The CMB: A case study in Bayesian probability
The CMB: A Case Study

- Our underlying theories are statistical. How do we learn about cosmology from CMB observations?
  - predictions of power spectra (and higher moments): (quantum) noise
  - expand to include polarization

- Inferences in cosmology

- Measuring the spectrum, $C_\ell$
  - temperature and polarization

- Measuring cosmological parameters

- Beyond the power spectrum
  - anisotropy [not small scales…]
    - sub-case study: topology
  - non-Gaussianity

In the notes, but probably won’t have time
Data analysis as Radical Data Compression

- Radical Compression
  - Trillions of bits of data
  - Billions of measurements at 9 frequencies
  - 50 million pixel map of whole sky
  - 2 million harmonic modes measured
  - 2500 $C_\ell$ variances
    - 2000$\sigma$ detection of CMB anisotropy power
  - Fit with just 6 parameters
    - Baryon density, CDM density, angular scale of sound horizon, reionization optical depth, slope and amplitude of primordial $P(k)$
    - $\Omega_b h^2, \Omega_c h^2, \theta_{MC}, \tau, n_s, A_s$
  - With no significant evidence for a 7th
Parameters & $C_\ell$

- What we really want:
  - $P(\text{theory} \mid \text{data})$
    - theory = the parameters of LCDM
      - or perhaps even an indication of which overall theory is correct
    - data = our CMB data and any other information ("priors") we might consider.
  - Data compression
    - $P(\text{theory} \mid \text{raw TOI data}) \approx P(\text{theory} \mid \text{noisy CMB map}) \approx P(\text{theory} \mid \text{estimated } \hat{C}_\ell)$
    - Also need error bars (and/or full covariance matrix)
      - Even then, this is only approximate
        - effect of foreground removal on maps
        - $\hat{C}_\ell$ dist’n depends on more than just central value & covariance
Bayesian methods: hierarchical models

- Timestream ($d_t$)
  ⇒ Map ($T_p \sim d_p$)
  ⇒ Spectrum ($C_l \sim d_l$)
  ⇒ cosmology

- without loss of information? (~Sufficient Statistics)

- $P(\text{Cosmology}|d_t N_{tt'}) = P(\text{Cosmology} | \text{Map}_{\text{pix}}, N_{pp'})$
  $\approx P(\text{Cosmology} | D_l N_{ll'}, x_l)$
  (Bond, AJ, Knox; WMAP)

- (assume that we can calculate $P(\text{Cosmology}|D_l N_{ll'}, x_l)$ even from non-Bayes estimators)

Posterior $\propto$ Prior $\times$ Likelihood

\[
P(H | DI) = \frac{P(H | I)P(D | HI)}{P(D | I)}
\]
CMB Data Analysis: mapmaking

- Model: data = signal + noise, as a function of time
  \[ d_t = A_{tp} T_p + n_t \]
  \[ \langle n_t n_{t'} \rangle = N_{tt'} \quad \sim \text{stationary} \]

- Step 1: mapmaking (estimate \( T_p \))
  - Gaussian noise ⇒ General least squares
    \[ \bar{T}_p = (A^T N^{-1} A)^{-1} A^T N^{-1} d \]
    \[ \langle \delta T_p \delta T_{p'} \rangle = (A^T N^{-1} A)^{-1} \]

  - If we stop here, uniform prior gives a Gaussian posterior for the map with this mean and variance.
    - aside: Gaussian \( C_\ell \) prior gives Wiener filter

  - But it is also a sufficient statistic

- Algorithms:
  - Rely on simplicity/sparseness of \( A_{tp} \)
  - FFT methods to apply timestream (\( t \)) operations
  - Conjugate gradient least-squares solution (nb. doesn’t give corr’n matrix)
  - Further simplifications for specific cases (1/f noise, observations in rings)
NB. pixelization on sphere non-trivial. CMB uses “HEALPix”
CMB Data Analysis:
Spectrum estimation

- **Step 2:** Don’t need to go back to the timeline to estimate the power spectrum, $C_\ell$.

- Model the sky as a correlated, statistically isotropic Gaussian random field

\[
\frac{T(\hat{x}) - \bar{T}}{\bar{T}} = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{x}) \quad \langle a_{\ell m}^* a_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{mm'} C_\ell
\]

- Many practical issues in calculating this explicitly.

- At low $\ell$, use sampling (usu. Gibbs), Newton-Raphson, Copula

- At high $\ell$, approximate by a function of estimated (ML) $C_\ell$ and errors & some other information $X_\ell$
Expected errors

- Estimating the error (variance\(^{1/2}\)) on a variance \((C_\ell)\)
  \[
  \langle \delta C_\ell \delta C_\ell \rangle = \langle a_{\ell m} a_{\ell m} a_{\ell m} a_{\ell m} \rangle - \langle a_{\ell m} a_{\ell m} \rangle \langle a_{\ell m} a_{\ell m} \rangle
  \]
  - Wick’s theorem: \(\langle a^4 \rangle = 3 \langle a^2 \rangle^2\)
- CMB case: Knox 95, Hobson & Magueijo 96
  - need to account for \((2\ell + 1)f_{\text{sky}}\) measurements of each \(\ell\)
    \[
    (\delta C_\ell)^2 \approx \frac{2}{(2\ell + 1)f_{\text{sky}}} (C_\ell + N_\ell)^2 \\
    N_\ell \approx w^{-1} = (\theta_p \sigma_p)^{-2}
    \]
    - # of modes
    - Sample (cosmic) Variance
    - Noise variance

- Bandpowers: bin in \(\ell\) (weighted for specific \(C_\ell\) shape) to reduce errors and decrease covariance
Figure 19. The temperature angular power spectrum of the primary CMB from Planck, showing a precise measurement of seven acoustic peaks, that are well fit by a simple six-parameter CDM theoretical model (the model plotted is the one labelled [Planck + WP + highL] in Planck Collaboration XVI (2013)). The shaded area around the best-fit curve represents cosmic variance, including the sky cut used. The error bars on individual points also include cosmic variance. The horizontal axis is logarithmic up to $\theta = 50$, and linear beyond. The vertical scale is $C_\ell / (\ell + 1)$. The measured spectrum shown here is exactly the same as the one shown in Fig. 1 of Planck Collaboration XVI (2013), but it has been rebinned to show better the low-$\ell$ region.

Figure 20. The temperature angular power spectrum of the CMB, estimated from the SMICA Planck map. The model plotted is the one labelled [Planck + WP + highL] in Planck Collaboration XVI (2013). The shaded area around the best-fit curve represents cosmic variance, including the sky cut used. The error bars on individual points do not include cosmic variance. The horizontal axis is logarithmic up to $\theta = 50$, and linear beyond. The vertical scale is $C_\ell / (\ell + 1)$. The binning scheme is the same as in Fig. 19.

8.1.1. Main catalogue

The Planck Catalogue of Compact Sources (PCCS, Planck Collaboration XXVIII (2013)) is a list of compact sources detected by Planck over the entire sky, and which therefore contains both Galactic and extragalactic objects. No polarization information is provided for the sources at this time. The PCCS differs from the ERCSC in its extraction philosophy: more effort has been made on the completeness of the catalogue, without reducing notably the reliability of the detected sources, whereas the ERCSC was built in the spirit of releasing a reliable catalog suitable for quick follow-up (in particular with the short-lived Herschel telescope). The greater amount of data, different selection process and the improvements in the calibration and map-making processing help the PCCS to improve the performance (in depth and numbers) with respect to the previous ERCSC.

The sources were extracted from the 2013 Planck frequency maps (Sect. 6), which include data acquired over more than two sky coverages. This implies that the flux densities of most of the sources are an average of three or more different observations over a period of 15.5 months. The Mexican Hat Wavelet algorithm (López-Caniego et al. 2006) has been selected as the baseline method for the production of the PCCS. However, one additional method, MTXF (González-Nuevo et al. 2006) was implemented in order to support the validation and characterization of the PCCS.

The source selection for the PCCS is made on the basis of Signal-to-Noise Ratio (SNR). However, the properties of the background in the Planck maps vary substantially depending on frequency and part of the sky. Up to 217 GHz, the CMB is the

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Error band: cosmic variance estimate
error bars: cosmic + noise variance
A toy model

- Consider all-sky observations with uniform white noise
  \[ d_p = T_p + n_p \]
  \[ \langle T_p T_p' \rangle = S_{pp'} = \sum_{\ell} \frac{2\ell + 1}{4\pi} C_\ell B_\ell^2 P_\ell (\hat{x}_p \cdot \hat{x}_p') \]

- Pixel-space likelihood
  \[ \langle n_p n_p' \rangle = N_{pp'} = \sigma^2 \delta_{pp'} \]
  \[ P(d_p|C_\ell) = \frac{1}{2\pi(S+N)|^{1/2}} \exp -\frac{1}{2} d^T (S+N)^{-1} d \]

- Work in harmonic space
  \[ d_{\ell m} \sim \int d^2 \hat{x}_p \, d(\hat{x}_p) Y_{\ell m}(\hat{x}_p) \]

- White noise equiv to const. noise spectrum, \( N_\ell = N \propto \sigma^2 \)
  \[ \langle n_{\ell m} n_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{mm'} N \]

- Likelihood separates
  \[ P(d_{\ell m}|C_\ell) = \prod_{\ell} \frac{1}{2\pi(C_\ell + N)|^{\ell+1/2}} \exp \left( -\frac{2\ell + 1}{2} \frac{\hat{C}_\ell}{C_\ell + N} \right) \]
  \[ \text{with pseudo spectrum} \quad \hat{C}_\ell \equiv \frac{1}{2\ell + 1} \sum_m |d_{\ell m}|^2 \]
Toy model

\[ P(d_{\ell m}|C_\ell) = \prod_\ell \frac{1}{|2\pi(C_\ell + N)|^{\ell+1/2}} \exp \left( -\frac{2\ell + 1}{2} \frac{\hat{C}_\ell}{C_\ell + N} \right) \]

\[ \hat{C}_\ell \equiv \frac{1}{2\ell + 1} \sum_m |d_{\ell m}|^2 \]

- Likelihood (as a function of \( C_\ell \)) maximized at \( C_\ell = \hat{C}_\ell - N \)

- with curvature

\[ \left. \frac{d^2 \ln P}{dC_\ell^2} \right|_{C_\ell = \hat{C}_\ell - N} = - \left( \frac{2\hat{C}_\ell}{2\ell + 1} \right)^{-1} \]

- cf. Gaussian

\[ \frac{d^2 \ln P}{dx^2} = - (\sigma^2)^{-1} \]

and Fisher information

\[ F \equiv - \left\langle \frac{d^2 \ln P}{dx^2} \right\rangle \]

- Skew positive likelihood
- more Gaussian as \( \ell \to \infty \)
Bayesian methods: MADCAP/MADspec

- (quasi-)Newton-Raphson iteration to Likelihood maximum
- Algorithm driven by matrix manipulation (iterated quadratic):
  \[ \delta C_\ell = \frac{1}{2} F_{ll'}^{-1} \text{Tr} \left[ (dd^T - C) \left( C^{-1} \frac{\partial C}{\partial C_\ell} C^{-1} \right) \right] \]
  \[ F_{ll'} = \frac{1}{2} \text{Tr} \left[ C^{-1} \frac{\partial C}{\partial C_\ell} C^{-1} \frac{\partial C}{\partial C_{l'}} \right] \]
  Fisher matrix
  \[ C = S + N \]
- Fisher = approx. Likelihood curvature
- full polarization: signal matrix \( S_{xx'}^{pp'} \)
- Arbitrary (precomputed) noise spectrum
- Arbitrary linear filters
  - Stompor et al; Jaffe et al; Slosar et al
- \( O(N^3) \) operations naively (matrix manipulations), speedup to \( \sim O(N^2) \) for spectrum estimates (potentially large prefactor)
  - Fully parallelized (MPI, SCALAPACK)
  - do calculations in the natural basis
  - no explicit need for full \( N_{pp'} \) matrix in pixel basis (just noise spectrum or autocorrelation)
- e.g., MAXIMA, BOOMERANG

\[ \delta C_\ell = \frac{1}{2} F_{ll'}^{-1} \text{Tr} \left[ (dd^T - C) \left( C^{-1} \frac{\partial C}{\partial C_\ell} C^{-1} \right) \right] \]
\[ F_{ll'} = \frac{1}{2} \text{Tr} \left[ C^{-1} \frac{\partial C}{\partial C_\ell} C^{-1} \frac{\partial C}{\partial C_{l'}} \right] \]
Fisher matrix
\[ C = S + N \]
Frequentist Monte Carlo methods

- **MASTER:** quadratic pseudo-\( C_\ell \) estimate (Hivon et al)
  \[
d_{\ell m} = \sum_p d_p w_p \Omega_p Y_{\ell m}(\hat{x}_p)
\]
  \[
  \hat{C}_\ell = \frac{1}{2\ell + 1} \sum_m |d_{\ell m}|^2
\]
  \[
  \hat{C}_\ell \approx \langle \hat{C}_\ell \rangle = \sum_{\ell'} C_{\ell' M_{\ell' \ell}} F_{\ell} B_{\ell}^2 + N_{\ell}
\]

  where

  - \( N \) is noise bias
  - \( M \) is mode coupling depending on sky coverage
  - \( F \) is experimental filter

- **SPICE:** transform of correlation function estimate (Szapudi et al)
  \[
  \hat{C}_\ell \approx \langle \hat{C}_\ell \rangle = \sum_{\ell'} C_{\ell' M_{\ell' \ell}} F_{\ell} B_{\ell}^2 + N_{\ell}
\]

- **Issues:** filters, weights, noise estimation/iteration, input maps

  - Optimal or naïve?
Hybrid Methods: FASTER

- Key insight: MASTER covariance formalism allows calculation of diagonal part of pseudo-$a_{lm}$ covariance — use for likelihood maximization
  - (nb. this has maximum entropy and so is conservative!)
  - $Diagonal$ likelihood:

\[
P(d_{lm} \mid C_{\ell} I) = \frac{1}{\left[ 2\pi \langle \hat{C}_{\ell} + N_{\ell} \rangle \right]^{1/2}} \exp\left[ -\frac{1}{2} \frac{|d_{lm}|^2}{\langle \hat{C}_{\ell} + N_{\ell} \rangle} \right]
\]

- MC evaluation of means;
- Newton-Raphson iteration towards maximum
- Easy calculation of Likelihood shape parameters

B98, CBI; Contaldi et al
(related suggestions from Delabrouille et al)
Take advantage of uncorrelated noise between different detectors

\[
\langle d_p^1 d_{p'}^2 \rangle = \langle (s_p^1 + n_p^1)(s_{p'}^2 + n_{p'}^2) \rangle = S_{pp}^{12} + N_{pp}^{12} = S_{pp},
\]

Monte Carlo method — without need for noise bias removal
Comparisons

B98: Ruhl et al 2003

FASTER: Contaldi et al 2004

**FASTER**
**MASTER**
*(MC avg)*
Timing and efficiency

- Time
  - optimal/bayes: $N_p^3$
  - monte carlo: $N^{1.5}$
  - prefactors: $N_{MC}, N_{bin}, \ldots$

- Space
  - TOI: 50 GB/yr @200Hz
  - maps: 384 Mb @ $N_{side}=2048$
  - noise matrix: $N^2/2$ entries
    - $\sim 9$ petabytes @ $N_{side}=2048$

resource management will become an issue even for cheapest methods

SPICE: Szapudi et al
Bayesian/Frequentist Correspondence

- Why do both methods seem to work?
  - frequentist mean $\sim$ likelihood maximum
  - frequentist variance $\sim$ likelihood curvature

- Correspondence is exact for
  - linear gaussian models (mapmaking)
  - variance estimation with no correlations and “iid” noise — simple version of $C_l$ problem
    - e.g., all sky, uniform noise
    - likelihood only function of $d_{lm}^2$
    - breaks down in realistic case of correlations, finite sky, varying noise
  - “asymptotic limit”
    - $\sim$ high $l$ iff noise correlations not “too strong”

- But we still want to bootstrap from point estimates to the full likelihood function
Formally the same problem:

\[ \mathbf{d}_p \Rightarrow (i,q,u)_p = \mathbf{d}_{i,p} = \mathbf{d}_q \]

\[ \langle \mathbf{d}_q \mathbf{d}_{q'} \rangle = N_{qq'} + S_{qq'} \]

low S/N, large systematics

complicated correlations:

- \( N_{qq'} \): pixel differences
- \( S_{qq'} = S_{qq'}^i \): linearly dependent on all of \( C_l^{XX'} \) (\( X = T, E, B \))

- e.g., Seljak, Zaldarriaga; Kamionkowski, Kosowsky, Stebbins; &c.

E/B leakage (= T/E/B correlation)

- in principle, don’t need extra separation step if full correlations/distributions is known

- in practice, E/B characteristics impose specific correlation structure — easier to “separate”

- Wiener filter for map from \( C_l \)
Formally the same problem:

\[ d_p \Rightarrow (i,q,u)_p = d_{i,p} = d_q \]

\[ \langle d_q d_{q'} \rangle = N_{qq'} + S_{qq'} \]

- low S/N, large systematics
- complicated correlations:
  - \( N_{qq} \): pixel differences
  - \( S_{qq'} = S_{ij}^{qq} \): linearly dependent on all of \( C_{l_{XX'}}^X \) (X=T,E,B)

- e.g., Seljak, Zaldarriaga; Kamionkowski, Kosowsky, Stebbins; &c.

E/B leakage (= T/E/B correlation)

- in principle, don’t need extra separation step if full correlations/distributions is known
- in practice, E/B characteristics impose specific correlation structure — easier to “separate”
  - Wiener filter for map from \( C_l \).
From $C_\ell$ to cosmology

- **Step 3**: Calculate & characterize posterior prob over some space of cosmological models and imposed priors

- For simplest $[?]$ theories, $C_\ell$ is a deterministic function of the cosmological parameters $\theta = \{H_0, n_s, \Omega_m, \Omega_{DE}, \ldots\}$

- $P(\theta|DI) = \int dC_\ell P(\theta|I) P(C_\ell|\theta I) P(C_\ell|DI)$
  
  $= P(\theta|I) P(C_\ell[\theta]|DI)$
  
  $= P(\theta|I) P(C_\ell[\theta]|\hat{C}_\ell, \sigma_\ell, \text{shape}, I)$

- So est’d $C_\ell$ is [approximately] a **sufficient statistic**

- Only approximate, so not really a separate step

  - $P(\theta|d_i) = P(\theta|T_p) \approx P(\theta|C_\ell)$

  - can explore the likelihood — or finally assign meaningful priors on $\theta$ and calculate the posterior

- MCMC, etc.
The shape of the likelihood function

\[ P(T|C_\ell) = \frac{1}{|2\pi(S+N)|} \exp \left( -\frac{1}{2} T^T (S+N)^{-1} T \right) \]

- Complicated function of \( C_\ell \) [through \( S(C_\ell) \)]
- not a Gaussian in \( C_\ell \)
  - big effect at low \( \ell \)
  - ~Offset lognormal (BJK 00)
- Gaussian in \( \ln(C_\ell + x_\ell) \)
- Other approximations better at moderate \( \ell \)
  - e.g., Hamimeche & Lewis
    - include polarization
    - treat \( T, Q, U \) on same footing
Sampling from the posterior

- Infeasible to directly explore $P(\theta|\text{data})$ for many parameters $\theta$
  - e.g., even the 6-parameter base LCDM model would require $\sim 100^6 = 10^{12}$ evaluations for 100 grid points in each direction...

- Instead, generate samples $\theta_i$ from the distribution.
  - Easy to evaluate moments (means, variances)

  \[ \langle \theta \rangle = \frac{1}{N} \sum_i \theta_i \quad \text{or, more generally} \quad \langle f(\theta) \rangle = \frac{1}{N} \sum_i f(\theta_i) \]
MCMC

- Generate samples from posterior $P(x)$
- Most methods require being able to generate samples from some simpler distribution
- e.g., Markov Chain Monte Carlo
  - Start with proposal distribution $Q(x^*|x)$: probability of proposing point $x^*$ if starting at point $x$
  - often $Q(x|y) = Q(|x-y|)$ (Metropolis)
  - Metropolis Algorithm:
    - given point $x^{(i)}$, generate $x^*$ from $Q(x^*|x^{(i)})$
    - accept $x^*$ as $x^{(i+1)}$ with probability $\min[1,P(x^*)/P(x^{(i)})]$;
    - otherwise $x^{(i+1)} = x^{(i)}$
    - repeat…
prop Sig[μ] = 0.2
prop Sig[σ] = 0.2
acceptance = 27.6%
μ = 0.0208 ± 0.0988
σ = 0.985 ± 0.074

prop Sig[μ] = 0.1
prop Sig[σ] = 0.1
acceptance = 54.6%
μ = −0.128 ± 0.497
σ = 1.2 ± 0.575

prop Sig[μ] = 0.8
prop Sig[σ] = 0.8
acceptance = 4.3%
μ = −0.0193 ± 0.0887
σ = 0.974 ± 0.0513
Monte Carlo methods for the CMB

- Markov Chain Monte Carlo: A. Lewis’ CosmoMC
  - coupled with fast deterministic calculation of power spectrum as fn of cosmological parameters
  - e.g. CMBFAST, CAMB, CLASS

- Other techniques
  - e.g., Skilling’s “nested sampling” which also allows fast calc’n of model likelihoods (“evidence”)
Aside: Gibbs Sampling

- Combine parametric models of foregrounds with power spectrum estimation
  - Jewell et al; Wandelt et al; Eriksen et al; Larson et al;
  - draw [full-sky] map realization given $C_\ell$ and foreground parameter (Wiener filter)
  - draw foreground realization given $C_\ell$ and map
  - draw $C_\ell$ realization given map (Wishart, Gamma dists)
- Output is sample maps and samples of $C_\ell$
  - not always useful for subsequent parameter estimation
  - construct approx. likelihood by averaging over samples
- Blackwell-Rao estimator
The Planck likelihood

- **High $\ell$**
  - Start with pseudo-$C_\ell$ of each detector, with conservative masks
    - for cosmology, consider
      100x100, 143x143, 217x217, 143x217
  - Foregrounds:
    - Use 353 GHz as a dust template
    - Explicit power spectral templates for unresolved point sources, SZ, CIB
  - Instrument:
    - relative calibration between 100, 143, 217
    - beam errors
  - Use Gaussian approximation assuming a fiducial models gives the signal covariances (Hamimeche & Lewis)

- **low $\ell$**
  - Temperature: Planck 30-353 GHz
  - polarization: WMAP
    - needed to fix optical depth $\tau$
Measuring the geometry of the Universe

\[ \Omega_{\text{tot}} = \Omega_m + \Omega_\Lambda = 1 \]

Observations of distant supernova

CMB Observations

Flat Universe

\[ \Omega_{\text{tot}} = \Omega_m + \Omega_\Lambda = 1 \]

Amount of “dark energy” (cosmological constant)

Amount of “matter” (normal + dark)
Measuring the geometry of the Universe

Fig. 25. The Planck+WP+highL data combination (samples; colour-coded by the value of $H_0$) partially breaks the geometric degeneracy between $\Omega_m$ and $\Omega_\Lambda$ due to the effect of lensing in the temperature power spectrum. These limits are significantly improved by the inclusion of the Planck lensing reconstruction (black contours). Combining also with BAO (right; solid blue contours) tightly constrains the geometry to be nearly flat.
Fig. 3. Constraints in the $\Omega_m - H_0$ plane. Points show samples from the Planck-only posterior, coloured by the corresponding value of the spectral index $n_s$. The contours (68% and 95%) show the improved constraint from Planck+lensing+WP. The degeneracy direction is significantly shortened by including WP, but the well-constrained direction of constant $\Omega_m h^3$ (set by the acoustic scale), is determined almost equally accurately from Planck alone.

Fig. 2. Comparison of the base $\Lambda$CDM model parameters for Planck+lensing only (colour-coded samples), and the 68% and 95% constraint contours adding WMAP low-$\ell$ polarization (WP; red contours), compared to WMAP-9 (Bennett et al. 2012; grey contours).
Hierarchical Models

- So we have a **hierarchical model**
  - ask progressively more complicated questions of the data, with (approximately) no dependence on the details of previous results
  - Timelines $\Rightarrow$ maps $\Rightarrow$ spectra $\Rightarrow$ parameters

- Each is a “nuisance parameter” for the next step w/ an uncontroversial prior defining that step
  - e.g., $\langle T_p T_p' \rangle = S_{pp'}(C_\ell) \quad P(C_\ell|\theta) = \delta[C_\ell - C_\ell(\theta)]$

- But in the realistic case there may be other nuisance parameters for which the priors are relevant:
  - timeline systematics, foregrounds, &c.
Testing assumptions

- We have been calculating the posterior
  \[ P(\Omega_b h^2, \Omega_c h^2, \theta_{MC}, \tau, n_s, A_s | \text{Planck, I}) \]

- Background information “I”
  - (testable) assumptions:
    - LCDM in general
    - gaussianity, isotropy
    - by some measures, it obeys these assumptions very well

Very difficult to test these assumptions absent a specific alternative, in a Bayesian way
Low power on large scales
Anomalies?

Less structure

“cold spot”

More structure

Small (but statistically significant) difference between the power in the hemispheres

Overall low amplitude at large scales

nb. there is also a known asymmetry from CMB dipole aberration.
Large-scale anisotropy

- Hemispherical differences: how can we arrange anisotropy on the scale of the horizon?
  - initial conditions: anisotropic inflation?
  - the large-scale structure of spacetime
    - change the geometry: Bianchi
      - homogeneous + anisotropic spacetimes
    - change the topology
The shape of the Universe

- General relativity determines the curvature of the Universe, but not its topology (holes and handles).
- Most theories of quantum gravity (and quantum cosmology) predict topological change on small scales and at early times.
- Does this have cosmological implications?
  - E.G., small universe $\Rightarrow$ fewer large-scale modes available $\Rightarrow$ low power on large scales?
Topology in a flat “universe”

Don’t need to “embed” the square to have a connected topology.

“tiling the plane”
Topology in a flat “universe”

Don’t need to “embed” the square to have a connected topology.
Topology in a flat “universe”

Don’t need to “embed” the square to have a connected topology.

“tiling the plane”
Don’t need to “embed” the square to have a connected topology.
Tile the 2-sphere with different fundamental domains
   (Each of these has a 3-sphere analogy)
Can also tile the hyperbolic universe:
   (Bond, Pogosyan, etc.)

http://www.scienpecnews.org/pages/sn_arc98/2_21_98/bob1.htm
## Multiply-connected Spherical Topologies

<table>
<thead>
<tr>
<th>Space</th>
<th>Fundamental group</th>
<th>Order</th>
<th>Elements</th>
<th>F.P.</th>
</tr>
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<td>Quaternionic</td>
<td>Binary Dihedral</td>
<td>8</td>
<td>order 2 rotations about 2 perpendicular axes</td>
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<td>Binary Tetrahedral</td>
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<tr>
<td>Poincaré</td>
<td>Binary Icosahedral</td>
<td>120</td>
<td>symmetries of r. icosahedron</td>
<td></td>
</tr>
</tbody>
</table>
Measuring Topology with the CMB

- Perfect correlation [of SW]
- “circles in the sky”
- finite-lag correlation

Us!

Last Scattering Surface
Topology in the CMB

- Look for repeated patterns
- **Generic** & specific methods
- matching patches (e.g., Levin et al)
  - method of images (e.g., Bond et al)
  - assumes infinitely thin LSS
  - mostly open Universes
- Circles in the sky (Cornish, Spergel, Starkman)
  - looks for LSS structure; ignores different views of the same point
  - nb. generic methods work as frequentist null tests but need comparison w/ specific topologies to get statistics
    - even Bayesians need to do exploratory statistics
  - Cornish et al '04: “fewer than 1 in 100 random skies generate a false match” [??]: limit out to 24 Gpc
Topology: methods

- When topological scale $\lesssim$ Horizon scale, induce anisotropic correlations (and suppress power) on large scales.

- Direct search for matched circles
  - sensitive to topology with parallel matched surfaces

- Explicit Likelihood
  - calculate correlation matrix for specific topologies.
  - 3d Gaussian with $\langle \delta_k \delta_{k'} \rangle = (2\pi)^3 \delta_D(k+k')P(k)$ w/ $k$ restricted to fundamental domain with boundary conditions
  - induced CMB correlations depend on topology (incl. orientation)
Simulated Maps ($\Omega_k = -0.063$)
Lowest multipoles

<table>
<thead>
<tr>
<th>Quaternionic</th>
<th>Octahedral</th>
<th>Truncated cube</th>
<th>Poincaré</th>
<th>WMAP</th>
</tr>
</thead>
<tbody>
<tr>
<td>ℓ=2</td>
<td></td>
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<tr>
<td>ℓ=3</td>
<td></td>
<td></td>
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<tr>
<td>ℓ=4</td>
<td></td>
<td></td>
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<tr>
<td>all</td>
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</tbody>
</table>
Bayesian topology

\[ P(a|C) = \frac{1}{|2\pi C|^{1/2}} \exp \left( -\frac{1}{2} a^T C^{-1} a \right) \]

- **Full correlation matrix:**
  - \( C = \langle a_{\ell m} a_{\ell' m'} \rangle = C_{\ell \ell' mm'} = C(\text{cosmology, topology}) \)
  - \( C_{\ell \ell' mm'} \propto \int d^3 k \Delta_{\ell}(k, \Delta \eta) \Delta_{\ell'}(k, \Delta \eta) P(k) \Rightarrow \sum_n \Delta_{\ell}(k_n, \Delta \eta) \Delta_{\ell'}(k_n, \Delta \eta) P(k_n) Y_{\ell m}(\hat{n}) Y_{\ell' m'}^{*}(\hat{n}) \)
  - \( a = a_{\ell m} \)
    - (Noise irrelevant on scales of interest)
    - Suppressed power \( \Rightarrow \) stronger correlations

**Planck**

Scientists of the temperature on the sky when

(\( \delta \) is the Kronecker delta symbol and exceeds the interme-

(\( k (2 \hat{p} p / 5.1 \) on the point correlation function of the source

(``

\[ \chi (p^2 m R p S p / a, = = \) and \( n ) \) by \( i \) correlation matrix \( m x \))``

The strongest constraints are imposed on \( p \) \( R \), \( \) rec \( / i m \). Weaker constraints are imposed on topologies \( p k \) \( R \), \( m C y 40 \) both methods have been tested and were found to

\[ \text{on the source function unless all the terms} \]

\( \hat{Y} p (p \hat{Y} p k \hat{Y} p a \) \( \hat{Y} p m \) \( \hat{Y} p p \) \( k \).

Thus, we can constrain all topologies predicting one pair

\( \) and the three spherical cases \( S R m C \).

\( C_{\ell \ell' mm'} \) refers to the need for sum regularization in the models with

\( \text{diagonal, nor is it} \)

\( T \)\text{ical harmonic coe}

\( \) limits consideration to Sachs-Wolfe terms) when the action

\( \) in the source function are scalar quantities (which is the case if

\( \) with all back-to-back circles centred on a great circle of the ce-

\( \) and the three spherical cases

\( \) turn spaces, as well as Klein and chimney spaces. The statistic

\( \) is still modified versus the singly-connected limit. The e

\( \) large scales are not present, although the

\( \) domain. Note that when

\( \)
Pixel correlations

- Octahedral:
  - $h=0.64$, $\Omega_k = -0.017$

Rows of the correlation matrix:
Pixel correlations

- Poincaré:
  - \( h = 0.52, \Omega_k = -0.063 \)

Rows of the correlation matrix:
“Matched circles” in a simulated Universe:

Alas, not found... we can limit the size of the “fundamental cube” to be greater than the size of the surface we observe with the CMB:

- side $L \geq 26$ Gpc
Topology: results

- No strong evidence for topology on the scale of the last-scattering surface
Bianchi Models

- Homogeneous, anisotropic spaces
- $\text{VII}_h$: global shear and rotation
  - parameter $h$ relates vorticity $\omega_i$ to shear $\sigma_{ij}$, $\Omega_{\text{tot}}$
    \[
    \left(\frac{\omega}{H}\right)_0 = \frac{(1 + h)^{1/2}(1 + 9h)^{1/2}}{6h} \frac{1 - \Omega_{\text{tot}}}{\Omega_{\text{tot}}} \sqrt{\left(\frac{\sigma_{12}}{H}\right)_0^2 + \left(\frac{\sigma_{13}}{H}\right)_0^2}
    \]
- Focusing induces specific pattern of temperature anisotropy on large scales
- Full likelihood calculation (Gaussian added to deterministic template)
  - consistent cosmology very low likelihood
### Bianchi Models

#### Flat-decoupled

<table>
<thead>
<tr>
<th>Bianchi Parameter</th>
<th>MAP</th>
<th>SMICA Mean</th>
<th>MAP</th>
<th>SEVEM Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_k^0$</td>
<td>0.38</td>
<td>0.32 ± 0.12</td>
<td>0.35</td>
<td>0.31 ± 0.15</td>
</tr>
<tr>
<td>$\Omega_m^0$</td>
<td>0.20</td>
<td>0.31 ± 0.20</td>
<td>0.22</td>
<td>0.30 ± 0.20</td>
</tr>
<tr>
<td>$x$</td>
<td>0.63</td>
<td>0.67 ± 0.16</td>
<td>0.66</td>
<td>0.62 ± 0.23</td>
</tr>
<tr>
<td>$(\omega/H_0)$</td>
<td>$8.8 \times 10^{-10}$</td>
<td>$(7.1 \pm 1.9) \times 10^{-10}$</td>
<td>$9.4 \times 10^{-10}$</td>
<td>$(5.9 \pm 2.4) \times 10^{-10}$</td>
</tr>
</tbody>
</table>

**Fig. 20:**
- (a) Flat-decoupled-Bianchi model.

#### Open-coupled

<table>
<thead>
<tr>
<th>Bianchi Parameter</th>
<th>MAP</th>
<th>SMICA Mean</th>
<th>MAP</th>
<th>SEVEM Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_k$</td>
<td>0.05</td>
<td>0.07 ± 0.05</td>
<td>0.09</td>
<td>0.08 ± 0.04</td>
</tr>
<tr>
<td>$\Omega_m^0$</td>
<td>0.41</td>
<td>0.33 ± 0.07</td>
<td>0.41</td>
<td>0.32 ± 0.07</td>
</tr>
<tr>
<td>$x$</td>
<td>0.55</td>
<td>0.60 ± 0.07</td>
<td>0.50</td>
<td>0.59 ± 0.07</td>
</tr>
<tr>
<td>$(\omega/H_0)$</td>
<td>$5.9 \times 10^{-10}$</td>
<td>$(4.0 \pm 2.4) \times 10^{-10}$</td>
<td>$9.3 \times 10^{-10}$</td>
<td>$(4.5 \pm 2.8) \times 10^{-10}$</td>
</tr>
</tbody>
</table>

**Fig. 20:**
- (b) Open-coupled-Bianchi model.

### Flat-coupled:

$\omega_0/H_0 < 8.1 \times 10^{-10}$ (95%)
The Bianchi VII\textsubscript{h} uncoupled “model” accounts for much of the hemispherical asymmetry.
Non-gaussianity

- Another way to go beyond (and check) the simple assumptions
  - In general, can’t write down “the” distribution for a parameter with some specific set of higher-order moments. (cf. yesterday’s discussion of Gaussian as MaxEnt)
  - Use frequentist estimators...
- In the absence of a specific model, want to determine phenomenological parameters describing departure from an isotropic multivariate gaussian distribution.
  - e.g., moments — but not unique (there is no distribution that has mean, variance, skewness, but no higher moments)
    - for (suitably defined) small non-gaussianity, third-order moments should dominate
    - full determination of 3-pt function is computationally infeasible (and we lack sufficient S/N)
  - parameterize non-gaussianity
non-Gaussianity: $f_{\text{NL}}$

- Heuristically \( \phi = \phi_G + f_{\text{NL}}(\phi_G^2 - \langle \phi_G^2 \rangle) \) for a Gaussian \( \phi_G \) (e.g., multi-field inflation)
- This is the (spatially) local model for non-Gaussianity
- Induces specific 3-d correlations
  \[
  \langle \phi\phi\phi \rangle \sim 3 f_{\text{NL}} (\langle \phi_G\phi_G\phi_G\phi_G \rangle - \langle \phi_G\phi_G \rangle \langle \phi_G\phi_G \rangle) + O(f_{\text{NL}}^2)
  \]
  \[
  \sim 6 f_{\text{NL}} \langle \phi_G\phi_G \rangle \langle \phi_G\phi_G \rangle + O(f_{\text{NL}}^2)
  \]
  and hence 2-d correlations in the CMB
- Corresponds to Fourier bispectrum \( B(k_1, k_2, k_3) \) which peaks in squeezed case \( k_1 \ll k_2 \approx k_3 \)
  - modulate small-scale structure by large-scale modes
    - cf. galaxy bias
- More generally, consider other shapes (e.g., equilateral) motivated by specific theories
Estimating non-Gaussianity

\[ \langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_2 m_2} \rangle = G_{m_1 m_2 m_3}^{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3} = G_{m_1 m_2 m_3}^{\ell_1 \ell_2 \ell_3} h_{\ell_1 \ell_2 \ell_3}^{-2} B_{\ell_1 \ell_2 \ell_3} \]

- Expect to be able to estimate the third moment by taking some weights average over cubic products of data
  - (cf. quadratic estimators of power spectra)
    - “optimal” (min-var) weights computationally infeasible (Heavens 1998) — average over all triples of data
    - ignoring off-diagonal covariance gives somewhat more tractable case (Creminelli et al. 2006).
    - further simplify for “separable” shapes (Komatsu et al=KSW) and linear combinations thereof (Fergusson & Shellard)
    - generalize to \( S_\ell = \text{skew-} C_\ell \), retains shape information in one \( \ell \) direction (Heavens & Munshi)
Non-Gaussianity from Planck

- Planck detects (non-Primordial) non-Gaussianity...

<table>
<thead>
<tr>
<th></th>
<th>Independent</th>
<th>ISW-lensing subtracted</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KSW</td>
<td>Binned</td>
</tr>
<tr>
<td><strong>SMICA</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local</td>
<td>9.8 ± 5.8</td>
<td>9.2 ± 5.9</td>
</tr>
<tr>
<td>Equilateral</td>
<td>-37 ± 75</td>
<td>-20 ± 73</td>
</tr>
<tr>
<td>Orthogonal</td>
<td>-46 ± 39</td>
<td>-39 ± 41</td>
</tr>
<tr>
<td><strong>NILC</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local</td>
<td>11.6 ± 5.8</td>
<td>10.5 ± 5.8</td>
</tr>
<tr>
<td>Equilateral</td>
<td>-41 ± 76</td>
<td>-31 ± 73</td>
</tr>
<tr>
<td>Orthogonal</td>
<td>-74 ± 40</td>
<td>-62 ± 41</td>
</tr>
<tr>
<td><strong>SEVEM</strong></td>
<td></td>
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</tr>
<tr>
<td>Local</td>
<td>10.5 ± 5.9</td>
<td>10.1 ± 6.2</td>
</tr>
<tr>
<td>Equilateral</td>
<td>-32 ± 76</td>
<td>-21 ± 73</td>
</tr>
<tr>
<td>Orthogonal</td>
<td>-34 ± 40</td>
<td>-30 ± 42</td>
</tr>
<tr>
<td><strong>C-R</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local</td>
<td>12.4 ± 6.0</td>
<td>11.3 ± 5.9</td>
</tr>
<tr>
<td>Equilateral</td>
<td>-60 ± 79</td>
<td>-52 ± 74</td>
</tr>
<tr>
<td>Orthogonal</td>
<td>-76 ± 42</td>
<td>-60 ± 42</td>
</tr>
</tbody>
</table>
Non-Gaussianity from Planck

We see ISW lensing...

... but nothing primordial

- Local

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The CMB: A Case Study

- Hierarchical Bayesian formalism
  - raw-data ⇒ maps ⇒ spectra ⇒ parameters
  - radical data compression
  - need to keep track of likelihood function details

- Checking assumptions
  - “anomalies”?
    - No obvious solution by changing the large-scale structure of spacetime (topology, Bianchi)
  - non-Gaussianity
    - lensing, point sources, correlations detected in Planck
    - no evidence yet for primordial non-Gaussianity