Physics is a science
- We have a model/theory of how something in nature works
- The model makes a prediction of some quantity which we can observe in nature
- We make an observation and compare the result to the prediction
- The observation will have limited accuracy/precision

There is no Truth!
- Our theories are only valid to the level at which the predictions they make are verifiable.
- Understanding how well we measure the agreement between a prediction and a measurement tells us at what level we can believe the predictions of our theory.
Today’s objectives

This lecture is an introduction to the material

You will follow this up with exercises in the computer suite during weeks 2-3 of the term

- Understand the nature of experimental uncertainties
- Learn about the types of errors
- Learn about error distributions
  - The normal distribution
- Learn how to calculate from data
  - Mean, error on the mean
  - Standard Deviation
- Learn how to calculate
  - Propagation of errors
  - Combinations of errors
Task: Measure the width of a room

I have this really clever device to measure the distance between two points.

It sends a laser beam from the device to a distant wall, and then provides a number which tells me the distance between the device and the laser spot which appears on the distant wall.
Measure width of my living room

Place device against wall
Point laser at opposite wall
Read the measurement
Repeat (many times)

Living room G02
Exg timber board

3.211 m
The output of this exercise is a single measurement
- 3.211 m
- Notice it has a unit (first requirement for a measurement)

I want to use this measurement to make a test of a theory
- In this case to see if the architectural drawings are correct

I need to know if my measurement is a **significant** test of the theory

I need to know if my measurement is a **reliable** test of the theory

I can quantify this by quoting an **uncertainty** or **error** for the measurement
Does the measurement agree with the theory? My drawings say the width of the room is 3.4 m

<table>
<thead>
<tr>
<th>Measurement</th>
<th>Compatible/Incompatible</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.211 ±0.005 m</td>
<td>Incompatible</td>
</tr>
<tr>
<td>3.2 ±0.3 m</td>
<td>Compatible</td>
</tr>
</tbody>
</table>

- This measurement is not compatible with the 3.4 m prediction from the drawings
- This measurement is compatible with the drawings

Unless we know the error estimate on our measurement, we can’t compare it to our theoretical expectations.

A measurement without an error estimate is meaningless. Be careful with significant figures 3.211±0.3 m is not sensible. Quote 1 or 2 significant figures.
How can I estimate the error?

The usual method is to repeat the measurement and look at the distribution of measurements in order to extract an estimate of the error.

Here is a Histogram of a series of measurements I made of the width of my room.

It is constructed by plotting how many times each different measurement appeared.

I made 34 measurements of the width of the wall.


Types of Errors

Random or Statistical Errors

These can arise from

- Random fluctuations in the apparatus
- Noise in electrical circuits
- Inaccuracies in positioning the device
- Inaccuracies in reading the device

We can improve the result by making more measurements – errors average out as they are random

The data seem to fluctuate around here
Types of Errors

Mistakes

- Obviously incorrect measurements
  - In this case the measuring device was not placed firmly against the wall
  - Be careful when making measurements
  - Check everything that you write down
  - Watch out for “rogue points” on graphs – but be careful to understand where they might come from!

This measurement could be an outlier
Types of Errors

Systematic Errors

Example - Something which affects all or a group of measurements in a coherent way

- Measuring the width of the room in a different place gives a different measurement
- Calibration errors, zero offsets
- No improvement by making more measurements

3.213 ± 0.001 m

3.281 ± 0.001 m

Histogram of data with bin width = 1 mm
Accuracy and Precision

An accurate measurement is one with a small systematic error.

A precise measurement is one with a small random error.

Quite precise

But overall not very accurate!
The Normal distribution

Many repeated measurements of a randomly distributed error can be well described by a normal distribution.

The distribution is characterized by two parameters:

- A mean ($\bar{x}$) and a standard deviation ($\sigma$)

68% of measurements lie within 1 standard deviation of the mean.

95% of measurements lie within 2 standard deviations of the mean.

$P(x)dx$ is the probability that a measurement lies in the range $(x, x+dx)$

General form:

$$ P(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma^2} \right] $$

$\int_{-\infty}^{\infty} P(x)dx = 1$

i.e. a measurement is certain to lie somewhere in the range $(-\infty, \infty)$

The Normal distribution
Normal distributions do a good job of describing randomly distributed data.

We can see how far away “outlier” points really are.

The bad data point appears to be fairly “unlikely” to come from this distribution.

We need to be able to calculate the properties of the normal distribution which describes our data.

**Mean**

**Standard Deviation**

A normal distribution overlaid on our measurements.
A large dataset with a mean of about 0.44V and a standard deviation (\(s\)) of about 0.15 V
Calculating the mean from a set of data

<table>
<thead>
<tr>
<th>Properties of the data set</th>
<th>Mean: The best estimate of the true value</th>
</tr>
</thead>
<tbody>
<tr>
<td>• For data with $n$ independent measurements $x_i$ of a quantity $x$</td>
<td>$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$</td>
</tr>
<tr>
<td>• We are trying to determine what is the true value of $x$</td>
<td></td>
</tr>
<tr>
<td>• The best estimate of the true value is given by the arithmetic mean</td>
<td></td>
</tr>
</tbody>
</table>
Sample Standard Deviation (s)

In order to estimate the standard deviation of our distribution (what we would calculate with an infinite number of measurements) we calculate the *sample standard deviation*.

This is the best estimate of the standard deviation from our sample of n data points.

\[
s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2
\]

\[
s^2 = \frac{1}{n-1} \left[ \sum_{i=1}^{n} x_i^2 - \frac{1}{n} \left( \sum_{i=1}^{n} x_i \right)^2 \right]
\]
Calculating mean/standard deviation

The data ...


The result: mean = 3.214 m, standard deviation = 0.0062 m
How much does removing the “bad” data point change the result?

The result: mean = 3.214 m, standard deviation=0.0062 m

The result: mean = 3.213 m, standard deviation=0.0044 m

Small change of mean, bigger change of standard deviation
How well have we measured the mean value?

- The estimate of the random error on the mean value given for \( n \) independent measurements is given by the standard error of the measurement

\[
\sigma_m = \frac{s}{\sqrt{n}}
\]

- The more measurements we make, the better our estimate of the mean value becomes
  - This is only true if other sources of error are smaller
  - An infinite number of measurements would in principle give us the true value we are trying to measure ...
Taking the data from our measurements of the width of the room we can estimate how well we have measured the width.

(I have chosen here to use the data where we have removed the badly measured point)

This value has an error of less than 1 mm.

The probability is 68% that the true value for the width should lie within \( \pm \sigma_m \) of the mean value we have calculated.

\[
\bar{x} = 3.213 \text{ m}, \ s = 0.0044 \text{ m}, \ n = 33
\]

\[
\sigma_m = \frac{s}{\sqrt{n}} = \frac{0.0044}{\sqrt{33}} \text{ m}
\]

\[\implies \sigma_m = 0.0008 \text{ m}\]

\[
\bar{x} = 3.213 \pm 0.0008 \text{ m}
\]

Remember we quote the value with an error and a unit.
The above is only an estimate of the statistical (random) error.
We must also consider systematic errors.
The measuring device has an accuracy of ±2 mm, there is no point in taking more measurements!

There are very often other sources of error which are larger than the random errors which we measure.

There are many different ways to find out about these.

Specification sheets for instruments are one resource which can give you some guidance about the accuracy you can achieve when using the device.

Sometimes it is possible to control some of these sources of error and get a better result, sometimes not!
Imagine we need to know the error on a quantity which is a function of something which we have measured.

Say we have measured a quantity $x$ with an error $\sigma_x$.

For example – convert from cm to inches

Convert from measurement of a radius to the measurement of the area of a circle.

For example

$z = 2.54 \ x$

$z = \pi x^2$

In general for $z = f(x)$

$$\sigma_z = \left| \frac{dz}{dx} \right| \sigma_x$$

So

$$z = 2.54 \ x \quad \Rightarrow \quad \sigma_z = 2.54 \sigma_x$$

$$z = \pi x^2 \quad \Rightarrow \quad \sigma_z = 2\pi x \sigma_x$$

**Propagation of errors**
Combination of errors

When we combine two quantities which are added together

\[ z = x + y \]
\[ z = x - y \]

We calculate the total error by adding the errors in quadrature

\[ \sigma_z^2 = \sigma_x^2 + \sigma_y^2 \]
\[ \sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2} \]

Combine the measurement from the Laser with a Ruler measurement

\[ T = (L + R) \]

\[ L = 3.211 \pm 0.002 \, m \]
\[ R = 0.023 \pm 0.003 \, m \]

\[ \sigma_T = \sqrt{\sigma_L^2 + \sigma_R^2} \]
\[ = \sqrt{(0.002)^2 + (0.003)^2} \]
\[ = 0.004 \, m \]

\[ T = (L + R) \pm \sigma_T \]
\[ = 3.234 \pm 0.004 \, m \]
Combination of errors

If a measurement is a product of two quantities

\[ z = xy \]

or a quotient

\[ z = \frac{x}{y} \]

The fractional errors add in quadrature

\[ \frac{\sigma_z}{z} = \frac{\sigma_x}{x^2} + \frac{\sigma_y}{y^2} \]

\[ \sigma_z = \sqrt{\frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}} \]

\[ A = (LW) \quad \frac{\sigma_A}{A} = \sqrt{\frac{\sigma_L^2}{L^2} + \frac{\sigma_W^2}{W^2}} \]

\[ L = 3.211 \pm 0.002 \, m \]

\[ W = 2.951 \pm 0.003 \, m \]

\[ \sigma_A = A \sqrt{\frac{\sigma_L^2}{L^2} + \frac{\sigma_W^2}{W^2}} \]

\[ = (9.48) \sqrt{\frac{(0.002)^2}{(3.211)^2} + \frac{(0.003)^2}{(2.951)^2}} \, m^2 \]

\[ = 0.01 \, m^2 \]

\[ A = (LW) \pm \sigma_A \]

\[ = 9.48 \pm 0.01 \, m^2 \]
In general if \( z = f(x, y) \)

\[
\sigma_z^2 = \left( \frac{\partial z}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial z}{\partial y} \right)^2 \sigma_y^2
\]

so for \( A = LW \)

\[
\sigma_A^2 = \left( \frac{\partial A}{\partial L} \right)^2 \sigma_L^2 + \left( \frac{\partial A}{\partial W} \right)^2 \sigma_W^2
\]

\[
= W^2\sigma_L^2 + L^2\sigma_W^2
\]

Where we used

\[
\frac{\partial A}{\partial L} = W \quad \frac{\partial A}{\partial W} = L
\]

And we can write

\[
\frac{\sigma_A^2}{A^2} = \frac{W^2\sigma_L^2 + L^2\sigma_W^2}{A^2} = \frac{W^2\sigma_L^2 + L^2\sigma_W^2}{(LW)^2}
\]

\[
= \frac{\sigma_L^2}{L^2} + \frac{\sigma_W^2}{W^2}
\]
Estimating errors properly is difficult work.

Error estimates too small? [or constants have been varying?!]
Errors you might not have thought about

LEP/LHC is constructed in a tunnel 27km in circumference, about 100 m below the surface near the French-Swiss border.
In the 1990’s experiments went on to measure the production of Z Bosons with the LEP accelerator.
The mass of the Z Boson was determined to within a few parts per million
This depended on knowing the energy of the Beams of particles to very high accuracy
- Need to know the length of the tunnel
- Need to know the current in the magnets
The Energy is subject to phase of the moon!

The moon distorts the earth’s surface

This changes the length of the accelerator and hence the energy is changed!

Comparing the energy shift in the accelerator to that expected by the motion of the earth’s surface because of the tides.
The French TGV passes near the accelerator ring

The current in the rails couples to the ring through the ground changing the energy!
Measurements and Uncertainties session in the computing suite

- Practice in calculating errors
- Presenting results
- Remember to bring
  - Your Calculator
  - Your lab book

In exams you will be supplied with a Casio fx-83GT Plus

Cost around £8.
[no obligation on you to own one but you should know how to use one.]
Resources - Online

- **Lab Handbook:**
  - [http://www3.imperial.ac.uk/physicsuglabs/firstyearlab/labmanual](http://www3.imperial.ac.uk/physicsuglabs/firstyearlab/labmanual)
    - Measurement and uncertainties lab-script
    - Summary sheet

- **Blackboard** ([bb.imperial.ac.uk](http://bb.imperial.ac.uk))
  - Measurements and Errors on Blackboard
    - This lecture
    - Datasets for the lab exercises
Textbooks

• Measurements and their uncertainties. I.G. Hughes and T.P Hase, OUP, 2010
• More Advanced
Final Words

- Every time you make a measurement, think about the error you need to assign to it.
  - Think about how you can determine/measure the error
  - Think about what might contribute to the error
  - Be sensible and realistic when assigning errors
- Even theorists need to understand the errors associated with measurements!