Mass at Zero and Small-Strike Implied Volatility Expansion in the SABR Model

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The SABR Model

Consider

$$\begin{aligned} dX_t &= Y_t X_t^{\beta}\, dW_t, & X_0 &= x_0 > 0, \\ dY_t &= \nu Y_t\, dZ_t, & Y_0 &= y_0 > 0, \\ d\langle Z, W\rangle_t &= \rho\, dt, \end{aligned}$$

$$\nu > 0,\ \rho \in [-1, 1],\ \beta \in [0, 1],$$

and W and Z are Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

The SABR Initial Value Problem

The geometric idea

Relate the SABR Kolmogorov equation

$$\partial_s K_{X,Y}(s,x,y) = \frac{1}{2} \underbrace{y^2 \left(x^{2\beta} \partial^2_{xx} + 2\rho\nu x^{\beta} \partial^2_{xy} + \nu^2 \partial^2_{yy} \right)}_{\mathcal{A}_{SABR}(x,y)} K_{X,Y}(s,x,y)$$

to a heat equation

$$\partial_s K^{g}_{X,Y}(s,x,y) = \frac{1}{2} \Delta_{g(x,y)} \, K^{g}_{X,Y}(s,x,y)$$

on a manifold with an appropriately chosen Riemannian metric g.

The Riemannian metric

Consider $\mathcal{A}$ uniformly elliptic second order operator,
$\xi^{ij}(x)$ highest order coefficients of $\mathcal{A}$,
Matrix of coefficients $\Xi(x) := (\xi^{ij}(x))_{i,j}$
$g_{ij}(x)$ coefficients of the inverse Ξ^{-1}. Then

Riemannian metric

$$\sum_{i,j=1}^{n} g_{ij}(x)\, dx^i \otimes dx^j$$

is then a symmetric covariant tensor field on the state space S, the pair (S, g) is a Riemannian manifold.

The SABR Initial Value Problem

Heat equation on a manifold

The operator $\mathcal{A}_{SABR}$, of the SABR model, differs from the manifold's Laplace operator $\Delta_{g(x,y)}$ only by a lower order term $b(x,y)\partial_x$:

$$\mathcal{A}_{SABR}(x,y) + \underbrace{\nu^2 y^2 x^{2\beta-1}}_{b(x,y)} \partial_x = \Delta_{g(x,y)}.$$

The regular perturbation

The following asymptotic relation holds for the fundamental solutions of the related PDEs

$$K_{X,Y}(s,x,y) = (\mathrm{Id} + \lambda b(x,y)\partial_x) K^{g}_{X,Y}(s,x,y) + O(\lambda^2)$$

for all $s > 0$ and $X, Y, x, y > 0$, where the expansion is in $\lambda = \epsilon s$.

Small-Time Asymptotics

For $\partial_t u = \frac{1}{2}\mathcal{A}u$, when $\mathcal{A}$ is *uniformly elliptic*,
we have the following short time asymptotic limit

Varadhan's Formula

$$\lim_{t\to 0} t \log p_t(z_1, z_2) = -\frac{d(z_1, z_2)^2}{2}$$

$p_t(z_1, z_2)$ denotes the fundamental solution of $\partial_t u = \frac{1}{2}\mathcal{A}u$,
$d(z_1, z_2)$ the Riemannian distance from the metric $g_{ij} = (\Xi)^{-1}_{i,j}$.

The Geometric Viewpoint

Degeneracy at $x = 0, y = 0$.

The matrix of highest order coefficients of $\mathcal{A}_{SABR}$ is

$$\Xi_{SABR}(x, y) = \begin{pmatrix} y^2 x^{2\beta} & y^2 \rho x^{\beta} \\ y^2 \rho x^{\beta} & y^2 \end{pmatrix}.$$

Riemannian metric at $x = 0, y = 0$ not defined.

$$g(x, y) = (\Xi_{SABR})^{-1}(x, y) = \begin{pmatrix} \frac{1}{(1-\rho^2) y^2 x^{2\beta}} & \frac{-\rho}{(1-\rho^2) y^2 x^{\beta}} \\ \frac{-\rho}{(1-\rho^2) y^2 x^{\beta}} & \frac{1}{(1-\rho^2) y^2} \end{pmatrix}.$$

Problem?

Varadhan's formula does not always fail when Ξ does not fulfill the uniform ellipticity condition

Normal SABR model: $\beta = 0$, $\rho = 0$

$$\Xi_{SABR}(x, y) = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix},$$

for all $\{(x, y) \in \mathbb{R}^2 : x, y > 0\}$, in fact for all $\{(x, y) \in \mathbb{R}^2 : y > 0\}$.

The Riemannian metric is the well known Poincaré metric

$$\frac{1}{y^2} dx \otimes dx + \frac{1}{y^2} dy \otimes dy.$$

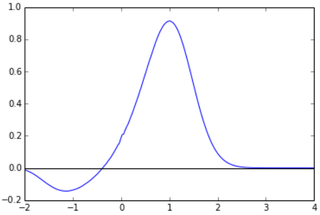
Consider the SABR Formula (HLW)

$$\sigma_{SABR} \approx \frac{\alpha \log(x/K)}{\frac{x^{1-\beta} - K^{1-\beta}}{1-\beta}} \left(\frac{\zeta}{\hat{\xi}(\zeta)} \right)$$

$$\left\{ 1 + \left[\frac{2\gamma_2 - \gamma_1^2 + x_{av}^{-2}}{24} \alpha^2 x^{2\beta} + \frac{\rho\nu\alpha\gamma_1 x_{av}^{\beta}}{4} + \frac{(2-3\rho^2)\nu^2}{24} \right] \epsilon^2 T + \dots \right\}$$

Expansion for the implied volatility in $\epsilon = \nu T$

The degeneracy at $x = 0$ matters.



Small-Strike Asymptotics

If $\mathbb{P}(X_T \leq K) - \mathbb{P}(X_T = 0) = \mathcal{O}(K^{\varepsilon})$ as $K \downarrow 0$,
and X is a true martingale then

Small-Strike Expansion with Positive Mass (de Marco et al. '13)

$$I_T(K) = \sqrt{\frac{2|\log K|}{T}} + \frac{\mathcal{N}^{-1}(\mathrm{m}_T)}{\sqrt{T}} + \frac{(\mathcal{N}^{-1}(\mathrm{m}_T))^2}{2\sqrt{2T|\log K|}} + \Phi(K),$$

$\mathrm{m}_T := \mathbb{P}(X_T = 0)$ is the mass at the origin,
$\mathcal{N}$ the Gaussian cumulative distribution function,
$\Phi : (-\infty, 0) \to \mathbb{R}$ satisfies $\limsup_{K\downarrow 0} \sqrt{2T|\log K|}|\Phi(K)| \leq 1$.
See also Gulisashvili '15.

The Mass at Zero in the SABR Model

Uncorrelated case: decompose SABR by time-change

$$\mathbb{P}(X_t = 0) = \int_0^\infty \mathbb{P}\left(\widetilde{X}_r = 0\right) \mathbb{P}\left(\int_0^t Y_s^2 \mathrm{d}s \in \mathrm{d}r\right) \mathrm{d}r,$$

where the mass at zero for the CEV model $\widetilde{X}$ is

$$\mathbb{P}\left(\widetilde{X}_r = 0\right) = 1 - \Gamma\left(\frac{1}{2(1-\beta)}, \frac{x_0^{2(1-\beta)}}{2r(\beta-1)^2}\right),$$

with $\Gamma(v, z) \equiv \Gamma(v)^{-1} \int_0^z u^{v-1} \mathrm{e}^{-u} \mathrm{d}u$.

Tractable formula for the mass $\mathbb{P}(X_t = 0)$?

The density of the time-change

$$\mathbb{P}\left(\int_0^t Y_s^2 \mathrm{d}s \in \mathrm{d}r\right)$$

- familiar: appears when pricing Asian options
- related to the Hartman-Watson density
- highly oscillating expressions, double integral, ...

$\Rightarrow$ Numerical difficulties

$$= \frac{2^{1/4}\sqrt{\nu}}{r^{3/4}} \exp\left(-\frac{\nu^2 t}{8} - \frac{1}{4\nu^2 r}\right) m_{2\nu^2 t}\left(-\frac{3}{4}, \frac{1}{4\nu^2 r}\right) \mathrm{d}r$$

Small-time Asymptotics

Oscillating parts:

$$m_y(\mu, z) \equiv \frac{8z^{3/2}\Gamma(\mu + \frac{3}{2})\mathrm{e}^{\frac{\pi^2}{4y}}}{\pi\sqrt{2\pi y}} \times$$
$$\int_0^\infty \mathrm{e}^{-z\cosh(2u)-\frac{1}{y}u^2}\mathrm{M}\left(-\mu, \frac{3}{2}, 2z\sinh(u)^2\right)\sinh(2u)\sin\left(\frac{\pi u}{y}\right)\mathrm{d}u$$

M is the Kummer function:

$$\mathrm{M}(a, b, x) \equiv 1 + \sum_{k=1}^{\infty} \frac{a(a+1)\ldots(a+k-1)x^k}{b(b+1)\ldots(b+k-1)k!}.$$

Way out: Direct inverse Laplace transform approach inspired by Gerhold '11. $\Rightarrow$ Obtain small-time asymptotics.

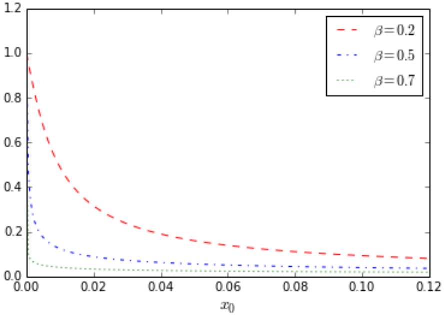
Large-time Asymptotics

$$\lim_{t\uparrow\infty} \mathbb{P}(X_t = 0)$$
$$= 1 - \frac{y_0}{\nu\sqrt{2\pi}} \int_0^\infty \Gamma\left(\frac{1}{2(1-\beta)}, \frac{x_0^{2(1-\beta)}}{2r(\beta-1)^2}\right) r^{-3/2} \exp\left(-\frac{y_0^2}{2\nu^2 r}\right) \mathrm{d}r.$$

Fairly regular $\Rightarrow$ Numerics, asymptotic expansion

Accumulation of Mass in the SABR Model

Influence of the initial value x_0 on the large-time mass at zero



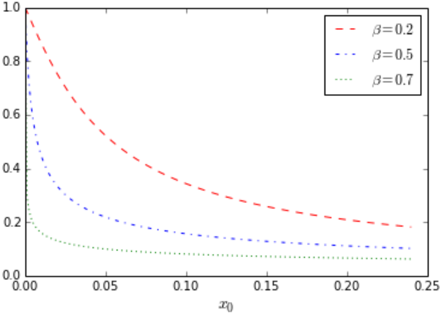
with $(y_0, \nu) = (0.015, 0.6)$

"not feeling the boundary"

volatility process for these parameters fairly well-behaved

Accumulation of Mass in the SABR Model

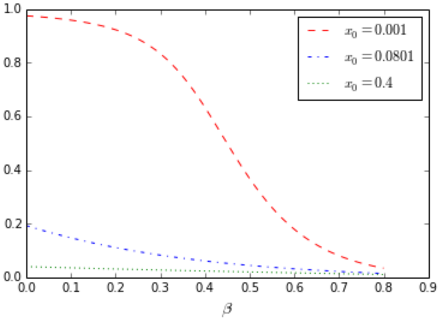
Influence of the initial value x_0 on the large-time mass at zero



with $(y_0, \nu) = (0.1, 1)$

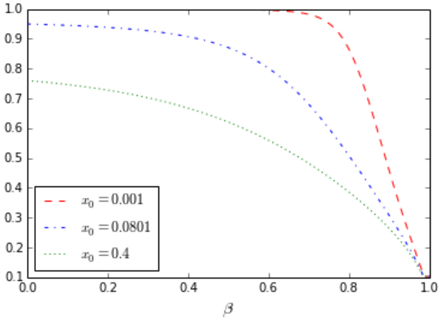
Accumulation of Mass in the SABR Model

Influence of the parameter β on the large-time mass at zero



with $(y_0, \nu) = (0.015, 0.6)$

Influence of the parameter β on the large-time mass at zero



with $(y_0, \nu) = (0.1, 1)$

Application: Comparing Implied Volatilities

Recall de Marco et al. '13:

$$I_T(K) = \sqrt{\frac{2|\log K|}{T}} + \frac{\mathcal{N}^{-1}(\mathrm{m}_T)}{\sqrt{T}} + \frac{(\mathcal{N}^{-1}(\mathrm{m}_T))^2}{2\sqrt{2T|\log K|}} + \Phi(K) \quad (1)$$

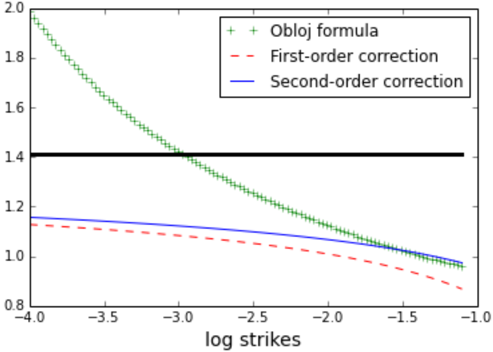
- model independent
- by definition arbitrage-free

we plot the functions $k := \log K \in \mathbb{R} \mapsto I_T(\mathrm{e}^k)\sqrt{T/|k|}$

we compare SABR formula (Obłój refinement) with (1)

in order to avoid arbitrage, has to be bounded by $\sqrt{2}$

Application: Comparing Implied Volatilities



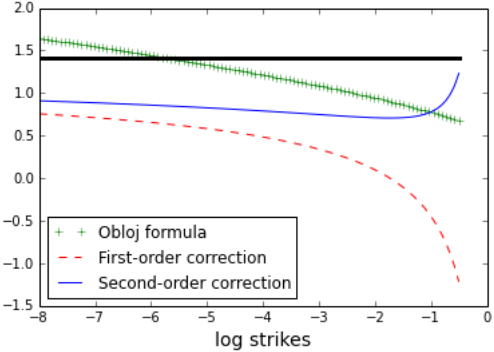
We plot $k \in \mathbb{R} \mapsto I_T(\mathrm{e}^k)\sqrt{T/|k|}$.

The black line marks the level $\sqrt{2}$.

Parameters are $(\nu, \beta, \rho, x_0, y_0, T) = (0.3, 0, 0, 0.35, 0.05, 10)$

The large-time mass is equal to 28.3%

Application: Comparing Implied Volatilities



We plot $k \in \mathbb{R} \mapsto I_T(\mathrm{e}^k)\sqrt{T/|k|}$.
The black line marks the level $\sqrt{2}$.

Parameters are $(\nu, \beta, \rho, x_0, y_0, T) = (0.6, 0.6, 0, 0.08, 0.015, 10)$

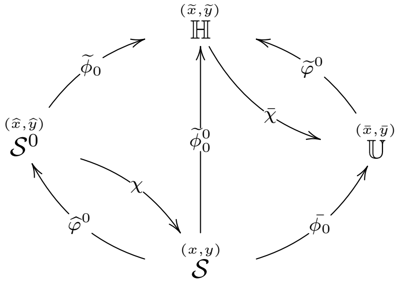
The large-time mass is equal to 3.1%

Correlated Case

We consider the associated heat equation

$$\begin{aligned} \mathrm{d}X_t &= Y_t X_t^{\beta}\mathrm{d}W_t + \frac{\beta}{2} Y_t^2 X_t^{2\beta-1}\mathrm{d}t, & X_0 = x_0 > 0,\\ \mathrm{d}Y_t &= \nu Y_t \mathrm{d}Z_t, & Y_0 = y_0 > 0,\\ \mathrm{d}\langle Z, W\rangle_t &= \rho \mathrm{d}t, \end{aligned}$$

with $\nu > 0$, $\rho \in (-1, 1)$, $\beta \in [0, 1)$.
Particular interest in the cases $\beta = 0$ and $\rho = 0$.
See also: Hobson '10, Döring-H. '15.



Thank you for your attention!