}$ be *progressive*, that is, X is progressively measurable with respect to its natural filtration and its natural backward filtration. For example, this is the case if X is left- or right-continuous.

Whenever necessary, we will require that $\mathscr{X}$ is *(right-hand) Feller*, which means that the function $[0,t]\times S\to\mathbb{R}$, $(r,x)\mapsto E_{r,x}[\varphi(X_t)]$ is (right-)continuous for all $t\in[0,T]$ and each continuous $\varphi\in B_b(S,\mathbb{R})$. In this case, it follows that the map

$$[0,t]\times S\to\mathbb{R}^k,\quad (r,x)\mapsto E_{r,x}\bigg[\int_r^t \varphi(s,X_s)\,\mu(ds)\bigg] \tag{2.2}$$

is (right-)continuous for each $t\in[0,T]$ and every μ-dominated $\varphi\in B([0,t]\times S,\mathbb{R}^k)$ for which $\varphi(s,\cdot)$ is continuous for μ-a.e. $s\in[0,t]$, by dominated convergence.

Example 2.6. Let the setting of Example 2.1 hold, then X is the *path process* of a process $Y:[0,T]\times\Omega\to\mathbb{R}^d$ in the sense that $X_t=Y^t$ for all $t\in[0,T]$ if and only if

$$X_s(\omega)|[0,r]=X_r(\omega)\quad\text{for all } r,s\in[0,T] \tag{2.3}$$

with $r\leq s$ and each $\omega\in\Omega$. In this case, Y is uniquely determined, $(\mathscr{F}_t)_{t\in[0,T]}$-adapted, and continuous. Further, (i) is equivalent to $Y^s=y^s$ $P_{s,y}$-a.s. for all $(s,y)\in[0,T]\times S$.

The class of non-anticipative progressive Markov processes $\mathscr{X}$ fulfilling condition (2.3) is used in [12] to construct path-dependent diffusion processes, which extend standard Markovian diffusions in the context of semilinear parabolic PPDEs. In particular, conditions granting the (right-hand) Feller property of $\mathscr{X}$ are provided there.

2.4 The Markovian terminal value problem

We let $D \in \mathscr{B}(\mathbb{R}^k)$ have non-empty interior, $f : [0,T] \times S \times D \to \mathbb{R}^k$ be measurable with respect to $\mathscr{B}([0,T] \times S) \otimes \mathscr{B}(D)$, and $g \in B(S,D)$ be consistent in the sense that $g(x) = g(y)$ for all $x, y \in S$ with $d_S((T,x),(T,y)) = 0$. Let us assume initially that

$$E_{r,x}[|g(X_T)|] < \infty \quad \text{for all } (r,x) \in [0,T] \times S.$$

Further, we let $\varepsilon \in B([0,T] \times S, \mathbb{R}_+)$ be μ-dominated and define an interval I in $[0,T]$ to be *admissible* if it is of the form $I = (t,T]$ or $I = [t,T]$ for some $t \in [0,T)$. This allows us to introduce the *Markovian terminal value problem* (M), by defining ε-approximate solutions.

Definition 2.7. An *ε-approximate solution* to (M) on an admissible interval I is a consistent map $u \in B(I \times S, D)$ for which both $|u(t,X_t)|$ and $\int_r^t |f(s,X_s,u(s,X_s))|\,\mu(ds)$ are finite and $P_{r,x}$-integrable such that

$$\left| E_{r,x}[u(t,X_t)] - u(r,x) - E_{r,x}\left[\int_r^t f(s,X_s,u(s,X_s))\,\mu(ds)\right]\right| \leq E_{r,x}\left[\int_r^t \varepsilon(s,X_s)\,\mu(ds)\right]$$

and $u(T,x) = g(x)$ for all $r,t \in I$ with $r \leq t$ and each $x \in S$. Every 0-approximate solution is called a *solution*. If in addition $I = [0,T]$, then we will speak about a *global* solution.

For each admissible interval I, it follows from the Markov property of $\mathscr{X}$ that a map $u \in B(I \times S, D)$ is a solution to (M) on I if and only if $\int_r^T |f(s,X_s,u(s,X_s))|\,\mu(ds)$ is finite and $P_{r,x}$-integrable such that

$$u(r,x) = E_{r,x}[g(X_T)] - E_{r,x}\left[\int_r^T f(s,X_s,u(s,X_s))\,\mu(ds)\right]$$

for all $(r,x) \in I \times S$. Note that u is automatically consistent, as soon as these two conditions are valid. For our main results, we introduce admissibility and non-extendibility of solutions.

Definition 2.8. Assume that u is a solution to (M) on an admissible interval I.

(i) We say that u is *μ-admissible* if for each $r \in I$ there is a μ-null set $N \in \mathscr{B}([0,T])$ such that $u((N^c \cap [r,T]) \times S)$ is relatively compact in D. Moreover, u is called *admissible* if $u([r,T] \times S)$ is in fact relatively compact in D° for all $r \in I$.

(ii) Let u be an admissible solution to (M) on I. Then we call u *extendible* if there is another admissible solution $\tilde{u}$ to (M) on some admissible interval $\tilde{I}$ with $I \subsetneq \tilde{I}$ and $u = \tilde{u}$ on $I \times S$. Otherwise, u is *non-extendible* and I is called a *maximal interval of existence.*

2.5 The main results

We begin with non-extendibility and assume until the end of the paper that g is bounded, as this requirement is necessary for an admissible solution to exist.

Theorem 2.9. *Let D be convex, $f \in BC_\mu^{1-}([0,T] \times S \times D, \mathbb{R}^k)$, and g be bounded away from ∂D. Then there is a unique non-extendible admissible solution u_g to* (M) *on a maximal interval of existence I_g that is open in $[0,T]$. With $t_g^- := \inf I_g$ either $I_g = [0,T]$ or*

$$\liminf_{t \downarrow t_g^-} \inf_{x \in S} \min\left\{\mathrm{dist}(u_g(t,x),\partial D), \frac{1}{1 + |u_g(t,x)|}\right\} = 0. \tag{B}$$

Moreover, if $\mathscr{X}$ is (right-hand) Feller, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$, and g is continuous, then u_g is (right-)continuous.

Let us for the moment assume that the hypotheses of the theorem hold. If u_g is bounded away from ∂D, that is, if $\mathrm{dist}(u_g(t,x),\partial D) \geq \varepsilon$ for all $(t,x) \in I_g \times S$ and some $\varepsilon > 0$, and $I_g \neq [0,T]$, then from (B) it follows that

$$\limsup_{t \downarrow t_g^-} \sup_{x \in S} |u_g(t,x)| = \infty.$$

Let us instead suppose that u_g is bounded. For instance, this occurs whenever f is affine μ-bounded, by Lemma 3.5. Then the theorem says that either u_g is a global solution or

$$\liminf_{t \downarrow t_g^-} \inf_{x \in S} \mathrm{dist}(u_g(t,x),\partial D) = 0. \tag{2.4}$$

In particular, if u_g is not only bounded, but also its image $u_g(I_g \times S)$ is relatively compact in D°, then $I_g = [0,T]$. In the case $D = \mathbb{R}^k$, we combine these considerations with a Picard iteration to obtain the following result, which just requires local Lipschitz μ-continuity of f.

Proposition 2.10. *Let $D = \mathbb{R}^k$ and $f \in BC_\mu^{1-}([0,T] \times S \times \mathbb{R}^k, \mathbb{R}^k)$. Assume that f is affine μ-bounded, then $I_g = [0,T]$ and the sequence $(u_n)_{n \in \mathbb{N}_0}$ in $B_b([0,T] \times S, \mathbb{R}^k)$, defined recursively by $u_0(r,x) := E_{r,x}[g(X_T)]$ and*

$$u_n(r,x) := u_0(r,x) - E_{r,x}\left[\int_r^T f(s,X_s,u_{n-1}(s,X_s))\,\mu(ds)\right]$$

for all $n \in \mathbb{N}$, converges uniformly to u_g, the unique global bounded solution to (M)*.*

Let us at this place assume that $D = \mathbb{R}^k$ and f is an affine map in $w \in \mathbb{R}^k$. In other words, there are two maps $a : [0,T] \times S \to \mathbb{R}^k$ and $b : [0,T] \times S \to \mathbb{R}^{k \times k}$ such that

$$f(t,x,w) = a(t,x) + b(t,x)w$$

for all $(t,x,w) \in [0,T] \times S \times \mathbb{R}^k$. As a and b are necessarily Borel measurable, we infer from Examples 2.5 that f is affine μ-bounded and Lipschitz μ-continuous as soon as a and b are μ-dominated. Thus, we get a multidimensional Feynman-Kac formula, which for $k = 1$ follows from Dynkin [6, Theorem 4.1.1] provided $a = 0$ and $b \geq 0$.

Proposition 2.11. *Let $D = \mathbb{R}^k$ and suppose that $f(t,x,w) = a(t,x) + b(t,x)w$ for every $(t,x,w) \in [0,T] \times S \times \mathbb{R}^k$ and some μ-dominated $a \in B([0,T] \times S, \mathbb{R}^k)$ and $b \in B([0,T] \times S, \mathbb{R}^{k \times k})$. Then $I_g = [0,T]$ and*

$$u_g(r,x) = E_{r,x}[\Sigma_{r,T} g(X_T)] - E_{r,x}\left[\int_r^T \Sigma_{r,t} a(t,X_t)\,\mu(dt)\right] \tag{2.5}$$

for all $(r,x) \in [0,T] \times S$ and some map $\Sigma : [0,T] \times [0,T] \times \Omega \to \mathbb{R}^{k \times k}$, $(r,t,\omega) \mapsto \Sigma_{r,t}(\omega)$ with the following three properties:

(i) $\Sigma_{r,t}$ is $\sigma(X_s : s \in [r,t])$-measurable, $|\Sigma_{r,t}| \leq \sqrt{k} e^{\int_r^t |b(s,X_s)|\,\mu(ds)}$, and $\Sigma(\omega)$ is continuous for all $r,t \in [0,T]$ with $r \leq t$ and each $\omega \in \Omega$.

(ii) $\Sigma_{r,r} = \mathbb{I}_k$, $\Sigma_{r,s}\Sigma_{s,t} = \Sigma_{r,t}$, and $\Sigma_{r,t}(\omega)$ is an invertible matrix with $\Sigma_{r,t}(\omega)^{-1} = \Sigma_{t,r}(\omega)$ for all $r,s,t \in [0,T]$ and every $\omega \in \Omega$.

(iii) If $b(r,x)b(s,y) = b(s,y)b(r,x)$ for all $(r,x),(s,y) \in [0,T] \times S$, then $\Sigma_{r,t} = e^{-\int_r^t b(s,X_s)\,\mu(ds)}$ for all $r,t \in [0,T]$ with $r \leq t$.

Clearly, if there are a μ-dominated $c \in B([0,T]\times S,\mathbb{R})$ and $B \in \mathbb{R}^{k\times k}$ such that the map b in above proposition is of the form $b(t,x) = c(t,x)B$ for all $(t,x) \in [0,T]\times S$, then the commutation condition in (iii) holds. Hence, we may consider an example involving trigonometric functions.

Example 2.12. Let $k = 2$ and $a = 0$. Suppose that there are a μ-dominated $c \in B([0,T]\times S,\mathbb{R})$ and $\delta,\varepsilon \in \mathbb{R}\backslash\{0\}$ such that

$$b(r,x) = c(r,x)\begin{pmatrix} 0 & \delta \\ \varepsilon & 0 \end{pmatrix} \quad \text{for all } (r,x) \in [0,T]\times S.$$

We set $\rho := 1$, if $\delta\varepsilon > 0$, and $\rho := i \in \mathbb{C}$, otherwise. Then we can write u_g in the form

$$\begin{aligned}(u_g)_1(r,x) &= E_{r,x}\bigg[\cosh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\mu(ds)\bigg)g_1(X_T)\bigg] \\ &\quad + \rho\frac{\sqrt{|\delta\varepsilon|}}{\varepsilon}E_{r,x}\bigg[\sinh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\mu(ds)\bigg)g_2(X_T)\bigg], \\ (u_g)_2(r,x) &= \rho\frac{\sqrt{|\delta\varepsilon|}}{\delta}E_{r,x}\bigg[\sinh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\mu(ds)\bigg)g_1(X_T)\bigg] \\ &\quad + E_{r,x}\bigg[\cosh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\mu(ds)\bigg)g_2(X_T)\bigg]\end{aligned}$$

for all $(r,x) \in [0,T]\times S$.

Let us now restrict our attention to $k = 1$. While Proposition 2.10 covers the case $D = \mathbb{R}$, we can also derive global solutions if D is a non-degenerate interval.

Theorem 2.13. *Let D be a non-degenerate interval with $\underline{d} := \inf D$ and $\overline{d} := \sup D$. Assume that $f \in BC_\mu^{1-}([0,T]\times S\times D,\mathbb{R})$ and the following two conditions hold:*

(i) Whenever $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$), then f is both locally μ-bounded and locally Lipschitz μ-continuous at $\underline{d}$ (resp. $\overline{d}$) with $\lim_{w\downarrow\underline{d}} f(\cdot,x,w) \le 0$ (resp. $\lim_{w\uparrow\overline{d}} f(\cdot,x,w) \ge 0$) for all $x \in S$ μ-a.s.

(ii) If $\underline{d} = -\infty$ (resp. $\overline{d} = \infty$), then f is affine μ-bounded from above (resp. from below).

Then there is a unique global bounded solution $\overline{u}_g$ to (M) *that agrees with u_g if g is bounded away from $\{\underline{d},\overline{d}\}\cap\mathbb{R}$. Moreover, if $\mathscr{X}$ is (right-hand) Feller, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$, and g is continuous, then $\overline{u}_g$ is (right-)continuous.*

In the case $D = \mathbb{R}_+$, global bounded solutions to (M) can be expressed via the log-Laplace functionals of superprocesses provided f admits the representation required below.

Example 2.14. Let $D = \mathbb{R}_+$ and $b,c \in B_b([0,T]\times S,\mathbb{R}_+)$. We let n be a kernel from $[0,T]\times S$ to $(0,\infty)$ for which $\int_0^\infty u\wedge u^2\,n(\cdot,\cdot,du)$ is bounded. Assume that f is of the form

$$f(t,x,w) = b(t,x)w + c(t,x)w^2 + \int_0^\infty (e^{-uw} - 1 + uw)\,n(t,x,du) \tag{2.6}$$

for all $(t,x,w) \in [0,T]\times S\times\mathbb{R}_+$, then $f \in BC_\mu^{1-}([0,T]\times S\times\mathbb{R}_+,\mathbb{R}_+)$, due to Examples 2.5. Hence, Theorem 2.13 applies. For instance, let for the moment $n \in \mathbb{N}$, $\alpha_1,\ldots,\alpha_n \in (1,2)$, and $d_1,\ldots,d_n \in B_b([0,T]\times S,\mathbb{R}_+)$, then f could admit the representation

$$f(t,x,w) = b(t,x)w + c(t,x)w^2 + \sum_{i=1}^n d_i(t,x)w^{\alpha_i}$$

for each $(t,x,w) \in [0,T] \times S \times \mathbb{R}_+$. This follows from integration by parts and the choice $n(t,x,B) = \sum_{i=1}^n d_i(t,x)\alpha_i((\alpha_i - 1)/\Gamma(2-\alpha_i)) \int_B u^{-1-\alpha_i}\,du$ for all $(t,x) \in [0,T] \times S$ and each Borel set B in $(0,\infty)$, where Γ denotes the Gamma function.

In the general case (2.6), Theorem 1.1 in Dynkin [3] yields an $(\mathscr{X},\mu,f)$-superprocess, which is a consistent progressive Markov process $\mathscr{Z} = (Z,(\mathscr{G}_t)_{t\in[0,T]},\mathbb{Q})$ with state space $\mathscr{M}_f(S)$, the Polish space of all finite Borel measures on S, such that for each $t \in (0,T]$ and every $\tilde{g} \in B_b(S,\mathbb{R}_+)$, the function

$$[0,t] \times S \to \mathbb{R}_+, \quad (r,x) \mapsto -\log\left(E^Q_{r,\delta_x}\left[e^{-\int_S \tilde{g}(x)\,dZ_t(x)}\right]\right)$$

is Borel measurable and a global solution to (M) when T and g are replaced by t and $\tilde{g}$, respectively. Here, $\mathbb{Q}$ is of the form $\mathbb{Q} = \{Q_{r,\lambda} \,|\, (r,\lambda) \in [0,T] \times \mathscr{M}_f(S)\}$ and E^Q_{r,δ_x} denotes the expectation with respect to Q_{r,δ_x} for all $(r,x) \in [0,T] \times S$. Thus,

$$\overline{u}_g(r,x) = -\log\left(E^Q_{r,\delta_x}\left[e^{-\int_S g(x)\,dZ_T(x)}\right]\right)$$

for each $(r,x) \in [0,T] \times S$.

Finally, a combination of Theorem 2.13 with Proposition 2.11 gives the following result.

Corollary 2.15. *Suppose that D is a non-degenerate interval with $\underline{d} := \inf D$ and $\overline{d} := \sup D$, and there are two μ-dominated $a, b \in B([0,T] \times S, \mathbb{R})$ such that*

$$f(t,x,w) = a(t,x) + b(t,x)w \quad \text{for all } (t,x,w) \in [0,T] \times S \times D.$$

Additionally, for $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$) let $a(\cdot,x) + b(\cdot,x)\underline{d} \leq 0$ (resp. $a(\cdot,x) + b(\cdot,x)\overline{d} \geq 0$) for all $x \in S$ μ-a.s. Then

$$\overline{u}_g(r,x) = E_{r,x}\left[e^{-\int_r^T b(s,X_s)\,\mu(ds)} g(X_T)\right] - E_{r,x}\left[\int_r^T e^{-\int_r^t b(s,X_s)\,\mu(ds)} a(t,X_t)\,\mu(dt)\right] \tag{2.7}$$

for every $(r,x) \in [0,T] \times S$. Furthermore, whenever $\mathscr{X}$ is (right-hand) Feller, $a(s,\cdot)$ and $b(s,\cdot)$ are continuous for μ-a.e. $s \in [0,T]$, and g is continuous, then $\overline{u}_g$ is (right-)continuous.

3 Approach to the main results

3.1 Comparison, stability, and growth behavior of solutions

By using consistent boundedness and local dominance, we give a Markovian Gronwall inequality. A well-known result in this direction is provided by Dynkin [2, Lemma 3.2].

Lemma 3.1. *Let I be an admissible interval, $h \in B(S,\mathbb{R}_+)$ be such that $E_{r,x}[|h(X_T)|] < \infty$ for all $(r,x) \in [0,T] \times S$, and $a, b \in B(I \times S, \mathbb{R}_+)$ be locally μ-dominated. Suppose that $u \in B(I \times S, \mathbb{R}_+)$ is μ-suitably bounded and fulfills*

$$u(r,x) \leq E_{r,x}[h(X_T)] + E_{r,x}\left[\int_r^T a(s,X_s) + b(s,X_s)u(s,X_s)\,\mu(ds)\right]$$

for each $(r,x) \in I \times S$, then

$$u(r,x) \leq E_{r,x}\left[e^{\int_r^T b(s,X_s)\,\mu(ds)}\left(h(X_T) + \int_r^T a(s,X_s)\,\mu(ds)\right)\right]$$

for every $(r,x) \in I \times S$.

Proof. It follows inductively from the Markov property of $\mathscr{X}$ and integration by parts that

$$u(r,x) \leq \sum_{i=0}^{n} E_{r,x}\bigg[\frac{1}{i!}\int_r^T \Big(b(s,X_s)\,\mu(ds)\Big)^i \Big(h(X_T) + \int_r^T a(s,X_s)\mu(ds)\Big)\bigg]$$
$$+ E_{r,x}\bigg[\int_r^T \Big(\int_r^t b(s,X_s)\,\mu(ds)\Big)^n \frac{b(t,X_t)}{n!} u(t,X_t)\,\mu(dt)\bigg]$$

for all $(r,x) \in I \times S$ and each $n \in \mathbb{N}$. Since u is μ-suitably bounded, dominated convergence yields that

$$\lim_{n\uparrow\infty} E_{r,x}\bigg[\int_r^T \Big(\int_r^t b(s,X_s)\,\mu(ds)\Big)^n \frac{b(t,X_t)}{n!} u(t,X_t)\,\mu(dt)\bigg] = 0$$

for each $(r,x) \in I \times S$. Hence, monotone convergence gives the asserted estimate. ☐

Let us compare approximate solutions.

Lemma 3.2. *Assume that $f|([0,T]\times S\times W)$ is Lipschitz μ-continuous for some set W in D. That is, there is a μ-dominated $\lambda \in B([0,T]\times S,\mathbb{R}_+)$ such that*

$$|f(t,x,w) - f(t,x,w')| \leq \lambda(t,x)|w-w'| \quad \textit{for all } (t,x) \in [0,T]\times S$$

and each $w,w' \in W$. Let $\varepsilon,\tilde{\varepsilon} \in B([0,T]\times S,\mathbb{R}_+)$ be μ-dominated, $\tilde{g} \in B_b(S,D)$ be consistent, and I be an admissible interval. Then every ε-approximate solution u to (M) *on I and each $\tilde{\varepsilon}$-approximate solution $\tilde{u}$ to* (M) *on I, where g is replaced by $\tilde{g}$, satisfy*

$$|u-\tilde{u}|(r,x) \leq E_{r,x}\bigg[e^{\int_r^T \lambda(s,X_s)\,\mu(ds)}\Big(|g-\tilde{g}|(X_T) + \int_r^T (\varepsilon+\tilde{\varepsilon})(s,X_s)\,\mu(ds)\Big)\bigg]$$

for all $(r,x) \in I\times S$ provided u, $\tilde{u}$ are μ-suitably bounded and $u(\cdot,y), \tilde{u}(\cdot,y) \in W$ for each $y \in S$ μ-a.s. on I.

Proof. The triangle inequality yields that

$$|u-\tilde{u}|(r,x) \leq E_{r,x}[|g-\tilde{g}|(X_T)] + E_{r,x}\bigg[\int_r^T (\varepsilon+\tilde{\varepsilon})(s,X_s) + \lambda(s,X_s)|u-\tilde{u}|(s,X_s)\,\mu(ds)\bigg]$$

for each $(r,x) \in I\times S$, since $|f(s,X_s,u(s,X_s)) - f(s,X_s,\tilde{u}(s,X_s))| \leq \lambda(s,X_s)|u-\tilde{u}|(s,X_s)$ for μ-a.e. $s \in [r,T]$. Hence, Lemma 3.1 leads us to the asserted estimate. ☐

From the comparison we get an uniqueness result provided f belongs to (2.1). Note that the procedure of the proof originates from Theorem 6.7 in Amann [1].

Corollary 3.3. *Suppose that $f \in BC_\mu^{1-}([0,T]\times S\times D,\mathbb{R}^k)$. Then there is at most a unique μ-admissible solution to* (M) *on every admissible interval I.*

Proof. Suppose that u and $\tilde{u}$ are two μ-admissible solutions to (M) on I and let $r \in I$. Then there is a compact set K in D such that $u(\cdot,y), \tilde{u}(\cdot,y) \in K$ for all $y \in S$ μ-a.s. on $[r,T]$. As K is compact, it follows despite of minor modifications from Proposition 6.4 in Amann [1] that there is a neighborhood W of K in D such that $f|([0,T]\times S\times W)$ is Lipschitz μ-continuous. Hence, $u = \tilde{u}$ on $[r,T]\times S$, by Lemma 3.2. The assertion follows. ☐

Now, we consider stability.

Proposition 3.4. *Let $f \in BC_\mu^{1-}([0,T]\times S\times D,\mathbb{R}^k)$ and I be an admissible interval. For each $n \in \mathbb{N}$ let $\varepsilon_n \in B([0,T]\times S,\mathbb{R}_+)$ be μ-dominated, $g_n \in B_b(S,D)$ be consistent, and u_n be an ε_n-approximate solution to* (M) *on I with g replaced by g_n. Assume that the following three conditions hold:*

(i) $(g_n)_{n\in\mathbb{N}}$ *and* $\left(\int_0^T \varepsilon_n(t,X_t)\,\mu(dt)\right)_{n\in\mathbb{N}}$ *converge uniformly to* g *and* 0*, respectively.*

(ii) The closure of $\{u_n(r,x)\,|\,n\in\mathbb{N}\}$ *is included in* D *for each* $(r,x)\in I\times S$.

(iii) For each $r\in I$ *there is a compact set* K *in* D *such that* $u_n(\cdot,y)\in K$ *for all* $n\in\mathbb{N}$ *and each* $y\in S$ μ*-a.s. on* $[r,T]$.

Then $(u_n)_{n\in\mathbb{N}}$ *converges locally uniformly in* $t\in I$ *and uniformly in* $x\in S$ *to the unique* μ*-admissible solution to* (M) *on* I.

Proof. As uniqueness is covered by Corollary 3.3, we turn directly to the existence claim. Let $r\in I$ and K be a compact set K in D so that $u_n(\cdot,y)\in K$ for all $n\in\mathbb{N}$ and each $y\in S$ μ-a.s. on $[r,T]$. Then there is a neighborhood W of K in D and a μ-dominated $\lambda\in B([r,T]\times S,\mathbb{R}_+)$ with $|f(t,x,w)-f(t,x,w')|\leq\lambda(t,x)|w-w'|$ for all $(t,x)\in[r,T]\times S$ and each $w,w'\in W$. Thus, Lemma 3.2 ensures that

$$|u_m-u_n|(s,x)\leq E_{s,x}\bigg[e^{\int_s^T\lambda(t,X_t)\,\mu(dt)}\bigg(|g_m-g_n|(X_T)+\int_s^T(\varepsilon_m+\varepsilon_n)(t,X_t)\,\mu(dt)\bigg)\bigg]$$

for all $m,n\in\mathbb{N}$ and every $(s,x)\in[r,T]\times S$. From (i) we infer that $(u_n)_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence on $[r,T]\times S$. As (ii) holds and $r\in I$ has been arbitrarily chosen, this shows that $(u_n)_{n\in\mathbb{N}}$ converges locally uniformly in $t\in I$ and uniformly in $x\in S$ to some map $u\in B(I\times S,D)$.

We now check that u is a μ-admissible solution to (M) on I. Let as before $r\in I$ and K be a compact set in D with $u_n(\cdot,y)\in K$ for all $n\in\mathbb{N}$ and each $y\in S$ μ-a.s. on $[r,T]$, which gives $u(\cdot,y)\in K$ for all $y\in S$ μ-a.s. on $[r,T]$. Let us pick a μ-dominated $\lambda\in B([r,T]\times S,\mathbb{R}_+)$ with $|f(t,x,w)-f(t,x,w')|\leq\lambda(t,x)|w-w'|$ for all $(t,x)\in[r,T]\times S$ and every $w,w'\in K$, then

$$\begin{aligned}&\bigg|u_n(s,x)-E_{s,x}[g(X_T)]-E_{s,x}\bigg[\int_s^T f(t,X_t,u(t,X_t))\,\mu(dt)\bigg]\bigg|\\&\qquad\leq E_{s,x}[|g_n-g|(X_T)]+E_{s,x}\bigg[\int_s^T\lambda(t,X_t)|u_n-u|(t,X_t)+\varepsilon_n(t,X_t)\,\mu(dt)\bigg]\end{aligned}$$

for all $n\in\mathbb{N}$ and each $(s,x)\in[r,T]\times S$. This entails that $(u_n)_{n\in\mathbb{N}}$ also converges locally uniformly in $t\in I$ and uniformly in $x\in S$ to the map

$$I\times S\to\mathbb{R}^k,\quad (r,x)\mapsto E_{r,x}[g(X_T)]-E_{r,x}\bigg[\int_r^T f(s,X_s,u(s,X_s))\,\mu(ds)\bigg],$$

which proves the proposition. □

We conclude with a growth estimate.

Lemma 3.5. *Assume that* f *is affine* μ*-bounded. In other words, there are two* μ*-dominated* $a,b\in B([0,T]\times S,\mathbb{R}_+)$ *with* $|f(t,x,w)|\leq a(t,x)+b(t,x)|w|$ *for all* $(t,x,w)\in[0,T]\times S\times D$. *Then every* μ*-suitably bounded solution* u *to* (M) *on* I *fulfills*

$$|u(r,x)|\leq E_{r,x}\bigg[e^{\int_r^T b(s,X_s)\,\mu(ds)}\bigg(|g(X_T)|+\int_r^T a(s,X_s)\,\mu(ds)\bigg)\bigg]\tag{3.1}$$

for each $(r,x)\in I\times S$.

Proof. We see that $|u(r,x)|\leq E_{r,x}[|g(X_T)|]+E_{r,x}\big[\int_r^T a(s,X_s)+b(s,X_s)|u(s,X_s)|\,\mu(ds)\big]$ for every $(r,x)\in I\times S$. In consequence, Lemma 3.1 gives the claimed estimate. □

3.2 Local existence in time

We aim to construct an approximate solution locally in time. Once this is achieved, we apply the stability result of the previous section to deduce a solution as uniform limit of a sequence of approximate solutions. This is a common approach in the classical theory of ODEs (see for instance Section 7 in Amann [1]).

For each $\beta > 0$ we define $N_{\mathscr{X},b}(g)$ to be the set of all $w \in \mathbb{R}^k$ such that $|w - E_{r,x}[g(X_T)]| < \beta$ for some $(r,x) \in [0,T] \times S$. Because we are dealing with the transition probabilities $\mathbb{P}$, the convexity of D should be required, as the lemma below indicates.

Lemma 3.6. *Let D be convex and g be bounded away from ∂D, that means, there is $\varepsilon > 0$ such that $\mathrm{dist}(g(x), \partial D) \geq \varepsilon$ for all $x \in S$. Then there exists $\beta > 0$ such that*

$$N_{\mathscr{X},\beta}(g) \text{ is relatively compact in } D^\circ. \tag{3.2}$$

Proof. Let K be a compact set in D° such that $g(S) \subset K$, then $\int_S g(x)\, P(dx)$ belongs to the convex hull of K for each probability measure P on $(S, \mathscr{B}(S))$. As the convexity of D entails that of D°, it follows from Carathéodory's Convex Hull Theorem that along with K the convex hull of K is a compact set in D°. Hence, there is $\beta > 0$ so that $\inf_{(r,x)\in[0,T]\times S} \mathrm{dist}(E_{r,x}[g(X_T)], \partial D) > \beta$. Since $N_{\mathscr{X},\beta}(g)$ is simply the β-neighborhood of $\{E_{r,x}[g(X_T)] \,|\, (r,x) \in [0,T] \times S\}$, the asserted condition (3.2) follows. ☐

Until the end of this section, let D be convex, f be locally μ-bounded, and g be bounded away from ∂D. Due Lemma 3.6, we can choose $\beta > 0$ satisfying (3.2). Let $a \in B([0,T] \times S, \mathbb{R}_+)$ be μ-dominated such that $|f(t,x,w)| \leq a(t,x)$ for all $(t,x) \in [0,T] \times S$ and each $w \in \overline{N}_{\mathscr{X},\beta}(g)$, the closure of $N_{\mathscr{X},\beta}(g)$. Then

$$E_{r,x}\left[\int_r^T a(s, X_s)\, \mu(ds)\right] \leq \beta \tag{3.3}$$

for all $(r,x) \in [T-\alpha, T] \times S$ and some $\alpha \in (0,T]$. The choices of β and α such that (3.2) and (3.3) hold, respectively, are used to construct a $\overline{N}_{\mathscr{X},\beta}(g)$-valued solution to (M) on $[T-\alpha, T]$.

Proposition 3.7. *Suppose that $\varepsilon \in B([0,T] \times S, \mathbb{R}_+)$ is μ-dominated and there is $\delta > 0$ so that $|f(t,x,w) - f(t,x,w')| \leq \varepsilon(t,x)$ for all $(t,x) \in [0,T] \times S$ and each $w, w' \in \overline{N}_{\mathscr{X},\beta}(g)$ with $|w - w'| < \delta$. Then there is an $\overline{N}_{\mathscr{X},\beta}(g)$-valued ε-approximate solution u to* (M) *on $[T-\alpha, T]$. In addition, if $\mathscr{X}$ is (right-hand) Feller, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$, and g is continuous, then u is (right-)continuous.*

Proof. At first, since a is μ-dominated, there is $\eta \in (0, \alpha]$ such that $E_{r,x}\left[\int_r^t a(s, X_s)\, \mu(ds)\right] < \delta$ for all $r, t \in [T-\alpha, T]$ with $r \leq t < r + \eta$ and each $x \in S$. Given η, we choose $n \in \mathbb{N}$ and $t_0, \ldots, t_n \in [T-\alpha, T]$ such that

$$T - \alpha = t_n < \cdots < t_0 = T \quad \text{and} \quad \max_{i\in\{1,\ldots,n\}} (t_i - t_{i-1}) < \eta.$$

Starting with $u_0 : [T-\alpha, T] \times S \to \overline{N}_{\mathscr{X},\beta}(g)$ given by $u_0(r,x) := E_{r,x}[g(X_T)]$, we recursively introduce a sequence $(u_i)_{i\in\{1,\ldots,n\}}$ of consistent Borel measurable maps, by letting for each $i \in \{0, \ldots, n-1\}$ the map $u_{i+1} : [t_{i+1}, t_i] \times S \to \overline{N}_{\mathscr{X},\beta}(g)$ be defined via

$$u_{i+1}(r,x) := E_{r,x}[u_i(t_i, X_{t_i})] - E_{r,x}\left[\int_r^{t_i} f(s, X_s, E_{s,X_s}[u_i(t_i, X_{t_i})])\, \mu(ds)\right].$$

It follows by induction over $i \in \{1, \dots, n\}$ that u_i is indeed a well-defined consistent Borel measurable map taking all its values in $\overline{N}_{\mathscr{X},\beta}(g)$ such that

$$|E_{r,x}[u_i(t, X_t)] - u_i(r, x)| \leq E_{r,x}\left[\int_r^t a(s, X_s)\,\mu(ds)\right] \tag{3.4}$$

for all $r, t \in [t_i, t_{i-1}]$ with $r \leq t$ and each $x \in S$. This is an immediate consequence of the facts that $u_i(t_i, x) = u_{i-1}(t_i, x)$ and $|E_{r,x}[u_i(t, X_t)] - u_0(r, x)| \leq E_{r,x}\left[\int_t^{t_0} a(t', X_{t'})\,\mu(dt')\right]$ for each $r, t \in [t_i, t_{i-1}]$ with $r \leq t$ and every $x \in S$.

The crucial outcome of this construction is that if we define $u : [T - \alpha, T] \times S \to \overline{N}_{\mathscr{X},\beta}(g)$ by $u(r, x) := u_i(r, x)$ with $i \in \{1, \dots, n\}$ such that $r \in [t_i, t_{i-1}]$, then u is an ε-approximate solution to (M) on $[T - \alpha, T]$. To see this, let $i \in \{1, \dots, n\}$, then

$$\begin{aligned}
&\left|E_{r,x}[u(t, X_t)] - u(r, x) - E_{r,x}\left[\int_r^t f(s, X_s, u(s, X_s))\,\mu(ds)\right]\right| \\
&\qquad = \left|E_{r,x}\left[\int_r^t f(s, X_s, E_{s,X_s}[u_{i-1}(t_{i-1}, X_{t_{i-1}})]) - f(s, X_s, u_i(s, X_s))\,\mu(ds)\right]\right| \\
&\qquad \leq E_{r,x}\left[\int_r^t \varepsilon(s, X_s)\,\mu(ds)\right]
\end{aligned}$$

for every $r, t \in [t_i, t_{i-1}]$ with $r \leq t$ and each $x \in S$, since $u_{i-1}(t_{i-1}, X_{t_{i-1}}) = u_i(t_{i-1}, X_{t_{i-1}})$ and from $t_{i-1} - s \leq \eta$ in combination with (3.4) we infer that $|E_{s,X_s}[u_i(t_{i-1}, X_{t_{i-1}})] - u_i(s, X_s)| < \delta$ for all $s \in [t_i, t_{i-1}]$. Hence, the first assertion follows.

Let us now suppose that $\mathscr{X}$ is (right-hand) Feller, $f(s, \cdot, \cdot)$ is continuous for μ-a.e. $s \in [0, T]$, and g is continuous. Then for each non-degenerate interval I in $[0, T]$ and every right-continuous $\tilde{u} \in B(I \times S, D)$, we see readily that $f(s, \cdot, \tilde{u}(s, \cdot))$ is continuous for μ-a.e. $s \in [0, T]$. In combination with (2.2), it follows inductively that $u_1, \dots, u_n$ are (right-)continuous, which yields the (right-)continuity of u. $\square$

By constructing a suitable sequence of approximate solutions, a local existence result can be derived.

Proposition 3.8. *Let $f \in BC_\mu^{1-}([0, T] \times S \times D, \mathbb{R}^k)$, then there is a unique admissible solution u to* (M) *on $[T - \alpha, T]$, which is $\overline{N}_{\mathscr{X},\beta}(g)$-valued. Moreover, if $\mathscr{X}$ is (right-hand) Feller, $f(s, \cdot, \cdot)$ is continuous for μ-a.e. $s \in [0, T]$, and g is continuous, then u is (right-)continuous.*

Proof. The uniqueness assertion follows directly from Corollary 3.3. To establish existence, we note that, as $\overline{N}_{\mathscr{X},\beta}(g)$ is compact, there exists a μ-dominated $\lambda \in B([T - \alpha, T] \times S, \mathbb{R}_+)$ such that

$$|f(t, x, w) - f(t, x, w')| \leq \lambda(t, x)|w - w'|$$

for all $(t, x) \in [T - \alpha, T] \times S$ and each $w, w' \in \overline{N}_{\mathscr{X},\beta}(g)$. Thus, Proposition 3.7 provides some $\overline{N}_{\mathscr{X},\beta}(g)$-valued (λ/n)-approximate solution u_n to (M) on $[T - \alpha, T]$ for each $n \in \mathbb{N}$. Additionally, if $\mathscr{X}$ is (right-hand) Feller, $f(s, \cdot, \cdot)$ is continuous for μ-a.e. $s \in [0, T]$, and g is continuous, then u_n is (right-)continuous.

Next, Proposition 3.4 entails that $(u_n)_{n \in \mathbb{N}}$ converges uniformly to a $\overline{N}_{\mathscr{X},\beta}(g)$-valued solution u to (M) on $[T - \alpha, T]$, which proves the first claim. Since the uniform limit of a sequence of $\mathbb{R}^k$-valued (right-)continuous maps on $[T - \alpha, T] \times S$ is again (right-)continuous, the second assertion follows directly from what we have just shown. $\square$

Now, we prove a fixed-point result, which we need later on.

Lemma 3.9. *Let I be a compact admissible interval, $\mathscr{H}$ be a closed set in $B_b(I \times S, \mathbb{R}^k)$, and $\Psi : \mathscr{H} \to \mathscr{H}$ be a map for which there is a μ-dominated $\lambda \in B(I \times S, \mathbb{R}_+)$ such that*

$$|\Psi(u) - \Psi(v)|(r,x) \leq E_{r,x}\bigg[\int_r^T \lambda(s, X_s)|u - v|(s, X_s)\,\mu(ds)\bigg] \tag{3.5}$$

for all $u, v \in \mathscr{H}$ and each $(r,x) \in I \times S$. Then for every $u_0 \in \mathscr{H}$, the sequence $(u_n)_{n\in\mathbb{N}_0}$, recursively given by $u_n := \Psi(u_{n-1})$ for all $n \in \mathbb{N}$, converges uniformly to the unique fixed-point of Ψ.

Proof. Because the uniqueness assertion can be easily inferred from Lemma 3.1, we just show that $(u_n)_{n\in\mathbb{N}_0}$ converges uniformly to some fixed-point of Ψ. By induction,

$$|u_{n+1} - u_n|(r,x) \leq E_{r,x}\bigg[\int_r^T \bigg(\int_r^t \lambda(s, X_s)\,\mu(ds)\bigg)^{n-1} \frac{\lambda(t, X_t)}{(n-1)!}\Delta(t, X_t)\,\mu(dt)\bigg]$$

for all $n \in \mathbb{N}$ and every $(r,x) \in I \times S$, where $\Delta := |\Psi(u_0) - u_0|$. From the triangle inequality and integration by parts we obtain that

$$|u_m - u_n|(r,x) \leq \sum_{i=n}^{m-1} \frac{1}{i!} E_{r,x}\bigg[\bigg(\int_r^T \lambda(s, X_s)\,\mu(ds)\bigg)^i\bigg] \sup_{(s,y)\in[r,T]\times S} \Delta(s,y)$$

for all $m, n \in \mathbb{N}$ with $m > n$ and each $(r,x) \in I \times S$. This shows that $(u_n)_{n\in\mathbb{N}_0}$ is a uniformly Cauchy sequence. Since $\mathscr{H}$ is closed in $B_b(I \times S, \mathbb{R}^k)$, it converges uniformly to some $u \in \mathscr{H}$. As $(u_{n+1})_{n\in\mathbb{N}_0}$ also converges uniformly to $\Psi(u)$, we conclude that $u = \Psi(u)$. ☐

Let us indicate another local existence approach.

Remark 3.10. The set $\mathscr{H} := B_b([T - \alpha, T] \times S, \overline{N}_{\mathscr{X},\beta}(g))$ is closed in $B_b([T - \alpha, T] \times S, \mathbb{R}^k)$ and (3.3) guarantees that the map $\Psi : \mathscr{H} \to B([T - \alpha, T] \times S, \mathbb{R}^k)$ defined via

$$\Psi(u)(r,x) := E_{r,x}[g(X_T)] - E_{r,x}\bigg[\int_r^T f(s, X_s, u(s, X_s))\,\mu(ds)\bigg]$$

maps $\mathscr{H}$ into itself. So, let f be locally Lipschitz μ-continuous, then there is a μ-dominated $\lambda \in B([T - \alpha, T] \times S, \mathbb{R}_+)$ satisfying (3.5) for all $u, v \in \mathscr{H}$ and each $(r,x) \in [T - \alpha, T] \times S$. For this reason, Lemma 3.9 implies that Ψ has a unique fixed-point u, which is exactly the unique admissible solution to (M) on $[T - \alpha, T]$ that takes all its values in $\overline{N}_{\mathscr{X},b}(g)$.

Moreover, if $\mathscr{X}$ is (right-hand) Feller, $f(s, \cdot, \cdot)$ is continuous for μ-a.e. $s \in [0, T]$, and g is continuous, then from (2.2) we see that Ψ preserves (right-)continuity in the sense that $\Psi(\tilde{u})$ is (right-)continuous whenever $\tilde{u} \in \mathscr{H}$ is. Thus, in this case, u is (right-)continuous as uniform limit of a sequence of (right-)continuous maps in $\mathscr{H}$.

4 Proofs of the main results

4.1 Proof of Theorem 2.9

After having constructed solutions locally in time, we derive unique non-extendible admissible solutions and provide conditions ensuring their continuity. In this regard, the proof of Theorem 7.6 in Amann [1] has been be a good source for ideas.

Proof of Theorem 2.9. We begin with the first claim and define I_g to be the set consisting of $\{T\}$ and of all $t \in [0,T)$ for which (M) admits an admissible solution on $[t,T]$. By Proposition 3.8, we have $\{T\} \subsetneq I_g$ and hence, $t_g^- = \inf I_g < T$. Let $t \in (t_g^-, T]$, then there is $s \in I_g$ with $s < t$, which means that there is an admissible solution u to (M) on $[s,T]$. As $u|([t,T]\times S)$ is an admissible solution to (M) on $[t,T]$, we get that $t \in I_g$. Thus, I_g is an admissible interval.

To verify that I_g is open in $[0,T]$, we have to show that if $I_g \neq [0,T]$, then $t_g^- \notin I_g$. On the contrary, assume that $I_g \neq [0,T]$, but $t_g^- \in I_g$. Then $t_g^- > 0$ and there is an admissible solution u to (M) on $[t_g^-, T]$. Since $u(t_g^-,\cdot)$ is both bounded and bounded away from ∂D, Proposition 3.8 entails that the Markovian terminal value problem (M) with T and g replaced by t_g^- and $u(t_g^-,\cdot)$, respectively, has an admissible solution v on $[t_g^- - \alpha, t_g^-]$ for some $\alpha \in (0, t_g^-]$. Consequently, the map $w : [t_g^- - \alpha, T] \times S \to D^\circ$ given by $w(r,x) := u(r,x)$, if $r \geq t_g^-$, and $w(r,x) := v(r,x)$, otherwise, is another admissible solution to (M) on $[t_g^- - \alpha, T]$ extending u and v. We conclude that $t_g^- - \alpha \in I_g$, which contradicts the definition of t_g^-.

Let us now introduce the unique non-extendible admissible solution to (M). We recall that if $r,t \in I_g$ satisfy $r \leq t$, and u, v are two admissible solutions to (M) on $[r,T]$ and $[t,T]$, respectively, then $u = v$ on $[t,T]\times S$, due to Corollary 3.3. So, for each $r \in I_g$ we can mark the unique admissible solution to (M) on $[r,T]$ by u_r. Then

$$u_g : I_g \times S \to D^\circ, \quad u_g(r,x) := u_r(r,x)$$

is the unique non-extendible admissible solution to (M). In fact, if $t_g^- \in I_g$, which occurs if and only if $t_g^- = 0$ and $I_g = [0,T]$, then $u_g(r,x) = u_{t_g^-}(r,x)$ for all $(r,x) \in [0,T]\times S$. This in turn implies that u_g is well-defined and a global admissible solution. Now, let instead $t_g^- \notin I_g$, then $I_g = (t_g^-, T]$. In this case, we pick a strictly decreasing sequence $(t_n)_{n\in\mathbb{N}}$ in I_g with $\lim_{n\uparrow\infty} t_n = t_g^-$, then

$$u_g^{-1}(B) = \bigcup_{n\in\mathbb{N}} u_{t_n}^{-1}(B) \in \mathscr{B}(I_g \times S)$$

for all $B \in \mathscr{B}(D)$, since $u_{t_n}^{-1}(B) \in \mathscr{B}([t_n,T]\times S)$ for each $n \in \mathbb{N}$. Thus, u_g is Borel measurable. The representation $u_g|([r,T]\times S) = u_r$ for each $r \in I_g$ implies that u_g is an admissible solution to (M) on I_g. Finally, suppose that I is an admissible interval with $I_g \subsetneq I$ and u is an admissible solution to (M) on I, then there is $t \in I$ with $t \leq t_g^-$. By the definition of I_g, we obtain that $t \in I_g$, which is a contradiction to $I_g = (t_g^-, T]$. This justifies that u_g is non-extendible.

We turn to the second claim. By way of contradiction, assume that $I_g \neq [0,T]$, but (B) fails. Then $I_g = (t_g^-, T]$, and there are $\varepsilon \in (0, 1/\sqrt{2})$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in I_g with $\lim_{n\uparrow\infty} t_n = t_g^-$ such that

$$\inf_{x\in S} \min\left\{\operatorname{dist}(u_g(t_n,x),\partial D), \frac{1}{1+|u_g(t_n,x)|}\right\} \geq 2\varepsilon$$

for every $n \in \mathbb{N}$. As $D_\eta := \{w \in D \,|\, \operatorname{dist}(z,\partial D) \geq \eta \text{ and } |w| \leq 1/\eta\}$ is readily seen to be a convex compact set in D° for each $\eta \in (0, 2\varepsilon]$, it holds that $E_{r,x}[u_g(t_n, X_{t_n})] \in D_{2\varepsilon}$ for all $n \in \mathbb{N}$ and each $(r,x) \in [0,t_n]\times S$. Let $a \in B([t_g^-,T]\times S, \mathbb{R}_+)$ be μ-dominated and fulfill

$$|f(t,x,w)| \leq a(t,x)$$

for every $(t,x,w) \in [t_g^-,T]\times S \times D_\varepsilon$, then there exists some $\delta \in (0, T - t_g^-]$ such that $\sup_{x\in S} E_{r,x}\Big[\int_r^t a(s,X_s)\,\mu(ds)\Big] < \varepsilon$ for all $r,t \in [t_g^-,T]$ with $r \leq t < r+\delta$. This entails that

$$u_g(t,S) \text{ is relatively compact in } D_\varepsilon^\circ \tag{4.1}$$

for every $n \in \mathbb{N}$ and each $t \in (t_n - \delta_n, t_n]$, where $\delta_n := \delta \wedge (t_n - t_g^-)$. Indeed, suppose this is false, then there is $n \in \mathbb{N}$ for which $u_g(t,S)$ fails to be relatively compact in D_ε° for at least one $t \in (t_n - \delta_n, t_n]$. We set

$$s_n := \sup\{t \in (t_n - \delta_n, t_n] \,|\, u_g(t,S) \text{ is not relatively compact in } D_\varepsilon^\circ\},$$

then another application of Proposition 3.8 shows that $u_g(s_n, S)$ cannot be relatively compact in D_ε°. In particular, $s_n < t_n$, as $u_g(t_n, S) \subset D_{2\varepsilon}$. These considerations imply that

$$|E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| \leq E_{s_n,x}\bigg[\int_{s_n}^{t_n} a(s, X_s)\,\mu(ds)\bigg] < \varepsilon$$

for every $x \in S$, since $t_n - s_n < \delta_n \leq \delta$. From $E_{s_n,x}[u_g(t_n, X_{t_n})] \in D_{2\varepsilon}$ and $\varepsilon^2 < 1/2$ it follows that $|u_g(s_n, x)| < |E_{s_n,x}[u_g(t_n, X_{t_n})]| + \varepsilon \leq 1/(2\varepsilon) + \varepsilon < 1/\varepsilon$ for each $x \in S$. Moreover,

$$\begin{aligned} \mathrm{dist}(u_g(s_n, x), \partial D) &\geq \mathrm{dist}(E_{s_n,x}[u_g(t_n, X_{t_n})], \partial D) - |E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| \\ &\geq 2\varepsilon - |E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| > \varepsilon \end{aligned}$$

for all $x \in S$. In consequence, it follows that $u_g(s_n, S)$ is relatively compact in D_ε°, which is a contradiction. Therefore, condition (4.1) is valid.

Next, since $\lim_{n\uparrow\infty} t_n = t_g^-$, there is $n_0 \in \mathbb{N}$ such that $t_n - t_g^- \leq \delta$ and hence, $t_n - \delta_n = t_g^-$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Thus, (4.1) leads us to

$$|E_{t_g^-,x}[u_g(r, X_r)] - E_{t_g^-,x}[u_g(t, X_t)]| \leq E_{t_g^-,x}\bigg[\int_r^t a(s, X_s)\,\mu(ds)\bigg] < \varepsilon$$

for every $r, t \in (t_g^-, t_{n_0}]$ with $r \leq t$ and each $x \in S$. For this reason, the map $(t_g^-, T] \times S \to D^\circ$, $(t, x) \mapsto E_{t_g^-,x}[u_g(t, X_t)]$ is uniformly continuous in $t \in (t_g^-, T]$, uniformly in $x \in S$. Thus, there exists a unique map $\hat{w} \in B(S, D_\varepsilon)$ such that

$$\lim_{t\downarrow t_g^-} E_{t_g^-,x}[u_g(t, X_t)] = \hat{w}(x), \quad \text{uniformly in } x \in S.$$

At the same time, it follows from (4.1) together with dominated convergence that

$$\lim_{r\downarrow t_g^-} E_{t_g^-,x}\bigg[\int_r^T f(s, X_s, u_g(s, X_s))\,\mu(ds)\bigg] = E_{t_g^-,x}\bigg[\int_{(t_g^-,T]} f(s, X_s, u_g(s, X_s))\,\mu(ds)\bigg] \tag{4.2}$$

for every $x \in S$. Since the map $(t_g^-, T] \times S \to \mathbb{R}^k$, $(r, x) \mapsto E_{t_g^-,x}\big[\int_r^T f(s, X_s, u_g(s, X_s))\,\mu(ds)\big]$ is uniformly continuous in $r \in (t_g^-, T]$, uniformly in $x \in S$, the limit (4.2) holds in fact uniformly in $x \in S$. Thus, we define $u : [t_g^-, T] \times S \to D^\circ$ by

$$u(t, x) := u_g(t, x), \quad \text{if } t > t_g^-, \quad \text{and} \quad u(t, x) := \hat{w}(x), \quad \text{otherwise},$$

then it is immediate to see that u is another admissible solution to (M) on $[t_g^-, T]$. Hence, $t_g^- \in I_g$, which contradicts that I_g is open in $[0, T]$. This concludes the verification of the second claim.

At last, let $\mathscr{X}$ be (right-hand) Feller, $f(s, \cdot, \cdot)$ be continuous for μ-a.e. $s \in [0, T]$, and g be continuous. We define $\hat{I}_g$ to be the set consisting of $\{T\}$ and of all $t \in [0, T)$ for which (M) admits an admissible (right-)continuous solution on $[t, T]$ and set $\hat{t}_g^- := \inf \hat{I}_g$. Then Proposition 3.8 makes sure that $\{T\} \subsetneq \hat{I}_g$ and thus, $\hat{t}_g^- < T$. Using similar arguments as before, it follows that $\hat{I}_g$ is an admissible interval that is open in $[0, T]$.

By Corollary 3.3, the proof is complete, once we have shown that $\hat{t}_g^- = t_g^-$. Since $\hat{t}_g^- \geq t_g^-$, let us suppose that $\hat{t}_g^- > t_g^-$. Then $\hat{I}_g \neq [0, T]$ and hence, $\hat{I}_g = (\hat{t}_g^-, T]$. As u_g must be (right-)continuous on $\hat{I}_g \times S$ and

$$u_g(r, x) = E_{r,x}[g(X_T)] - E_{r,x}\bigg[\int_r^T f(s, X_s, u_g(s, X_s))\,\mu(ds)\bigg]$$

for all $(r, x) \in [\hat{t}_g^-, T] \times S$, we infer from (2.2) that u_g is in fact right-continuous on $[\hat{t}_g^-, T] \times S$. For this reason, we must face the contradiction that $\hat{t}_g^- \in \hat{I}_g$. This completes the proof. $\square$

4.2 Proofs of Propositions 2.10 and 2.11

Proof of Proposition 2.10. To establish the claim, we invoke Lemma 3.9. First, since f is affine μ-bounded, Lemma 3.5 implies that u_g is bounded, and as (2.4) cannot hold, we get that $I_g = [0,T]$. Hence, u_g is the unique global bounded solution to (M), by Theorem 2.9.

We choose two μ-dominated $a, b \in B([0,T]\times S, \mathbb{R}_+)$ such that $|f(t,x,w)| \leq a(t,x) + b(t,x)|w|$ for all $(t,x,w) \in [0,T] \times S \times \mathbb{R}^k$ and let $\mathscr{H}$ be the set of all $u \in B([0,T] \times S, \mathbb{R}^k)$ satisfying (3.1) for all $(r,x) \in [0,T] \times S$. Then $\mathscr{H}$ is closed in $B_b([0,T] \times S, \mathbb{R}^k)$ and $u_0, u_g \in \mathscr{H}$. We pick two μ-integrable $\overline{a}, \overline{b} \in B([0,T], \mathbb{R}_+)$ with $a(\cdot,y) \leq \overline{a}$ and $b(\cdot,y) \leq \overline{b}$ for all $y \in S$ μ-a.s., and set

$$c := e^{\int_0^T \overline{b}(s)\,\mu(ds)}\bigg(\sup_{y\in S}|g(y)| + \int_0^T \overline{a}(s)\,\mu(ds)\bigg).$$

Then each map $u \in \mathscr{H}$ satisfies $|u(r,x)| \leq c$ for each $(t,x) \in [0,T] \times S$. In addition, we introduce the mapping $\Psi : \mathscr{H} \to B_b([0,T] \times S, \mathbb{R}^k)$ defined via

$$\Psi(u)(r,x) := u_0(r,x) - E_{r,x}\bigg[\int_r^T f(s, X_s, u(s,X_s))\,\mu(ds)\bigg],$$

then a map $u \in \mathscr{H}$ is a global solution to (M) if and only if it coincides with u_g, the unique fixed-point of Ψ. From the Markov property of $\mathscr{X}$ and integration by parts we infer that Ψ maps $\mathscr{H}$ into itself. Finally, let $\lambda \in B([0,T] \times S, \mathbb{R}_+)$ be μ-dominated such that

$$|f(t,x,w) - f(t,x,w')| \leq \lambda(t,x)|w - w'|$$

for every $(t,x) \in [0,T] \times S$ and each $w, w' \in \mathbb{R}^k$ with $|w| \vee |w'| \leq c$. This guarantees that (3.5) is valid for all $u, v \in \mathscr{H}$ and each $(r,x) \in [0,T] \times S$. As this was the last condition we had to check, the claim follows from Lemma 3.9. $\square$

For the proof of Proposition 2.11 we consider an integral sequence of $\mathbb{R}^{k\times k}$-valued maps. To this end, we use the conventions that $[r,t] := [t,r]$ and $\int_r^t \overline{b}(s)\,\mu(ds) := -\int_t^r \overline{b}(s)\,\mu(ds)$ for all $r,t \in [0,T]$ with $t < r$, each $d \in \mathbb{N}$, and every μ-integrable $\overline{b} \in B([0,T], \mathbb{R}^{d\times d})$.

Lemma 4.1. *Assume that $b \in B([0,T] \times S, \mathbb{R}^{k\times k})$ is μ-dominated. Let the sequence $(\Sigma^{(n)})_{n\in\mathbb{N}_0}$ of $\mathbb{R}^{k\times k}$-valued maps on $[0,T] \times [0,T] \times \Omega$ be recursively given by $\Sigma_{r,t}^{(0)}(\omega) := \mathbb{I}_k$ and*

$$\Sigma_{r,t}^{(n)}(\omega) := \int_r^t b(s, X_s(\omega))\Sigma_{s,t}^{(n-1)}(\omega)\,\mu(ds) \quad \textit{for all } n \in \mathbb{N}.$$

Then $\Sigma_{r,t}^{(n)}$ is $\sigma(X_s : s \in [r,t])$-measurable, $|\Sigma_{r,t}^{(n)}| \leq \frac{\sqrt{k}}{n!}\Big(\Big|\int_r^t |b(s,X_s)|\,\mu(ds)\Big|\Big)^n$, and $\Sigma^{(n)}(\omega)$ is continuous for all $n \in \mathbb{N}_0$, each $r,t \in [0,T]$, and every $\omega \in \Omega$.

Proof. We prove the lemma by induction over $n \in \mathbb{N}_0$. In the initial induction step $n = 0$ the assignment $\Sigma^{(0)} = \mathbb{I}_k$ gives all results. Let us suppose that the claims are true for some $n \in \mathbb{N}_0$ and pick $r,t \in [0,T]$. Then, since $\mathscr{X}$ is progressive, the map $[r,t] \times \Omega \to \mathbb{R}^{k\times k}$, $(s,\omega) \mapsto b(s,X_s(\omega))\Sigma_{s,t}^{(n)}(\omega)$ is $\mathscr{B}([r,t]) \otimes \sigma(X_s : s \in [r,t])$-measurable, and as the Frobenius norm on $\mathbb{R}^{k\times k}$ is submultiplicative,

$$\begin{aligned}\bigg|\int_r^t |b(s,X_s)\Sigma_{s,t}^{(n)}|\,\mu(ds)\bigg| &\leq \sqrt{k}\bigg|\int_r^t \frac{|b(s,X_s)|}{n!}\bigg(\bigg|\int_s^t |b(s',X_{s'})|\,\mu(ds')\bigg|\bigg)^n \mu(ds)\bigg| \\ &= \frac{\sqrt{k}}{(n+1)!}\bigg(\bigg|\int_r^t |b(s,X_s)|\,\mu(ds)\bigg|\bigg)^{n+1}.\end{aligned}$$

Thus, $\Sigma_{r,t}^{(n+1)}$ is well-defined and the required estimate holds. In addition, an application of Fubini's theorem to each coordinate ensures that $\Sigma_{r,t}^{(n+1)}$ is $\sigma(X_s : s \in [r,t])$-measurable.

To show that $\Sigma^{(n+1)}(\omega)$ is continuous for all $\omega \in \Omega$, let again $r,t \in [0,T]$ and $(r_m,t_m)_{m\in\mathbb{N}}$ be a sequence in $[0,T]\times[0,T]$ that converges to (r,t), then $\lim_{m\uparrow\infty} \mathbb{1}_{[r_m,t_m]}(s)\Sigma_{s,t_m}^{(n)} = \mathbb{1}_{[r,t]}(s)\Sigma_{s,t}^{(n)}$ for μ-a.e. $s \in [0,T]$. Therefore, $\lim_{m\uparrow\infty} \Sigma_{r_m,t_m}^{(n+1)}(\omega) = \Sigma_{r,t}^{(n+1)}(\omega)$, by dominated convergence. $\square$

Proof of Proposition 2.11. The map f is affine μ-bounded and Lipschitz μ-continuous. Hence, Proposition 2.10 entails that the sequence $(u_n)_{n\in\mathbb{N}_0}$ in $B_b([0,T]\times S,\mathbb{R}^k)$, recursively given by $u_0(r,x) := E_{r,x}[g(X_T)]$ and

$$u_n(r,x) := u_0(r,x) - E_{r,x}\bigg[\int_r^T a(s,X_s) + b(s,X_s)u_{n-1}(s,X_s)\,\mu(ds)\bigg]$$

for all $n \in \mathbb{N}$, converges uniformly to u_g, the unique global bounded solution to (M). With the notation of Lemma 4.1, an induction proof shows that u_n is of the form

$$u_n(r,x) = E_{r,x}\bigg[\sum_{i=0}^{n}(-1)^i\Sigma_{r,T}^{(i)}g(X_T)\bigg] - E_{r,x}\bigg[\int_r^T \sum_{i=0}^{n-1}(-1)^i\Sigma_{r,t}^{(i)}a(t,X_t)\,\mu(dt)\bigg]$$

for all $n \in \mathbb{N}$ and each $(r,x) \in [0,T]\times S$. Because $\sum_{n=0}^{\infty}|(-1)^n\Sigma_{r,t}^{(n)}| \leq \sqrt{k}e^{|\int_r^t |b(s,X_s)|\,\mu(ds)|}$ for every $r,t \in [0,T]$, the series mapping $\sum_{n=0}^{\infty}(-1)^n\Sigma^{(n)}$ converges absolutely, uniformly in $(r,t,\omega) \in [0,T]\times[0,T]\times\Omega$. Lemma 4.1 together with the previous estimate imply that the limit map $\Sigma := \sum_{n=0}^{\infty}(-1)^n\Sigma^{(n)}$ fulfills (i). Hence, dominated convergence yields the representation formula (2.5).

Let us verify that (ii) holds as well. From $\Sigma_{r,r}^{(0)} = \mathbb{I}_k$ and $\Sigma_{r,r}^{(n)} = 0$ for all $n \in \mathbb{N}$ we get that $\Sigma_{r,r} = \mathbb{I}_k$ for each $r \in [0,T]$. By the Cauchy product for absolutely convergent matrix series, to verify that $\Sigma_{r,s}\Sigma_{s,t} = \Sigma_{r,t}$ for every $r,s,t \in [0,T]$, it is enough to show that

$$\sum_{i=0}^{n}\Sigma_{r,s}^{(i)}\Sigma_{s,t}^{(n-i)} = \Sigma_{r,t}^{(n)}$$

for all $n \in \mathbb{N}_0$, which follows inductively. Furthermore, from $\Sigma_{r,t}\Sigma_{t,r} = \Sigma_{r,r} = \mathbb{I}_k$ we conclude that $\Sigma_{r,t}(\omega)$ is invertible and $\Sigma_{r,t}(\omega)^{-1} = \Sigma_{t,r}(\omega)$ for all $r,t \in [0,T]$ and each $\omega \in \Omega$.

Regarding (iii), let b fulfill $b(r,x)b(s,y) = b(s,y)b(r,x)$ for every $(r,x),(s,y) \in [0,T]\times S$. Then the proposition follows as soon as we have proven that

$$\Sigma_{r,t}^{(n)} = \frac{1}{n!}\bigg(\int_r^t b(s,X_s)\,\mu(ds)\bigg)^n \tag{4.3}$$

for every $n \in \mathbb{N}$ and each $r,t \in [0,T]$ with $r \leq t$. Hence, we write S_n for the set of all permutations of $\{1,\dots,n\}$ and set $C_n^\sigma(r,t) := \{(s_1,\dots,s_n) \in [r,t]^n \,|\, s_{\sigma(1)} \leq \cdots \leq s_{\sigma(n)}\}$ for each $\sigma \in S_n$. From the measure transformation formula we obtain that

$$\begin{aligned}&\int_{C_n^\sigma(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\mu^n(s_1,\dots,s_n)\\ &\qquad = \int_{C_n(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\mu^n(s_1,\dots,s_n) = \Sigma_{r,t}^{(n)},\end{aligned}$$

where $C_n(r,t) := \{(s_1,\dots,s_n) \in [r,t]^n \,|\, s_1 \leq \cdots \leq s_n\}$. In the end, we utilize that $[r,t]^n = \bigcup_{\sigma\in S_n} C_n^\sigma(r,t)$. Then the hypothesis that μ is atomless and Fubini's theorem lead to

$$\bigg(\int_r^t b(s,X_s)\,\mu(ds)\bigg)^n = \sum_{\sigma\in S_n}\int_{C_n^\sigma(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\mu^n(s_1,\dots,s_n) = n!\Sigma_{r,t}^{(n)}.$$

That is, (4.3) is justified and the claim follows. $\square$

4.3 Proof of Theorem 2.13

We restrict our attention to $k = 1$. First, we use the Feynman-Kac formula (2.5) to represent the difference of two solutions. This idea is essentially based on Proposition 3.1 in Schied [18].

Lemma 4.2. *Let $f, \tilde{f} \in BC^{1-}_{\mu}([0,T]\times S\times D,\mathbb{R})$, $\tilde{g} \in B_b(S,D)$ be consistent, I be an admissible interval, u be a solution to* (M) *on I, and $\tilde{u}$ be a solution to* (M) *on I with $\tilde{f}$ and $\tilde{g}$ instead of f and g, respectively. Assume that u, $\tilde{u}$ are μ-admissible and define $a, b \in B(I \times S,\mathbb{R})$ by*

$$a(r,x) := (f-\tilde{f})(r,x,\tilde{u}(r,x)), \quad \textit{and} \quad b(r,x) := \frac{f(r,x,u(r,x)) - f(r,x,\tilde{u}(r,x))}{(u-\tilde{u})(r,x)},$$

if $u(r,x) \neq \tilde{u}(r,x)$, and $b(r,x) := 0$, otherwise. Then a, b are locally μ-dominated and

$$(u-\tilde{u})(r,x) = E_{r,x}\left[e^{-\int_r^T b(s,X_s)\,\mu(ds)}(g-\tilde{g})(X_T)\right] - E_{r,x}\left[\int_r^T e^{-\int_r^t b(s,X_s)\,\mu(ds)} a(t,X_t)\,\mu(dt)\right]$$

for each $(r,x) \in I \times S$. In particular, if $f \leq \tilde{f}$ and $g \geq \tilde{g}$, then $u \geq \tilde{u}$.

Proof. The second claim is a direct consequence of the first, since $a \leq 0$ whenever $f \leq \tilde{f}$. Thus, we merely have to prove the first assertion. To check that a and b are locally μ-dominated, it suffices to show that for each $r \in I$ there is a μ-integrable $\overline{c} \in B([r,T],\mathbb{R}_+)$ such that

$$|a(\cdot,y)| \vee |b(\cdot,y)| \leq \overline{c} \quad \text{for each } y \in S \quad \mu\text{-a.s. on } [r,T].$$

This condition follows readily from the local Lipschitz μ-continuity of f, the local μ-boundedness of $\tilde{f}$, and the hypothesis that u, $\tilde{u}$ are μ-admissible. By definition, $a(t,x) + b(t,x)(u-\tilde{u})(t,x) = f(t,x,u(t,x)) - \tilde{f}(t,x,\tilde{u}(t,x))$ for each $(t,x) \in I \times S$. Hence, we let $r \in I$ and choose $a_r, b_r \in B([0,T] \times S,\mathbb{R})$ so that $a_r(t,x) = a(t,x)$ and $b_r(t,x) = b(t,x)$, if $t \geq r$, and $a_r(t,x) = b_r(t,x) = 0$, otherwise. Then $f_r : [0,T] \times S \times \mathbb{R} \to \mathbb{R}$ given by

$$f_r(t,x,w) := a_r(t,x) + b_r(t,x)w$$

is affine μ-bounded and Lipschitz μ-continuous. In addition, the restriction of $u-\tilde{u}$ to $[r,T]\times S$ is a μ-admissible solution to (M) with f and g replaced by f_r and $g-\tilde{g}$, respectively. Thus, from Proposition 2.11 and Corollary 3.3 we infer the assertion. ☐

We suppose in the sequel that D is an interval, and set $\underline{d} := \inf D$ and $\overline{d} := \sup D$.

Lemma 4.3. *Let $\underline{d} > -\infty$ and f be affine μ-bounded from below, i.e., there are two μ-dominated $a, b \in B([0,T] \times S,\mathbb{R}_+)$ with $f(t,x,w) \geq -a(t,x) - b(t,x)|w|$ for all $(t,x,w) \in [0,T] \times S \times D$. Then every μ-suitably bounded solution u to* (M) *on an admissible interval I fulfills*

$$u(r,x) - \underline{d} \leq E_{r,x}\left[e^{\int_r^T b(s,X_s)\,\mu(ds)}\left(g(X_T) - \underline{d} + \int_r^T (a + b|\underline{d}|)(s,X_s)\,\mu(ds)\right)\right]$$

for all $(r,x) \in I \times S$.

Proof. It holds that

$$\begin{aligned} u(r,x) - \underline{d} \leq E_{r,x}[g(X_T) - \underline{d}] &+ E_{r,x}\left[\int_r^T (a + b|\underline{d}|)(s,X_s)\,\mu(ds)\right] \\ &+ E_{r,x}\left[\int_r^T \beta(s,X_s)(u(s,X_s) - \underline{d})\,\mu(ds)\right] \end{aligned}$$

for each $(r,x) \in I \times S$, because $|u(s,X^s)| \leq (u(s,X^s) - \underline{d}) + |\overline{d}|$ for all $s \in [r,T]$. By Lemma 3.1, the asserted estimate follows. ☐

Remark 4.4. Suppose instead that $\overline{d} < \infty$ and f is affine μ-bounded from above. To obtain a similar estimate in this case, we replace D by $-D = \{-w \,|\, w \in D\}$ and f by the function $[0,T] \times S \times (-D) \to \mathbb{R}$, $(t,x,w) \mapsto -f(t,x,-w)$, respectively, and apply the above lemma.

Next, we study the boundary behavior of solutions. To this end, we consider only the case $\underline{d} > -\infty$, as the case $\overline{d} < \infty$ can be treated similarly, by considering above remark.

Proposition 4.5. *Let $\underline{d} > -\infty$ and $f \in BC_\mu^{1-}([0,T]\times S \times D, \mathbb{R})$. Suppose that f is both locally μ-bounded and locally Lipschitz μ-continuous at $\underline{d}$ with $\lim_{w\downarrow \underline{d}} f(\cdot, x, w) \leq 0$ for all $x \in S$ μ-a.s., and let one of the following two conditions hold:*

(i) f is μ-bounded from above.

(ii) $\overline{d} = \infty$ and f is affine μ-bounded from below.

Then there is $c \in (0,1]$ such that each μ-admissible solution u to (M) *on an admissible interval I is subject to $u(r,x) - \underline{d} \geq c(E_{r,x}[g(X_T)] - \underline{d})$ for all $(r,x) \in I \times S$.*

Proof. Whenever $\underline{d} \notin D$, then we define the extension $\overline{f}$ of f to $[0,T] \times S \times (D \cup \{\underline{d}\})$ through $\overline{f}(t,x,\underline{d}) := \lim_{w\downarrow\underline{d}} f(t,x,w)$ for all $(t,x) \in [0,T] \times S$. Otherwise, we simply set $\overline{f} := f$, which gives $\overline{f} \in BC_\mu^{1-}([0,T] \times S \times (D \cup \{\underline{d}\}))$ in either case. Now, let u be a μ-admissible solution to (M) on an admissible interval I, then Lemma 4.2 implies that $a_u \in B(I \times S, \mathbb{R})$ defined via $a_u(r,x) := (\overline{f}(r,x,u(r,x)) - \overline{f}(r,x,\underline{d}))/(u(r,x) - \underline{d})$, if $u(r,x) > \underline{d}$, and $a_u(r,x) := 0$, otherwise, is locally μ-dominated and satisfies

$$u(r,x) - \underline{d} \geq E_{r,x}\left[e^{-\int_r^T a_u(s,X_s)\,\mu(ds)}(g(X_T) - \underline{d})\right]$$

for each $(r,x) \in I \times S$, since $\overline{f}(t, X_t, \underline{d}) \leq 0$ for μ-a.e. $t \in [r,T]$. We derive some μ-dominated $n \in B([0,T] \times S, \mathbb{R}_+)$ such that every μ-admissible solution u to (M) on an admissible interval I satisfies $a_u(r,x) \leq n(r,x)$ for each $(r,x) \in I \times S$. Once this is shown, the claim follows.

So, let us at first assume that (i) holds. Then there is a μ-dominated $a \in B([0,T] \times S, \mathbb{R}_+)$ with $\overline{f}(t,x,w) \leq a(t,x)$ for each $(t,x,w) \in [0,T] \times S \times D$. As f is locally Lipschitz μ-continuous at $\underline{d}$, there are $\delta > 0$ and a μ-dominated $\lambda \in B([0,T] \times S, \mathbb{R}_+)$ fulfilling $|\overline{f}(t,x,w) - \overline{f}(t,x,w')|$ $\leq \lambda(t,x)|w - w'|$ for every $(t,x) \in [0,T] \times S$ and all $w, w' \in [\underline{d}, \underline{d} + \delta) \cap D$. Hence,

$$a_u(r,x) \leq \lambda(r,x)\mathbb{1}_{[\underline{d},\underline{d}+\delta)}(u(r,x)) + \frac{a(r,x) - \overline{f}(r,x,\underline{d})}{\delta}\mathbb{1}_{[\underline{d}+\delta,\infty)}(u(r,x)) \leq n(r,x)$$

for every μ-admissible solution u to (M) on an admissible interval I and each $(r,x) \in I \times S$, where we have set $n := \max\{\lambda, (a - \overline{f}(\cdot,\cdot,\underline{d}))/\delta\}$. Since f locally μ-bounded at $\underline{d}$, we see easily that n is μ-dominated, as desired.

In place of assuming that f is μ-bounded from above, let (ii) be true. Then Lemma 4.3 yields $c > \underline{d}$ such that $u(I \times S) \subset [\underline{d}, c] \cap D$ for each μ-admissible solution u to (M) on an admissible interval I. Because $[\underline{d}, c]$ is compact, there is a μ-dominated $\lambda \in B([0,T] \times S, \mathbb{R}_+)$ such that $|\overline{f}(t,x,w) - \overline{f}(t,x,w')| \leq \lambda(t,x)|w - w'|$ for all $(t,x) \in [0,T] \times S$ and each $w, w' \in [\underline{d}, c]$. Hence, each μ-admissible solution u to (M) on an admissible interval I fulfills $|a_u(r,x)| \leq n(r,x)$ for all $(r,x) \in I \times S$ with $n := \lambda$. $\square$

Eventually, we are ready to establish the one-dimensional global existence- and uniqueness result.

Proof of Theorem 2.13. Let us verify the first claim. We begin with the case $\underline{d} > -\infty$ and $\overline{d} < \infty$. By using the function $[0,T]\times S\times(-D)\to\mathbb{R}$, $(t,x,w)\mapsto -f(t,x,-w)$, Proposition 4.5 yields that $I_{\tilde{g}} = [0,T]$ for every $\tilde{g}\in B_b(S,(\underline{d},\overline{d}))$ that is bounded away from $\{\underline{d},\overline{d}\}$. Thus, for all $n\in\mathbb{N}$ we define

$$g_n := (g\vee(\underline{d}+(\overline{d}-\underline{d})2^{-n}))\wedge(\overline{d}-(\overline{d}-\underline{d})2^{-n}), \tag{4.4}$$

then $g_n\in B_b(S,(\underline{d},\overline{d}))$ and $\mathrm{dist}(g_n,\{\underline{d},\overline{d}\})\geq(\overline{d}-\underline{d})2^{-n}$, which guarantees that $I_{g_n}=[0,T]$. Because $|g_n-g|\leq(\overline{d}-\underline{d})2^{-n}$ for all $n\in\mathbb{N}$, the sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly to g. If $D\subsetneq[\underline{d},\overline{d}]$, then we let $\overline{f}$ denote the unique extension of f to $[0,T]\times S\times[\underline{d},\overline{d}]$ such that

$$\overline{f}\in BC_{\mu}^{1-}([0,T]\times S\times[\underline{d},\overline{d}]).$$

Otherwise, we just set $\overline{f}:=f$. According to Proposition 3.4, the sequence $(u_{g_n})_{n\in\mathbb{N}}$ converges uniformly to the unique global bounded solution to (M) with $\overline{f}$ instead of f, which we denote by $\overline{u}_g$. By uniqueness, $\overline{u}_g = u_g$ whenever g is bounded away from $\{\underline{d},\overline{d}\}$. Since Proposition 4.5 also shows that $\overline{u}_g$ does not attain the value $\underline{d}$ (resp. $\overline{d}$) if the same is true for g, the function $\overline{u}_g$ is D-valued. Hence, $\overline{u}_g$ is the unique global bounded solution to (M).

Let us turn to the case $\underline{d}>-\infty$ and $\overline{d}=\infty$. Lemma 4.3 and Proposition 4.5 entail that $I_{\tilde{g}}=[0,T]$ for every $\tilde{g}\in B_b(S,(\underline{d},\infty))$ that is bounded away from $\underline{d}$. For each $n\in\mathbb{N}$ we set

$$g_n := g\vee(\underline{d}+2^{-n}), \tag{4.5}$$

then $g_n\in B_b(S,(\underline{d},\infty))$ and $\mathrm{dist}(g_n,\underline{d})\geq 2^{-n}$, which implies that $I_{g_n}=[0,T]$. In addition, $|g_n-g|\leq 2^{-n}$ and $g_n(x)-\underline{d}\leq(g(x)-\underline{d})\vee(1/2)$ for all $n\in\mathbb{N}$ and each $x\in S$. We can now infer from Lemma 4.3 and Proposition 3.4 that $(u_{g_n})_{n\in\mathbb{N}}$ converges uniformly to the unique global bounded solution to (M), denoted by $\overline{u}_g$. Once again, uniqueness forces $\overline{u}_g=u_g$ if g is bounded away from $\underline{d}$. From Proposition 4.5 we see that $\overline{u}_g$ cannot attain the value $\underline{d}$ if $g(x)>\underline{d}$ for all $x\in S$. For this reason, $\overline{u}_g$ is D-valued, which concludes the case $\underline{d}>-\infty$ and $\overline{d}=\infty$. The case $\underline{d}=-\infty$ and $\overline{d}<\infty$ is a consequence of the last case, by utilizing the familiar function $[0,T]\times S\times(-D)\to\mathbb{R}$, $(t,x,w)\mapsto -f(t,x,-w)$.

In the end, we note that for each $n\in\mathbb{N}$ the function g_n given either by (4.4) or (4.5), depending on which case occurs, is continuous if g is. Hence, as the uniform limit of a sequence of real-valued (right-)continuous functions on $[0,T]\times S$ is (right-)continuous, Theorem 2.9 implies the second assertion. ☐

Proof of Corollary 2.15. At first, Theorem 2.13 entails that (M) admits the unique global bounded solution $\overline{u}_g$, which is (right-)continuous if $\mathscr{X}$ is (right-hand) Feller, $a(s,\cdot)$ and $b(s,\cdot)$ are continuous for μ-a.e. $s\in[0,T]$, and g is continuous. Let us set

$$\overline{f}(t,x,w) := a(t,x)+b(t,x)w \quad \text{for all } (t,x,w)\in[0,T]\times S\times\mathbb{R},$$

then Proposition 2.11 implies that the unique global bounded solution $\tilde{u}_g$ to (M) with f replaced by $\overline{f}$ admits the required representation (2.7). However, $\overline{u}_g$ is also a global bounded solution to (M) when f is replaced by $\overline{f}$. Uniqueness gives $\overline{u}_g=\tilde{u}_g$. ☐

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