Linear Programs and Robust Hedging Problems

Sergey Badikov
joint work with Mark Davis and Antoine Jacquier

Imperial College London

Third Imperial-ETH Workshop on Mathematical Finance,
March 5, 2015

Introduction to Linear Programs

- A linear program (LP) is an optimization problem with linear objective function and linear constraints.
- Linear programs can be solved very efficiently (i.e. Simplex method by G.B. Dantzig 1947).
- There are different types of linear programs:
  - ▷ Finite dimensional LPs: finite number of decision variables and constraints;
  - ▷ Semi-infinite LPs: either number of decision variables or constraints is infinite (Polynomial approximation);
  - ▷ Infinite-dimensional LPs: both decision variables and constraints are infinitely dimensional (Optimal transport);
  - ▷ Continuous LPs: linear optimal control problem with linear state constraints (Bottleneck Problem - Bellman 1957).

Introduction to Linear Programs

- A linear program (LP) is an optimization problem with linear objective function and linear constraints.
- Linear programs can be solved very efficiently (i.e. Simplex method by G.B. Dantzig 1947).
- There are different types of linear programs:
  - ▷ Finite dimensional LPs: finite number of decision variables and constraints;
  - ▷ Semi-infinite LPs: either number of decision variables or constraints is infinite (Polynomial approximation);
  - ▷ Infinite-dimensional LPs: both decision variables and constraints are infinitely dimensional (Optimal transport);
  - ▷ Continuous LPs: linear optimal control problem with linear state constraints (Bottleneck Problem - Bellman 1957).

Introduction to Linear Programs

- A linear program (LP) is an optimization problem with linear objective function and linear constraints.
- Linear programs can be solved very efficiently (i.e. Simplex method by G.B. Dantzig 1947).
- There are different types of linear programs:
  - ▷ Finite dimensional LPs: finite number of decision variables and constraints;
  - ▷ Semi-infinite LPs: either number of decision variables or constraints is infinite (Polynomial approximation);
  - ▷ Infinite-dimensional LPs: both decision variables and constraints are infinitely dimensional (Optimal transport);
  - ▷ Continuous LPs: linear optimal control problem with linear state constraints (Bottleneck Problem - Bellman 1957).

Example of a Linear Program in Finite Dimensions

Optimal manufacturing: given a production facility where

- n is the number of production lines ($i = 1, \ldots, n$);
- m is the number of different products produced on each line ($j = 1, \ldots, m$);
- $x_i \geq 0$ is the level at which each line can be operated;
- w_j is the revenue collected from producing a unit of product;
- c_i is the cost of production per line if operated per level;
- $a_i j$ is the yield of each product on each line;
- b_j is the required output per product.

Example of a Linear Program in Finite Dimensions

- Objective is to produce a given number of products of each category at a minimal cost

$$\min \langle x, c \rangle_{\mathbb{R}^n} \quad \text{s.t.} \quad Ax = b, \quad x \geq 0,$$

where A is a $m \times n$ matrix;

Example of a Linear Program in Finite Dimensions

- Objective is to produce a given number of products of each category at a minimal cost

$$\min \langle x, c \rangle_{\mathbb{R}^n} \quad \text{s.t.} \quad Ax = b, \quad x \geq 0,$$

where A is a $m \times n$ matrix;

- Can ask a related question of maximizing revenue per unit of production given b units of different products

$$\max \langle b, w \rangle_{\mathbb{R}^m} \quad \text{s.t.} \quad A^* w - c \geq 0, \quad w \in \mathbb{R}^m,$$

where matrix $A^* = A^T$.

Definition of LP

Given two dual pairs of vector spaces (X, Y) and (Z, W) endowed with bilinear forms denoted by $\langle \cdot, \cdot \rangle_{XY}$ and $\langle \cdot, \cdot \rangle_{ZW}$

- Equality constrained problem (EP):

$$\widetilde{\mathcal{P}} := \inf \langle x, c \rangle_{XY} \quad \text{s.t.} \quad Ax = b, \quad x \geq 0,$$

  where $c \in Y$, $b \in Z$ are given and $A : X \to Z$ is a linear map.

- Dual equality constrained problem (EP*):

$$\widetilde{\mathcal{D}} := \sup \langle b, w \rangle_{ZW} \quad \text{s.t.} \quad A^* w - c \geq 0, \quad w \in W,$$

  where $A^* : W \to Y$ is the adjoint of A such that

$$\langle Ax, w \rangle_{ZW} = \langle x, A^* w \rangle_{XY}.$$

Definition of LP

Given two dual pairs of vector spaces (X, Y) and (Z, W) endowed with bilinear forms denoted by $\langle \cdot, \cdot \rangle_{XY}$ and $\langle \cdot, \cdot \rangle_{ZW}$

- Equality constrained problem (EP):

$$\widetilde{\mathcal{P}} := \inf \langle x, c \rangle_{XY} \quad \text{s.t.} \quad Ax = b, \quad x \geq 0,$$

  where $c \in Y$, $b \in Z$ are given and $A : X \to Z$ is a linear map.

- Dual equality constrained problem (EP*):

$$\widetilde{\mathcal{D}} := \sup \langle b, w \rangle_{ZW} \quad \text{s.t.} \quad A^* w - c \geq 0, \quad w \in W,$$

  where $A^* : W \to Y$ is the adjoint of A such that

$$\langle Ax, w \rangle_{ZW} = \langle x, A^* w \rangle_{XY} .$$

Weak and strong duality in LPs

- *Feasible solution*: if a decision variable satisfies constraints, it is feasible;
- *Weak duality*: if both the primal and the dual programs have feasible solutions then $\widetilde{\mathcal{P}} \leq \widetilde{\mathcal{D}}$;
- *Strong duality*: the primal program and its dual have the same value, i.e. $\widetilde{\mathcal{P}} = \widetilde{\mathcal{D}}$;
- Strong duality always holds for finite dimensional programs;
- Strong duality does not always hold in semi-infinite or infinite dimensional programs – *Duality gap*.

Weak and strong duality in LPs

- *Feasible solution*: if a decision variable satisfies constraints, it is feasible;
- *Weak duality*: if both the primal and the dual programs have feasible solutions then $\widetilde{\mathcal{P}} \leq \widetilde{\mathcal{D}}$;
- *Strong duality*: the primal program and its dual have the same value, i.e. $\widetilde{\mathcal{P}} = \widetilde{\mathcal{D}}$;
- Strong duality always holds for finite dimensional programs;
- Strong duality does not always hold in semi-infinite or infinite dimensional programs – *Duality gap*.

Weak and strong duality in LPs

- *Feasible solution*: if a decision variable satisfies constraints, it is feasible;
- *Weak duality*: if both the primal and the dual programs have feasible solutions then $\widetilde{\mathcal{P}} \leq \widetilde{\mathcal{D}}$;
- *Strong duality*: the primal program and its dual have the same value, i.e. $\widetilde{\mathcal{P}} = \widetilde{\mathcal{D}}$;
- Strong duality always holds for finite dimensional programs;
- Strong duality does not always hold in semi-infinite or infinite dimensional programs – *Duality gap*.

Weak and strong duality in LPs

- *Feasible solution*: if a decision variable satisfies constraints, it is feasible;
- *Weak duality*: if both the primal and the dual programs have feasible solutions then $\widetilde{\mathcal{P}} \leq \widetilde{\mathcal{D}}$;
- *Strong duality*: the primal program and its dual have the same value, i.e. $\widetilde{\mathcal{P}} = \widetilde{\mathcal{D}}$;
- Strong duality always holds for finite dimensional programs;
- Strong duality does not always hold in semi-infinite or infinite dimensional programs – *Duality gap*.

Weak and strong duality in LPs

- *Feasible solution*: if a decision variable satisfies constraints, it is feasible;
- *Weak duality*: if both the primal and the dual programs have feasible solutions then $\widetilde{\mathcal{P}} \leq \widetilde{\mathcal{D}}$;
- *Strong duality*: the primal program and its dual have the same value, i.e. $\widetilde{\mathcal{P}} = \widetilde{\mathcal{D}}$;
- Strong duality always holds for finite dimensional programs;
- Strong duality does not always hold in semi-infinite or infinite dimensional programs – *Duality gap*.

Interior point conditions for absence of duality gaps

Theorem (based on [And83], Theorem 8)

Suppose that the value of the primal program is finite. If b is in the interior of $\{Ax \in Z \mid x \geq 0\}$ and on the pre-image of some neighborhood of b in X the value function $\langle x, c \rangle_{XY}$ is bounded then there is no duality gap for (EP).

Brief Overview

[BHP13] Beiglböck et al. (2013) *Model-Independent Bounds for Option Prices: a Mass Transport Approach*;

[GHT14] Galichon et al. (2014) *Stochastic Control Approach to No-Arbitrage Bounds Given Marginals*;

[DS14] Dolinsky and Soner (2014) *Martingale Optimal Transport and Robust Hedging in Continuous Time*;

[Hen13] Henry-Labordère (2013), *Automated Option Pricing*;

[TT13] Tan and Touzi (2013), *Optimal Transportation under Controlled Stochastic Dynamics*;

Brief Overview

[BHP13] Beiglböck et al. (2013) *Model-Independent Bounds for Option Prices: a Mass Transport Approach*;

[GHT14] Galichon et al. (2014) *Stochastic Control Approach to No-Arbitrage Bounds Given Marginals*;

[DS14] Dolinsky and Soner (2014) *Martingale Optimal Transport and Robust Hedging in Continuous Time*;

[Hen13] Henry-Labordère (2013), *Automated Option Pricing*;

[TT13] Tan and Touzi (2013), *Optimal Transportation under Controlled Stochastic Dynamics*;

- Assumption is that marginals are uniquely determined and calibrated to the market;

Brief Overview

[BHP13] Beiglböck et al. (2013) *Model-Independent Bounds for Option Prices: a Mass Transport Approach*;

[GHT14] Galichon et al. (2014) *Stochastic Control Approach to No-Arbitrage Bounds Given Marginals*;

[DS14] Dolinsky and Soner (2014) *Martingale Optimal Transport and Robust Hedging in Continuous Time*;

[Hen13] Henry-Labordère (2013), *Automated Option Pricing*;

[TT13] Tan and Touzi (2013), *Optimal Transportation under Controlled Stochastic Dynamics*;

- Assumption is that marginals are uniquely determined and calibrated to the market;
- Relaxing the assumption of full marginals: Davis et al. (2013) *Arbitrage Bounds for Prices of Weighted Variance Swaps* [DOR13].

Problem set-up

- Assumptions on the market:
  - Market is frictionless;
  - No interest rates, no dividends;
  - Two time periods t_1 and t_2;
  - Allowed to trade dynamically in the underlying;
  - Buy and hold positions in other hedging instruments;
- European call options maturing at t_1 $\{k_{1,i}, p_{1,i}\}_{i=1}^{n_1}$ and t_2 $\{k_{2,i}, p_{2,i}\}_{i=1}^{n_2}$ with $n_1, n_2 < \infty$ satisfying no-arbitrage conditions [DH07, Theorem 4.2, p.9]

Problem set-up

- Assumptions on the market:
  - ▷ Market is frictionless;
  - ▷ No interest rates, no dividends;
  - ▷ Two time periods t_1 and t_2;
  - ▷ Allowed to trade dynamically in the underlying;
  - ▷ Buy and hold positions in other hedging instruments;
- European call options maturing at t_1 $\{k_{1,i}, p_{1,i}\}_{i=1}^{n_1}$ and t_2 $\{k_{2,i}, p_{2,i}\}_{i=1}^{n_2}$ with $n_1, n_2 < \infty$ satisfying no-arbitrage conditions [DH07, Theorem 4.2, p.9]

Problem set-up

- $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a Borel measurable function that denotes the pay-off of an exotic option at t_2;
- $\delta : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function in $C_b(\mathbb{R}_+)$ that denotes the delta hedge at time t_1;
- $c = (1, 1, p_{1,1}, \ldots, p_{1,n_1}, p_{2,1}, \ldots, p_{2,n_2})^T \in \mathbb{R}^m$ is a vector of today's prices with $m := 2 + n_1 + n_2$;
- $a(x_1, x_2) = (1, x_2, (x_1 - k_{1,1})_+, \ldots, (x_1 - k_{1,n_1})_+, \ldots, (x_2 - k_{2,n_2})_+)$ for all $(x_1, x_2) \in \mathbb{R}_+^2$ is a vector of pay-offs.

Problem set-up

- $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a Borel measurable function that denotes the pay-off of an exotic option at t_2;
- $\delta : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function in $C_b(\mathbb{R}_+)$ that denotes the delta hedge at time t_1;
- $c = (1, 1, p_{1,1}, \ldots, p_{1,n_1}, p_{2,1}, \ldots, p_{2,n_2})^T \in \mathbb{R}^m$ is a vector of today's prices with $m := 2 + n_1 + n_2$;
- $a(x_1, x_2) = (1, x_2, (x_1 - k_{1,1})_+, \ldots, (x_1 - k_{1,n_1})_+, \ldots, (x_2 - k_{2,n_2})_+)$ for all $(x_1, x_2) \in \mathbb{R}_+^2$ is a vector of pay-offs.

Problem set-up

- $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a Borel measurable function that denotes the pay-off of an exotic option at t_2;
- $\delta : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function in $C_b(\mathbb{R}_+)$ that denotes the delta hedge at time t_1;
- $c = (1, 1, p_{1,1}, \ldots, p_{1,n_1}, p_{2,1}, \ldots, p_{2,n_2})^T \in \mathbb{R}^m$ is a vector of today's prices with $m := 2 + n_1 + n_2$;
- $a(x_1, x_2) = (1, x_2, (x_1 - k_{1,1})_+, \ldots, (x_1 - k_{1,n_1})_+, \ldots, (x_2 - k_{2,n_2})_+)$ for all $(x_1, x_2) \in \mathbb{R}_+^2$ is a vector of pay-offs.

Problem set-up

- $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a Borel measurable function that denotes the pay-off of an exotic option at t_2;
- $\delta : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function in $C_b(\mathbb{R}_+)$ that denotes the delta hedge at time t_1;
- $c = (1, 1, p_{1,1}, \ldots, p_{1,n_1}, p_{2,1}, \ldots, p_{2,n_2})^T \in \mathbb{R}^m$ is a vector of today's prices with $m := 2 + n_1 + n_2$;
- $a(x_1, x_2) = (1, x_2, (x_1 - k_{1,1})_+, \ldots, (x_1 - k_{1,n_1})_+, \ldots, (x_2 - k_{2,n_2})_+)$ for all $(x_1, x_2) \in \mathbb{R}_+^2$ is a vector of pay-offs.

Primal Problem (sub-hedging)

$$\underline{\mathcal{P}} := \sup_{\pi \in \underline{\Pi}} \langle c, \pi \rangle ,$$

where

$$\underline{\Pi} := \left\{ \pi \in P_+^* \mid \exists \delta \in C_b(\mathbb{R}_+) \text{ s.t. } \Theta^{\pi,\delta}(x_1, x_2) \leq \Phi(x_1, x_2) \right\},$$

the inequality holds for $(x_1, x_2) \in \mathbb{R}_+^2$.

$$\Theta^{\pi,\delta}(x_1, x_2) := A(x_1, x_2)\pi + \delta(x_1)(x_2 - x_1).$$

The linear map A is defined by

$$A(x_1, x_2)\pi := a(x_1, x_2)^T \pi.$$

Primal Problem (super-hedging)

Equally can define a super-hedging problem

$$\overline{\mathcal{P}} := \inf_{\pi \in \overline{\Pi}} \langle c, \pi \rangle ,$$

where

$$\overline{\Pi} := \left\{ \pi \in P_+^* \mid \exists \delta \in C_b(\mathbb{R}_+) \text{ s.t. } \Theta^{\pi,\delta}(x_1, x_2) \geq \Phi(x_1, x_2) \right\},$$

the inequality holds for $(x_1, x_2) \in \mathbb{R}_+^2$.

Primal Problem

Decision variable π in the primal problem is in P_+^*

$$P_+^* = \{\pi \in \mathbb{R}^m \mid \langle x, \pi \rangle \geq 0, \text{ for all } x \in P_+\},$$

where P_+ is

$$P_+ = \{x \in \mathbb{R}^m \mid x = \lambda c, \text{ where } c \in P_+ \text{ for all } \lambda \in \mathbb{R}_+\}.$$

Lemma

The dual cone $P_+^* \in \mathbb{R}^m$ is closed in the usual topology on $\mathbb{R}^m$ if the prices $\{p_{1,i}\}_{i=1}^{n_1}$ and $\{p_{2,i}\}_{i=1}^{n_2}$ of European call options are consistent with absence of arbitrage.

Remark ([And83])

When the cone of primal decision variables P_+^* is closed the primal-dual program system becomes symmetric, i.e. the dual of the dual program is itself an LP and is equal to the primal problem.

Primal Problem

Decision variable π in the primal problem is in P_+^*

$$P_+^* = \{\pi \in \mathbb{R}^m \mid \langle x, \pi \rangle \geq 0, \text{ for all } x \in P_+\},$$

where P_+ is

$$P_+ = \{x \in \mathbb{R}^m \mid x = \lambda c, \text{ where } c \in P_+ \text{ for all } \lambda \in \mathbb{R}_+\}.$$

Lemma

The dual cone $P_+^* \in \mathbb{R}^m$ is closed in the usual topology on $\mathbb{R}^m$ if the prices $\{p_{1,i}\}_{i=1}^{n_1}$ and $\{p_{2,i}\}_{i=1}^{n_2}$ of European call options are consistent with absence of arbitrage.

Remark ([And83])

When the cone of primal decision variables P_+^* is closed the primal-dual program system becomes symmetric, i.e. the dual of the dual program is itself an LP and is equal to the primal problem.

Primal Problem

Decision variable π in the primal problem is in P_+^*

$$P_+^* = \{\pi \in \mathbb{R}^m \mid \langle x, \pi \rangle \geq 0, \text{ for all } x \in P_+\},$$

where P_+ is

$$P_+ = \{x \in \mathbb{R}^m \mid x = \lambda c, \text{ where } c \in P_+ \text{ for all } \lambda \in \mathbb{R}_+\}.$$

Lemma

The dual cone $P_+^* \in \mathbb{R}^m$ is closed in the usual topology on $\mathbb{R}^m$ if the prices $\{p_{1,i}\}_{i=1}^{n_1}$ and $\{p_{2,i}\}_{i=1}^{n_2}$ of European call options are consistent with absence of arbitrage.

Remark ([And83])

When the cone of primal decision variables P_+^* is closed the primal-dual program system becomes symmetric, i.e. the dual of the dual program is itself an LP and is equal to the primal problem.

Dual Problem

Sub-hedging dual problem:

$$\underline{\mathcal{D}} := \inf_{\mathbb{Q}\in\mathcal{M}} \int_{\mathbb{R}_+^2} \Phi(x_1, x_2)\, \mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2),$$

super-hedging dual problem:

$$\overline{\mathcal{D}} := \sup_{\mathbb{Q}\in\mathcal{M}} \int_{\mathbb{R}_+^2} \Phi(x_1, x_2)\, \mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2),$$

where

$$\mathcal{M} := \left\{ \mathbb{Q} \in Q_+ \mid A^*\mathbb{Q} = c,\ \int_{\mathbb{R}_+^2} \delta(x_1)(x_2 - x_1)\mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2) = 0 \right\},$$

and Q_+ denotes the set of all positive finite regular Borel measures on $\mathbb{R}_+^2$.

The adjoint map A^* is defined by

$$A^*\mathbb{Q} := \int_{\mathbb{R}_+^2} a(x_1, x_2)\mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2)$$

Martingale Condition

- A measure is a martingale measure if and only if it satisfies the condition

$$\int_{\mathbb{R}_+^2} \mathbf{1}_{\{x_1 \in A\}}(x_2 - x_1)\mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2) = 0,$$

for all Borel sets $A \in \mathcal{B}(\mathbb{R}_+)$.

- It can be extended to all functions $f \in C_b(\mathbb{R}_+)$

$$\int_{\mathbb{R}_+^2} f(x_1)(x_2 - x_1)\mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2) = 0.$$

Martingale Condition

- A measure is a martingale measure if and only if it satisfies the condition

$$\int_{\mathbb{R}_+^2} \mathbf{1}_{\{x_1 \in A\}}(x_2 - x_1)\mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2) = 0,$$

  for all Borel sets $A \in \mathcal{B}(\mathbb{R}_+)$.
- It can be extended to all functions $f \in C_b(\mathbb{R}_+)$

$$\int_{\mathbb{R}_+^2} f(x_1)(x_2 - x_1)\mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2) = 0.$$

Discretization

- Restrict the domain to $K = K_1 \times K_2 \subset \mathbb{R}^2_+$ s.t. K_1 and K_2 are compact;
- Let $\mathcal{S}_n$ be the set of all available delta hedge strategies such that $\mathcal{S}_n := \{f_1, \ldots, f_n\} \subset C(K_1)$ for $n \in \mathbb{N}$;
- $\delta \in \mathrm{Span}(\mathcal{S}_n)$ is a possible delta hedge strategy such that $\delta := \sum_{i=1}^n \lambda_i f_i$ for some $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, n$.

Discretization

- Restrict the domain to $K = K_1 \times K_2 \subset \mathbb{R}_+^2$ s.t. K_1 and K_2 are compact;
- Let $\mathcal{S}_n$ be the set of all available delta hedge strategies such that $\mathcal{S}_n := \{f_1, \ldots, f_n\} \subset C(K_1)$ for $n \in \mathbb{N}$;
- $\delta \in \mathrm{Span}(\mathcal{S}_n)$ is a possible delta hedge strategy such that $\delta := \sum_{i=1}^{n} \lambda_i f_i$ for some $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, n$.

Discretization

- Restrict the domain to $K = K_1 \times K_2 \subset \mathbb{R}_+^2$ s.t. K_1 and K_2 are compact;
- Let $\mathcal{S}_n$ be the set of all available delta hedge strategies such that $\mathcal{S}_n := \{f_1, \ldots, f_n\} \subset C(K_1)$ for $n \in \mathbb{N}$;
- $\delta \in \mathrm{Span}(\mathcal{S}_n)$ is a possible delta hedge strategy such that $\delta := \sum_{i=1}^n \lambda_i f_i$ for some $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, n$.

Discretized problems

Re-formulate the problem such that for each $n \in \mathbb{N}$ the primal programs read

$$\underline{\mathcal{P}}_n := \sup_{\pi \in \underline{\Pi}_n} \langle c, \pi \rangle , \tag{1}$$

where

$$\underline{\Pi}_n := \left\{ \pi \in P_+^* \mid \exists \delta \in \mathrm{Span}(\mathcal{S}_n) \text{ s.t. } \Theta^{\pi,\delta}(x_1, x_2) \leq \Phi(x_1, x_2) \right\} .$$

$$\overline{\mathcal{P}}_n := \inf_{\pi \in \overline{\Pi}_n} \langle c, \pi \rangle , \tag{2}$$

where

$$\overline{\Pi}_n := \left\{ \pi \in P_+^* \mid \exists \delta \in \mathrm{Span}(\mathcal{S}_n) \text{ s.t. } \Theta^{\pi,\delta}(x_1, x_2) \geq \Phi(x_1, x_2) \right\} .$$

Discretized problems

Re-formulate the problem such that for each $n \in \mathbb{N}$ the primal programs read

$$\underline{\mathcal{P}}_n := \sup_{\pi \in \underline{\Pi}_n} \langle c, \pi \rangle \,, \tag{1}$$

where

$$\underline{\Pi}_n := \left\{ \pi \in P_+^* \mid \exists \delta \in \mathrm{Span}(\mathcal{S}_n) \text{ s.t. } \Theta^{\pi,\delta}(x_1, x_2) \leq \Phi(x_1, x_2) \right\}.$$

$$\overline{\mathcal{P}}_n := \inf_{\pi \in \overline{\Pi}_n} \langle c, \pi \rangle \,, \tag{2}$$

where

$$\overline{\Pi}_n := \left\{ \pi \in P_+^* \mid \exists \delta \in \mathrm{Span}(\mathcal{S}_n) \text{ s.t. } \Theta^{\pi,\delta}(x_1, x_2) \geq \Phi(x_1, x_2) \right\}.$$

Discretized problems

The dual programs then read

$$\underline{\mathcal{D}}_n := \inf_{\mathbb{Q}\in\mathcal{M}_n} \int_K \Phi(x_1, x_2)\, \mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2), \tag{3}$$

$$\overline{\mathcal{D}}_n := \sup_{\mathbb{Q}\in\mathcal{M}_n} \int_K \Phi(x_1, x_2)\, \mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2), \tag{4}$$

where

$$\mathcal{M}_n := \left\{ \mathbb{Q} \in Q_+ \mid A^*\mathbb{Q} = c,\ \int_K f(x_1)(x_2 - x_1)\mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2) = 0 \right\},$$

for all $f \in \mathcal{S}_n$

Discretized problems

The dual programs then read

$$\underline{\mathcal{D}}_n := \inf_{\mathbb{Q}\in\mathcal{M}_n} \int_K \Phi(x_1, x_2)\, \mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2), \tag{3}$$

$$\overline{\mathcal{D}}_n := \sup_{\mathbb{Q}\in\mathcal{M}_n} \int_K \Phi(x_1, x_2)\, \mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2), \tag{4}$$

where

$$\mathcal{M}_n := \left\{ \mathbb{Q} \in Q_+ \mid A^*\mathbb{Q} = c,\ \int_K f(x_1)(x_2 - x_1)\mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2) = 0 \right\},$$

for all $f \in \mathcal{S}_n$

Strong duality - application of interior point condition

Theorem (Strong duality for sub-hedging problem)

Let the pay-off function $\Phi : K \to \mathbb{R}$ be a lower semi-continuous function and assume there exists a constant $C > 0$ such that

$$\Phi(x_1, x_2) \geq -C(1 + |x_1| + |x_2|), \quad \text{for all } x_1, x_2 \geq 0.$$

Assume that the value of dual program is finite and c lies in the interior of the set

$$V^n_{m+1} := \left\{ b \in \mathbb{R}^{m+1} \mid \int_K (a(x_1, x_2), f(x_1)(x_2 - x_1)) \, \mathbb{Q}(\mathrm{d}x_1, \mathrm{d}x_2) = b \right\},$$

where $\mathbb{Q} \in Q_+$ and for all $f \in \mathcal{S}_n$. Then the strong duality holds for each $n \in \mathbb{N}$, i.e. $\underline{\mathcal{P}}_n = \underline{\mathcal{D}}_n$. Moreover there exists an optimal portfolio π_*.

Strong duality - application of interior point condition

Corollary (Strong duality for super-hedging problem)

Let the pay-off function $\Phi : K \to \mathbb{R}$ be an upper semi-continuous function and assume there exists a constant $C > 0$ such that

$$\Phi(x_1, x_2) \leq C(1 + |x_1| + |x_2|), \quad \textit{for all } x_1, x_2 \geq 0.$$

Then the strong duality holds for each $n \in \mathbb{N}$, i.e. $\overline{\mathcal{P}}_n = \overline{\mathcal{D}}_n$. Moreover there exists an optimal portfolio π^*.

Limiting case

Lemma

As n tends to infinity, the following limits exist

$$\mathcal{M} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \mathcal{M}_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathcal{M}_k,$$

$$\underline{\Pi} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \underline{\Pi}_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \underline{\Pi}_k.$$

$$\overline{\Pi} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \overline{\Pi}_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \overline{\Pi}_k.$$

Attainment of optimal solutions

Lemma

Denoting the space of Borel probability measures on the set K as $P(K)$, $\mathcal{M}$ is closed subset of $P(K)$. Moreover it is compact.

Proposition

If the value of the dual problem $\underline{\mathcal{D}}$ ($\overline{\mathcal{D}}$ resp.) is finite, then the optimal value is attained and there exists an optimal measure $\mathbb{Q}_*$ ($\mathbb{Q}^*$ resp.).

Attainment of optimal solutions

Lemma

Denoting the space of Borel probability measures on the set K as $P(K)$, $\mathcal{M}$ is closed subset of $P(K)$. Moreover it is compact.

Proposition

If the value of the dual problem $\underline{\mathcal{D}}$ ($\overline{\mathcal{D}}$ resp.) is finite, then the optimal value is attained and there exists an optimal measure $\mathbb{Q}_*$ ($\mathbb{Q}^*$ resp.).

Further work

- Relaxing the assumption on compactness (i.e. setting $K = \mathbb{R}^2_+$);
- Extending the framework to multiple time periods and then to continuous time;
- Uniqueness of optimal solutions;
- Describing optimal portfolio weights explicitly.

References I

[And83] E. J. Anderson, *A Review of Duality Theory for Linear Programming over Topological Vector Spaces*, Journal of mathematical analysis and applications **97** (1983), 380–392.

[BHP13] M. Beiglböck, P. Henry-Labordère, and F. Penkner, *Model-independent bounds for option prices: a mass transport approach*, Finance and Stochastics **17** (2013), no. 3, 477–501.

[DH07] M. H.A. Davis and D. Hobson, *The range of traded option prices*, Mathematical Finance **17** (2007), 1–14.

[DOR13] M. H.A. Davis, J. Obłój, and V. Raval, *Arbitrage bounds for prices of weighted variance swaps*, Mathematical Finance (2013), published online.

References II

[DS14] Y. Dolinsky and H.M. Soner, *Martingale optimal transport and robust hedging in continuous time*, Probability Theory and Related Fields **160** (2014), no. 1-2, 391–427 (English).

[GHT14] A. Galichon, P. Henry-Labordère, and N. Touzi, *A Stochastic Control Approach to No-Arbitrage Bounds Given Marginals, with an Application to Lookback Options*, Annals of Applied Probability **24** (2014), no. 1, 312–336.

[Hen13] P. Henry-Labordère, *Automated option pricing: Numerical methods*, International Journal of Theoretical and Applied Finance **16** (2013), 135–162.

[TT13] K. Tanaka and A.A. Toda, *Discrete approximations of continuous distributions by maximum entropy*, Economic Letters **118** (2013), 445–450.

THANK YOU