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NONCOMMUTATIVE EXTENSION OF HYPOCOERCIVE DISSIPATIVE DYNAMICS

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BACKGROUND

2015-2018: BSc. Mathematics(Hons.), Lady Shri Ram, New Delhi, India
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PhD thesis title: *Coercive inequalities in noncommutative analysis*

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WHY DO WE NEED NC ANALOGUE?

Noncommutativity, in very basic words, means that the order of the operations matters. That is, for any two operations a, b:

$$\mathbf{ab \neq ba}$$

This phenomenon is found everywhere and has significant impact on mathematics and physics. During the early development of quantum mechanics, it was realized that the noncommutative framework was necessary to have an understanding of events that occur at the quantum scale. This led to developments in functional analysis in order to provide supporting framework for the physical theories.

RESEARCH PROBLEM

The idea of the project is to extend the existing classical theories to quantum scale(or in the noncommutative spaces). In noncommutative quantum mechanics, the elements that do not commute are the coordinate functions. For any two coordinate functions x^i, x^j, their commutator is nonzero, i.e,

$$[x^i, x^j] = x^i x^j - x^j x^i = i\theta^{ij}$$

which means that it is not possible to measure the position of a particle with respect to more than one axis. We want to find the noncommutative analogue of the well known Hörmander theorem and Hypocoercivity estimates.

HÖRMANDER THEOREM

The Hörmander theorem is useful in the theory of partial differential equations which states that every distributional solution to the PDE $Lu = f$ is smooth if f is smooth, i.e, the function is hypoelliptic in an open subset of $\mathbb{R}^N$ if the operator

$$L = \sum_{j=1}^{m} X_j^2 + Y$$

with fields X_i's and Y satisfy the famous Hörmander rank condition. The most basic example of a hypoelliptic operator is the Laplacian operator(Figure 1), given by the divergence of the gradient of a function.

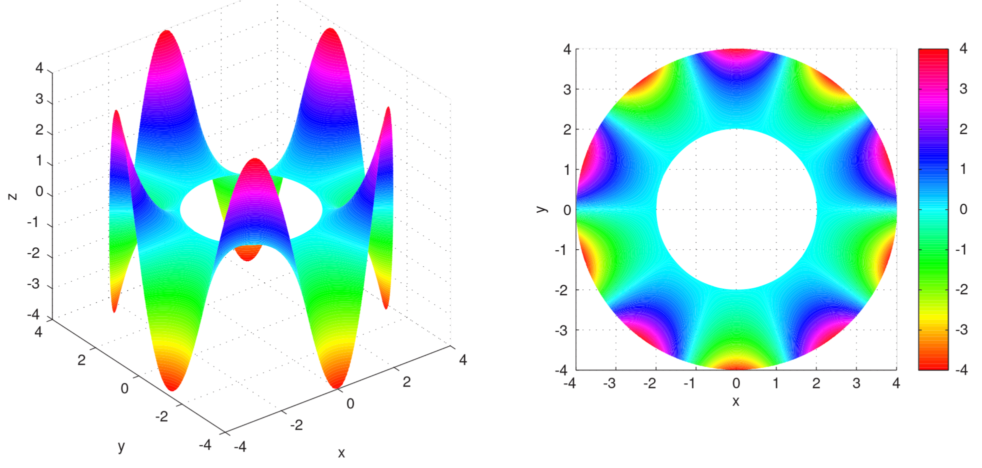
HYPOCOERCIVITY

The concept of Hypocoercivity was introduced by Villani which involves splitting of an operator into symmetric and anti symmetric parts. That is, an operator L can be split as

$$L = A^*A + B$$

where A^* is an adjoint of A, hence A^*A is always symmetric and $B = -B^*$. It is used to study the exponential convergence to equilibrium of partial differential equations.
The theorems of exponentially fast convergence to equilibrium has been established where both hypoellipticity and hypocoercivity occur together in the study of linear generators. These generators satisfy Hörmander's condition.

FIGURE 1



This graph (wiki) shows the solution of the Laplacian equation where f is zero and hence smooth. We see that the graph of the solution is also smooth given by the annulus. Hence, Laplacian is hypoelliptic operator.

NONCOMMUTATIVE FRAMEWORK

For the noncommutative spaces, we construct the generators of the form

$$L = \sum_{j=1}^{m} \delta_{X_j}^2 + \delta_Y$$

where δ_X are the derivations with respect to the fields which satisfy Hörmander condition. The derivations are the generalisation of the derivative function and they occur in noncommutative framework.
Similarly, we construct the generators

$$L = \delta_A^\star \delta_A + (\delta_B - \delta_B^\star)$$

to prove hypocoercive estimates similar to Villani in noncommutative spaces.
We have constructed various examples of both kinds of generators and established the other required components for the setup. The proof of the theorem and the estimates are in progress.

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