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Analysis of sensitivity and robustness of forced response for nonlinear dynamic structures

E.P. Petrov *

Mechanical Engineering Department, South Kensington Campus, London SW7 2AZ, UK

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ABSTRACT

An effective method is developed for calculation of the sensitivity and robustness of the forced response levels for strongly nonlinear structures. The sensitivity coefficients are determined with respect to parameters of friction contact interfaces, parameters of linear components of an assembled structure, frequency and level of excitation forces.

Equations for determination of first- and second-order sensitivity coefficients of the forced response are derived analytically from a nonlinear multiharmonic equation of motion. The analytical derivation allows accurate and fast evaluation of the sensitivity coefficients. The sensitivity coefficients are calculated in frequency domain for each excitation frequency over the frequency range analysed simultaneously with the force response levels. The developed highly efficient method allows calculation of sensitivity characteristics without a noticeable increase of the computation time in addition to the time required for the forced response calculation.

A measure of the forced response robustness is introduced. Sensitivity-based method for assessment of the forced response robustness for given ranges of uncertainty of structural and operating parameters is proposed.

The methodology developed is illustrated on a set of problems including cases of forced response analysis for realistic strongly nonlinear gas-turbine structures.

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1. Introduction

1.1. Sensitivity analysis for forced response

In order to make a justified choice of design parameters for jointed structures, there is a need to determine not only the levels of displacement and stresses which are caused by excitation forces but also how sensitive those forced response levels are to variations of the design parameters. Information about the sensitivity of the forced response facilitates choosing those parameters that can be changed in order to provide the required forced response level.

Furthermore, in many practical applications, there is often a need to find an optimum set of the design parameter values for a structure which gives a minimum of the forced response or satisfies other criteria formulated by the designer. The optimisation problems can be efficiently solved when sensitivity characteristics are determined and, hence, sensitivity analysis is an important and in many cases unavoidable part of the optimisation process. Due to importance of determination of sensitivity coefficients, efforts of many investigators were devoted to this problem which has taken

* Tel.: +44 20 7594 1920.
E-mail address: y.petrov@imperial.ac.uk

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on a special significance and was formed in an independent research area. Among reviews and a vast number of applications of the sensitivity characteristics for solution of various practical problems papers [1–6] can be noticed.

Moreover, the sensitivity coefficients can be used to construct accurate approximations for design merit function, which express the merit function explicitly as a function of design parameters and hence extensive multivariant calculations can be significantly reduced or avoided altogether (e.g. Ref. [7]).

Majority of machinery structures are assembled structures, which consist of two or more components assembled together and interacting at friction contact interfaces. For many practical problems, steady-state regimes of vibrations are of major interest and such regimes result usually in periodic vibrations. Analysis of periodic forced response is usually performed in frequency domain, which allows avoiding integration of the time-domain equations of motion and reduce the problem to solution of algebraic equations with respect to harmonic coefficients of the Fourier expansion of the periodic forced response. Some examples of analysis of forced response for gas-turbine engine structures, in particular of bladed discs with friction contact interfaces can be found in papers [8–17].

Accurate and fast analysis of periodic nonlinear forced response of structures with friction and gap contact interfaces is a challenging problem, especially when industrial problems have to be studied. These problems require using realistic large-scale finite element models including usually hundreds of thousand or millions degrees of freedom (DOFs) to capture accurately dynamic properties of components of a jointed structure. Moreover, accurate modelling of the contact stresses occurring under vibrations and accounting macro- and micro-slip at contact surfaces, slip–stick and contact–separation transitions, effects of variation of contact interfaces parameters and surface roughness, etc. An effective approach for calculation of multiharmonic periodic forced response for strongly nonlinear dynamic structures has been recently developed in Refs. [18–20]. This approach provides any accuracy desired for the periodic forced response calculation owing to fully analytical derivations for all nonlinear contact elements and the capability to include any number of harmonics in the multiharmonic expansion.

There are restricted number of investigations of sensitivity coefficients for structures with friction, for example in papers [21,22]. These investigations were performed mostly in time domain and with simplified friction models. Such calculations required very large computation time and therefore use of distributed-memory parallel computers was essential. In paper [23], analytical expressions for sensitivity coefficients of the multiharmonic friction forces to variation of such contact parameters as friction coefficient, gap values, normal load, etc. have been derived. These expressions allow exact and fast determination of the sensitivity characteristics for the contact interfaces, which provides a basis for the sensitivity analysis of the forced response. A pioneer method aimed at direct calculation of sensitivity of the forced response with respect to contact interface parameters has been proposed in Ref. [7].

In this paper, a method proposed in Ref. [7] is developed further adding formulations allowing calculation of the sensitivity of the forced response with respect to variation of excitation forces, excitation frequency and parameters of linear components of an assembled structure comprising essentially nonlinear contact interfaces. In contrast to papers [21] and [22], this method is aimed at analysis of steady-state forced response, and its efficiency allows evaluation of the sensitivity characteristics without a noticeable increase of the computation in addition to time required for forced response analysis.

1.2. Robustness of the forced response

Choice of a design concept and design parameters of a machinery structure is usually made to achieve the highest performance, reliability, lowest weight, and other criteria. However, in practice, there is significant and, in some cases, high level of uncertainty in operating conditions. Moreover, values of the design parameters are inevitably uncertain due to manufacture imperfections and tolerances, they vary during life-time cycle of the structure, and, moreover, different operating regimes and service history introduce additional scatter of design parameters for different specimens of nominally the same structure. Therefore, practical structures operate in conditions which can differ from the conditions for which their design concept and values of design parameters were selected and optimised.

Because of these inevitable uncertainties, for practical structures, a very important requirement is an ability of the design to be 'robust', i.e. to function correctly and reliably not only for some nominal operating conditions but also in the conditions that differ from normal, sometimes in the presence of invalid inputs resulting from inevitable design parameter scatters and variations of operating conditions. Robust design generally means that a structure subjected to realistic scatters of parameter values is capable of functioning without catastrophic failures under a large variety of operating conditions and for a sufficiently long time. In most cases, the robustness is a necessary quality for a structure, which has to be satisfied in first place, sometimes even, by compromising many other criteria.

In the past, the robustness requirement was tried to be satisfied by simple introducing some aggregate estimate for robustness, such as, for example, a safety factor, but modern design technology requires determination of effects of each of design parameter on the robustness of a structure, combined effects of variations of multitude of parameters, and optimisation of structure parameters with respect to robustness, may be together with the optimisation of other very important characteristics such as performance, weight, etc.

The robustness is closely associated with the sensitivity of state parameters of a structure, e.g. forced response levels in the case considered in this paper. Generally, recognised definitions of the robustness make direct correspondence between the sensitivity and the robustness. Such definitions include the definition, provided in Ref. [25]: "Robustness is the degree to which a system is insensitive to effects that are not considered in the design" and descriptions provided in Ref. [26]: "Robustness signifies insensitivity against small deviations in the assumptions … Robust statistical procedures are designed to reduce the sensitivity of the parameter estimates to failures in the assumption of the model".

Owing to the close association between the robustness and the sensitivity, the sensitivity coefficients of the forced response can be used as some measure of the robustness. Comparison of sensitivity coefficients with respect to different parameters helps to select parameters which are responsible for reduction of the robustness. They can also allow determination of operating conditions, e.g. excitation frequency ranges where robustness can be high and low.

However, the robustness has a broader meaning and its measure is not exactly equal to the sensitivity coefficients. For a qualified decision about design robustness, it is necessary to explore behaviour of a structure for all possible values of design and excitation parameters within uncertainty ranges, while the sensitivity coefficients give quantitative characteristics for the rate of variation of the forced response with variation of design parameters, strictly speaking, only in infinitesimal vicinity of the design parameter values for which the sensitivity coefficient values are determined.

1.3. Objectives of this paper

In this paper, an effective method is described for calculation of the sensitivity of the forced response levels for strongly nonlinear structures to variation of parameters of friction contact interfaces, parameters of linear components of an assembled structure, the period of excitation forces and their levels. The sensitivity characteristics are calculated in frequency domain for each excitation frequency over the frequency range analysed simultaneously with the force response levels. Equations for determination of first- and second-order sensitivity coefficients of the forced response are derived analytically from a nonlinear multiharmonic equation of motion. The analytical derivation allows accurate and fast evaluation of the sensitivity coefficients. There is no a significant increase of the computation time in addition to the time needed for the forced response calculation.

The sensitivity coefficients are used for constructing explicit expressions for forced response levels as a function of parameters of contact interface parameters, parameters of linear components of an assembled structure and parameters of excitation forces.

An approach for determination of uncertainty in the forced response levels caused by uncertainty of design and operating condition parameters is proposed.

A measure of the forced response robustness is introduced. Sensitivity-based method for assessment of the forced response robustness for given ranges of uncertainty of structural and operating parameters is proposed.

The methodology developed is illustrated on a set of problems including cases of forced response analysis for realistic gas-turbine structures.

2. A method for sensitivity analysis of nonlinear forced response

2.1. Multiharmonic equation of motion

A structure containing nonlinear contact interfaces between its different components and/or interfaces at boundaries of the structure is subjected to action of internal, linear and nonlinear forces and external, excitation forces. The linear forces usually comprise inertia forces, elastic deformation forces and viscous damping or material damping forces. Nonlinear forces considered here are forces of nonlinear interaction at the contact interfaces. The linear forces are expressed in the finite element method linearly through displacements using corresponding matrices (e.g. mass, stiffness or damping matrices). These matrices are constant and are independent of the displacements, for a linear system. Nonlinear forces in general cannot be represented in this form and are usually described by nonlinear equations depending on the displacements and on design parameters of the contact interfaces. The equation for motion for a structure with nonlinear interfaces can be written in the following form:

$$\boldsymbol{K}\boldsymbol{q}(t)+\boldsymbol{C}\dot{\boldsymbol{q}}(t)+\boldsymbol{M}\ddot{\boldsymbol{q}}(t)+\boldsymbol{f}(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),\boldsymbol{b})-p(t)=0 \tag{1}$$

where $\boldsymbol{q}(t)$ is a vector of displacements for all, N_0, degrees of freedom (DOFs) in the structure considered; $\boldsymbol{K}$, $\boldsymbol{C}$ and $\boldsymbol{M}$ are stiffness, viscous damping and mass matrices used for description of linear forces. For a structure analysed in a non-inertial coordinate system rotating with speed Ω, which are used, for example, for bladed discs and other components of gas-turbine engines mounted on a rotating rotor, the stiffness matrix can comprise not only conventional elastic stiffness matrix $\boldsymbol{K}_e$, but can also include terms accounting for the rotations effects, i.e. $\boldsymbol{K}=\boldsymbol{K}_e+\boldsymbol{K}_g(\Omega)-\Omega^2\boldsymbol{M}_\Omega$. Here $\boldsymbol{K}_g$ is a so-called geometric stiffness matrix reflecting stiffening effects of the centrifugal forces; $\boldsymbol{M}_\Omega$ is a spin-softening-matrix describing stiffness softening due to the changing direction of the centrifugal forces under vibration. $\boldsymbol{f}(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),\boldsymbol{b})$ is a vector of nonlinear interface forces, which is dependent on displacements, $\boldsymbol{q}(t)$, and velocities, $\dot{\boldsymbol{q}}(t)$, of DOFs at contact interface nodes. $\boldsymbol{p}(t)$ is a vector of periodic excitation forces with a period, T_P, and $\boldsymbol{b}$ is vector of design parameters for which the forced response sensitivities have to be determined. These parameters can be parameters of friction contact interfaces (such as, clearance or interference values, friction coefficient, contact stiffness, etc.), parameters of excitation force levels, the excitation frequency, parameters of linear components of the structure, etc.

For a search of the periodic vibration response, the displacements' variation in time is represented as a restricted Fourier series, which contains whichever harmonic components are necessary to approximate the sought solution, i.e.

$$\boldsymbol{q}(t) = \boldsymbol{Q}_0 + \sum_{j=1}^{n} \left(\boldsymbol{Q}_j^{(c)} \cos l_j\tau + \boldsymbol{Q}_j^{(s)} \sin l_j\tau \right) \tag{2}$$

where $\tau = \omega t$ is the dimensionless time, ω is the fundamental frequency of the solution, $\boldsymbol{Q}_j^{(c)}$ and $\boldsymbol{Q}_j^{(s)} (j = 1, \ldots, n)$ are vectors of cosine and sine harmonic coefficients for system DOFs, marked by superscripts (c) and (s) accordingly; $\boldsymbol{Q}_0$ is a vector of constant components of the displacements. l_j ($j = 1, \ldots, n$) are specific numbers of harmonics that are kept in the displacement expansion in addition to the constant component, these numbers can be selected to keep in the harmonic expansion only those harmonics which make significant contribution to the forced response.

The fundamental frequency is defined by a period, T, of the sought vibration response: $\omega = 2\pi/T$. When major and superharmonic resonance regimes are subject of interest, the vibration response period is equal to the period of excitation forces: $T = T_P$. For this case, owing to higher harmonics included in the multiharmonic representation of the forced response, the superharmonic resonances are determined along with major resonances. When subharmonic and/or combination resonance regimes are of interest, the forced response period is a multiple of period of excitation forces: $T = cT_P$, where c can be an integer number or a ratio of two integers. For this case, an appropriate choice of harmonic numbers, l_j, allows also determination of major (e.g. when there is a harmonic for which $l_j/c = 1$) and superharmonic resonances (e.g. when there are harmonics for which l_j/c are integers and larger 1).

The multiharmonic expansions for displacements, $\boldsymbol{q}(t)$, (see Eq. (2)) and a similar expansion for the nonlinear forces, $\boldsymbol{f}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), \boldsymbol{b})$, are substituted into the equation of motion, Eq. (1), and equating coefficients upon same harmonics gives a nonlinear multiharmonic equation of motion which contains all harmonic coefficients for all DOFs included in the model. As a result, nonlinear algebraic equation can be formulated with respect to a vector of harmonic coefficients for the displacements, $\boldsymbol{Q} = \left\{ \boldsymbol{Q}_0, \boldsymbol{Q}_1^{(c)}, \boldsymbol{Q}_1^{(s)}, \ldots, \boldsymbol{Q}_n^{(s)} \right\}^T$, in the following form:

$$\boldsymbol{R}(\boldsymbol{Q}, \omega, \boldsymbol{b}) = \boldsymbol{Z}(\omega, \boldsymbol{b})\boldsymbol{Q} + \boldsymbol{F}(\boldsymbol{Q}, \boldsymbol{b}) - \boldsymbol{P}(\boldsymbol{b}) = 0 \tag{3}$$

where $\boldsymbol{F}(\boldsymbol{Q}, \boldsymbol{b}) = \left\{ \boldsymbol{F}_0, \boldsymbol{F}_1^{(c)}, \boldsymbol{F}_1^{(s)}, \ldots, F_n^{(s)} \right\}^T$ is a vector of nonlinear contact forces, which is dependent on the harmonic coefficients of displacements, $\boldsymbol{Q}$, $\boldsymbol{P}(\boldsymbol{b}) = \left\{ \boldsymbol{P}_0, \boldsymbol{P}_1^{(c)}, \boldsymbol{P}_1^{(s)}, \ldots, P_n^{(s)} \right\}^T$ is the vector of harmonic coefficients of the excitation forces, $\boldsymbol{Z}(\omega, \boldsymbol{b})$ is a multiharmonic dynamic stiffness matrix of the linear components of the assembled structure. The multiharmonic dynamic stiffness matrix, $\boldsymbol{Z}(\omega, \boldsymbol{b})$, takes the form:

$$\underset{(N(2n+1)\times N(2n+1))}{\boldsymbol{Z}}(\omega, \boldsymbol{b}) = \mathrm{diag}[\boldsymbol{K}, \boldsymbol{Z}_1(\omega), \ldots, \boldsymbol{Z}_n(\omega)] \tag{4}$$

where

$$\underset{(2N\times 2N)}{\boldsymbol{Z}_j} = \begin{bmatrix} \boldsymbol{K}(\boldsymbol{b}) & 0 \\ 0 & \boldsymbol{K}(\boldsymbol{b}) \end{bmatrix} + (l_j\omega) \begin{bmatrix} 0 & \boldsymbol{C}(\boldsymbol{b}) \\ -\boldsymbol{C}(\boldsymbol{b}) & 0 \end{bmatrix} - (l_j\omega)^2 \begin{bmatrix} \boldsymbol{M}(\boldsymbol{b}) & 0 \\ 0 & \boldsymbol{M}(\boldsymbol{b}) \end{bmatrix}$$

2.2. Calculation of the nonlinear multiharmonic forced response

Eq. (3) represents a nonlinear set of equations with respect to vector of multiharmonic amplitudes, $\boldsymbol{Q}$. The Newton–Raphson method together with schemes of solution continuation/tracing is applied to solve this equation. An iterative Newton–Raphson solution process is expressed by the following formula:

$$\boldsymbol{Q}^{(k+1)} = \boldsymbol{Q}^{(k)} - \boldsymbol{J}^{-1}(\boldsymbol{Q}^{(k)})\boldsymbol{R}(\boldsymbol{Q}^{(k)}) \tag{5}$$

where superscript (k) indicates the iteration number and the Jacobian, $\boldsymbol{J}$, of Eq. (3) is determined as:

$$\boldsymbol{J}(\boldsymbol{Q}) = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{Q}} = \boldsymbol{Z}(\omega, \boldsymbol{b}) + \boldsymbol{K}_t \tag{6}$$

where $\boldsymbol{K}_t = \partial \boldsymbol{F}/\partial \boldsymbol{Q}$ is a tangent stiffness matrix of the nonlinear contact interfaces. The iterative solution is terminated when a required accuracy is achieved, i.e. $\|\boldsymbol{R}(\boldsymbol{Q})\| < \varepsilon$.

Solution of Eq. (3) is performed using the Newton–Raphson method and a set of continuation methods, as described in Ref. [23], is applied for the solution tracing.

2.3. Sensitivity of multiharmonic components of nonlinear forced response to design parameter variation

Sensitivities of the harmonic coefficients of the multiharmonic forced response are calculated for each found solution, $\boldsymbol{Q}$, with respect to a vector of design and excitation parameters, $\boldsymbol{b}$. The finite-difference calculation of the sensitivity coefficients, which is customarily used, cannot provide sufficient accuracy of the calculations, especially for the strongly nonlinear structures of non-smooth dynamics, which machinery structures with frictions, gaps and impacts are. Moreover, the finite-difference calculation usually requires large computational expense, especially for large-scale finite element models of practical structures, which can make sensitivity analysis unfeasible. In order to ensure high accuracy and speed of sensitivity calculation, all expressions are derived analytically in this paper including sensitivity of the nonlinear friction contact interface forces with respect to interface parameters, e.g. clearance and interference values, friction coefficients, contact stiffness, etc. In the paper, first- and second-order sensitivity sensitivities are determined. They provide information about rates of forced response variation when parameters of a structure are varied and as will be shown further, allow constructing approximations for the forced response explicitly as functions of design parameters.

2.3.1. General scheme for calculation of forced response sensitivity

In order to derive expressions for determination of the sensitivity coefficients for the nonlinear forced response with respect to parameters of the nonlinear contact interfaces, we can differentiate Eq. (3) with respect to jth parameter of interest, b_j:

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}b_j}=\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{Q}}\frac{\partial \boldsymbol{Q}}{\partial b_j}+\frac{\partial \boldsymbol{R}}{\partial b_j}=J\frac{\partial \boldsymbol{Q}}{\partial b_j}+\frac{\partial \boldsymbol{R}}{\partial b_j}=0 \tag{7}$$

and then solving Eq. (7) with respect to a vector of derivatives of the harmonic coefficients, $\partial\boldsymbol{Q}/\partial b_j$, we obtain:

$$\boldsymbol{J}\frac{\partial \boldsymbol{Q}}{\partial b_j}=-\frac{\partial \boldsymbol{R}}{\partial b_j}=-\boldsymbol{A}_1 \tag{8}$$

The Jacobian, $\boldsymbol{J}$, is obtained as a by-product of the Newton–Raphson solution process and moreover LR-factorisation of the Jacobian is made for the Newton–Raphson iterations. Owing to this calculation of the sensitivity, coefficients does not incur a significant computation cost since this calculation requires a solution of only one additional linear algebraic equation with already factorised matrix.

In order to derive an expression for second-order sensitivity coefficients of the forced response, we again differentiate Eq. (8) with respect to b_k, which results in the following equation with respect to a vector of second-order sensitivity, $\partial^2\boldsymbol{Q}/\partial b_k b_j$:

$$\boldsymbol{J}\frac{\partial^2 \boldsymbol{Q}}{\partial b_k\partial b_j}=-\frac{\partial^2 \boldsymbol{R}}{\partial b_k\partial b_j}-\frac{\partial \boldsymbol{J}}{\partial b_k}\frac{\partial \boldsymbol{Q}}{\partial b_j}-\frac{\partial \boldsymbol{J}}{\partial b_j}\frac{\partial \boldsymbol{Q}}{\partial b_k}-\frac{\partial}{\partial \boldsymbol{Q}}\left(\boldsymbol{J}\frac{\partial \boldsymbol{Q}}{\partial b_j}\right)\frac{\partial \boldsymbol{Q}}{\partial b_k}=-\boldsymbol{A}_2 \tag{9}$$

where $\partial\boldsymbol{Q}/\partial b_j$ and $\partial\boldsymbol{Q}/\partial b_k$ are determined from solution of Eq. (8). One can see that calculation of second-order sensitivities also requires solution of the linear algebraic equation with already factorised matrix and one additional right-hand vector.

2.3.2. Sensitivity with respect to parameters of contact interfaces

For this case, sensitivity characteristics are determined with respect to parameters of nonlinear interaction forces occurring at contact interfaces, such as friction coefficient, interference and clearance values, stiffness coefficients, etc. For this case, vector of nonlinear interaction forces, $\boldsymbol{F}(\boldsymbol{Q},\boldsymbol{b})$, and is dependent on such parameters in Eq. (3), while matrix, $\boldsymbol{Z}$, and vector, $\boldsymbol{P}$, are not and, therefore, expressions required for calculation of right-hand terms in Eqs. (8) and (9) for calculation of first- and second-order sensitivity coefficients take the form:

$$\boldsymbol{A}_1=\frac{\partial \boldsymbol{F}}{\partial b_j}\quad\text{and}\quad \boldsymbol{A}_2=\frac{\partial^2 \boldsymbol{F}}{\partial b_k\partial b_j}+\left[\frac{\partial}{\partial b_k}\left(\boldsymbol{K}_t\frac{\partial \boldsymbol{Q}}{\partial b_j}\right)+\frac{\partial}{\partial b_j}\left(\boldsymbol{K}_t\frac{\partial \boldsymbol{Q}}{\partial b_k}\right)+\frac{\partial}{\partial \boldsymbol{Q}}\left(\boldsymbol{K}_t\frac{\partial \boldsymbol{Q}}{\partial b_j}\right)\frac{\partial \boldsymbol{Q}}{\partial b_k}\right] \tag{10}$$

2.3.3. Sensitivity with respect to excitation forces

For this case, sensitivity characteristics are determined with respect to variation of excitation forces. A vector of excitation forces, $\boldsymbol{P}(\boldsymbol{b})$, in Eq. (3) is dependent on parameters b_j and, therefore, expressions required for calculation of first- and second-order sensitivity coefficients take the form:

$$\boldsymbol{A}_1=-\frac{\partial \boldsymbol{P}}{\partial b_j}\quad\text{and}\quad \boldsymbol{A}_2=\frac{\partial^2 \boldsymbol{P}}{\partial b_k\partial b_j}+\frac{\partial}{\partial \boldsymbol{Q}}\left(\boldsymbol{K}_t\frac{\partial \boldsymbol{Q}}{\partial b_j}\right)\frac{\partial \boldsymbol{Q}}{\partial b_k} \tag{11}$$

Eq. (11) can be used for the most general case, when all components of the multiharmonic excitation vector, $\boldsymbol{P}(\boldsymbol{b})$, are varied and every component of this vector is described individually by a known function of parameters. First and second derivatives of $\boldsymbol{P}(\boldsymbol{b})$ then can be calculated analytically or numerically with high accuracy.

In practice, very often, the forced response sensitivity has to be determined with respect to a value of amplitude of some selected harmonic applied to a selected DOF. For this particular case for the derivatives of the excitation vector take the following form:

$$\frac{\partial \boldsymbol{P}}{\partial b_j} = \boldsymbol{E}_r \quad \text{and} \quad \frac{\partial^2 \boldsymbol{P}}{\partial b_j \partial b_k} = 0 \tag{12}$$

where r is the position of the excitation harmonic of interest in the excitation vector, $\boldsymbol{P}$, and $\boldsymbol{E}_r = \{\underbrace{0, \ldots, 1}_{r}, \ldots 0\}^T$ is a vector with rth component equal to 1 and all the others are 0.

2.3.4. Sensitivity with respect to excitation frequency

For this case, sensitivity characteristics are determined with respect to excitation frequency, i.e. $b_j = \omega$. For this case, a vector of harmonic coefficients of excitation forces, $\boldsymbol{P}(\boldsymbol{b})$, is independent on parameter b_j in Eq. (3) and expressions required for calculation of first- and second-order sensitivity coefficients take the form:

$$A_1 = \frac{\partial \boldsymbol{Z}}{\partial \omega}\boldsymbol{Q} + \frac{\partial \boldsymbol{F}}{\partial \omega} \quad \text{and} \quad A_2 = \frac{\partial^2 \boldsymbol{Z}}{\partial \omega^2} + \left[2\left(\frac{\partial \boldsymbol{Z}}{\partial \omega} + \frac{\partial \boldsymbol{K}_t}{\partial \omega}\right) + \frac{\partial}{\partial \boldsymbol{Q}}\left(\boldsymbol{K}_t \frac{\partial \boldsymbol{Q}}{\partial \omega}\right)\right]\frac{\partial \boldsymbol{Q}}{\partial \omega} \tag{13}$$

The derivatives of the multiharmonic dynamic stiffness matrix with respect to the principal vibration frequency, ω, is determined analytically by differentiating Eq. (4):

$$\frac{\partial \boldsymbol{Z}}{\partial \omega} = \text{diag}\left[0, \frac{\partial \boldsymbol{Z}_1}{\partial \omega}, \ldots, \frac{\partial \boldsymbol{Z}_n}{\partial \omega}\right] \quad \text{and} \quad \frac{\partial^2 \boldsymbol{Z}}{\partial \omega^2} = \text{diag}\left[0, \frac{\partial^2 \boldsymbol{Z}_1}{\partial \omega^2}, \ldots, \frac{\partial^2 \boldsymbol{Z}_n}{\partial \omega^2}\right] \tag{14}$$

where

$$\frac{\partial \boldsymbol{Z}_j}{\partial \omega} = -2l_j^2\omega \begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & \boldsymbol{M} \end{bmatrix} + l_j \begin{bmatrix} 0 & \boldsymbol{C} \\ -\boldsymbol{C} & 0 \end{bmatrix} \quad \text{and} \quad \frac{\partial^2 \boldsymbol{Z}_j}{\partial \omega^2} = -2l_j^2 \begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & \boldsymbol{M} \end{bmatrix} \tag{15}$$

For a majority of practical contact interfaces, such as friction contact interfaces, clearances and interferences, unilateral contact, cubic nonlinearity and some others the nonlinear contact interface forces are not explicitly dependent on the vibration frequency, ω, and then Eq. (13) can be written in a simpler form:

$$A_1 = \frac{\partial \boldsymbol{Z}}{\partial \omega}\boldsymbol{Q} \quad \text{and} \quad A_2 = \frac{\partial^2 \boldsymbol{Z}}{\partial \omega^2} + \left[2\frac{\partial \boldsymbol{Z}}{\partial \omega} + \frac{\partial}{\partial \boldsymbol{Q}}\left(\boldsymbol{K}_t \frac{\partial \boldsymbol{Q}}{\partial \omega}\right)\right]\frac{\partial \boldsymbol{Q}}{\partial \omega} \tag{16}$$

2.3.5. Sensitivity with respect to modifications of linear components

For this case, the mass, stiffness and linear damping matrices are dependent on the vector of design parameters, $\boldsymbol{b}$, i.e. $\boldsymbol{M}(\boldsymbol{b})$, $\boldsymbol{K}(\boldsymbol{b})$, and $\boldsymbol{C}(\boldsymbol{b})$ and therefore:

$$A_1 = \frac{\partial \boldsymbol{Z}}{\partial b_j}\boldsymbol{Q} \quad \text{and} \quad A_2 = \frac{\partial^2 \boldsymbol{Z}}{\partial b_k \partial b_j} + \left[\frac{\partial \boldsymbol{Z}}{\partial b_k}\frac{\partial \boldsymbol{Q}}{\partial b_j} + \frac{\partial \boldsymbol{Z}}{\partial b_j}\frac{\partial \boldsymbol{Q}}{\partial b_k} + \frac{\partial}{\partial \boldsymbol{Q}}\left(\boldsymbol{K}_t \frac{\partial \boldsymbol{Q}}{\partial b_k}\right)\frac{\partial \boldsymbol{Q}}{\partial b_k}\right] \tag{17}$$

where first derivative of the multiharmonic dynamic stiffness matrix are calculated in the form:

$$\begin{aligned} \frac{\partial \boldsymbol{Z}}{\partial b_j} &= \text{diag}\left[\frac{\partial \boldsymbol{K}}{\partial b_j}, \frac{\partial \boldsymbol{Z}_1}{\partial b_j}, \ldots, \frac{\partial \boldsymbol{Z}_n}{\partial b_j}\right], \\ \frac{\partial \boldsymbol{Z}_j}{\partial b_j} &= \begin{bmatrix} \partial \boldsymbol{K}/\partial b_j - (l_j\omega)^2(\partial \boldsymbol{M}/\partial b_j) & (l_j\omega)(\partial \boldsymbol{C}/\partial b_j) \\ -(l_j\omega)(\partial \boldsymbol{C}/\partial b_j) & \partial \boldsymbol{K}/\partial b_j - (l_j\omega)^2(\partial \boldsymbol{M}/\partial b_j) \end{bmatrix} \end{aligned} \tag{18}$$

and second derivatives, $\partial^2 \boldsymbol{Z}/\partial b_k b_j$, are derived analogously. Eq. (18) can be used for the most general case, when every component of the dynamic stiffness matrix, $\boldsymbol{Z}_j$, can be described individually by a known function of the vector of parameters, $\boldsymbol{b}$. Owing to this, first and second derivatives of $\boldsymbol{Z}_j$ can be calculated analytically.

For an important particular case, when the forced response sensitivity has to be determined with respect to a value of an individual component of a stiffness, k_{lr}, mass, m_{lr}, or damping, c_{lr}, matrix (r and l are row and column of the matrix component, respectively) these derivatives can be obtained explicitly in the following form:

$$\frac{\partial \boldsymbol{Z}_j}{\partial k_{rl}} = \frac{\partial \boldsymbol{K}}{\partial k_{rl}} = \begin{bmatrix} \boldsymbol{E}_r\boldsymbol{E}_l^T & 0 \\ 0 & \boldsymbol{E}_r\boldsymbol{E}_l^T \end{bmatrix}, \quad \frac{\partial \boldsymbol{Z}_j}{\partial m_{rl}} = -(l_j\omega)^2 \begin{bmatrix} \boldsymbol{E}_r\boldsymbol{E}_l^T & 0 \\ 0 & \boldsymbol{E}_r\boldsymbol{E}_l^T \end{bmatrix}, \quad \frac{\partial \boldsymbol{Z}_j}{\partial c_{rl}} = l_j\omega \begin{bmatrix} 0 & \boldsymbol{E}_r\boldsymbol{E}_l^T \\ -\boldsymbol{E}_r\boldsymbol{E}_l^T & 0 \end{bmatrix} \tag{19}$$

and second derivatives are equal to zero:

$$\frac{\partial^2 \boldsymbol{Z}_j}{\partial^2 k_{rl}} = \frac{\partial^2 \boldsymbol{Z}_j}{\partial^2 m_{rl}} = \frac{\partial^2 \boldsymbol{Z}_j}{\partial^2 c_{rl}} = 0$$

2.4. Sensitivity for the maximum displacements

The spectrum of the forced response is generally multiharmonic for nonlinear structures, even when monoharmonic excitation is applied. Using the sensitivity coefficients obtained for each harmonic coefficient of the multiharmonic displacement, the sensitivity coefficients can be determined for other important characteristics of the forced response, such as sensitivity of the maximum value over vibration period for displacements selected at any point of a structure.

Variation in time of any DOF, $u(\tau)$, selected from vector $\boldsymbol{q}(t)$ can be expressed in the form:

$$u(\tau) = \boldsymbol{H}_{-}^{T}(\tau)\boldsymbol{U} \tag{20}$$

where $\boldsymbol{H}_{-}(\tau) = \{1, \cos l_1\tau, \sin l_1\tau, \ldots, \cos l_n\tau, \sin l_n\tau\}^T$ is a vector which includes all basis functions used in the multiharmonic expansion of Eq. (2), and $U = \{U_0, U_1^{(c)}, U_1^{(s)}, \ldots, U_n^{(s)}\}^T$ is the vector of harmonic coefficients selected from vector $\boldsymbol{Q}$ for the DOF analysed. Time instants, $\tau_{\max}$, when $u(\tau)$ reaches its extreme values are determined from the following equation:

$$\dot{u}(\tau) = \dot{\boldsymbol{H}}_{-}^{T}(\tau)\boldsymbol{U} = 0, \quad \tau \in [0, 2\pi] \tag{21}$$

This equation is solved over vibration period with respect and $\tau_{\max}$ is selected from all its roots as time providing maximum value for $|u(\tau)|$.

First-order sensitivity of the maximum coordinate displacement to parameter, b_j, variation can then be calculated in the following form:

$$\frac{\partial u(\tau_{\max})}{\partial b_j} = \dot{u}(\tau_{\max})\frac{\partial \tau_{\max}}{\partial b_j} + \boldsymbol{H}_{-}^{T}(\tau_{\max})\frac{\partial \boldsymbol{U}}{\partial b_j} = \boldsymbol{H}_{-}^{T}(\tau_{\max})\frac{\partial \boldsymbol{U}}{\partial b_j} \tag{22}$$

where $\partial \boldsymbol{U}/\partial b_j$ is a vector of the first-order sensitivity for harmonic coefficients selected from vectors $\partial \boldsymbol{Q}/\partial b_j$. The term containing $\dot{u}(\tau_{\max})$ is omitted in the final expression because $\dot{u}(\tau_{\max})$ is equal to zero at time $\tau_{\max}$.

Second-order sensitivity coefficients of the maximum displacement are calculated in the following way:

$$\frac{\partial^2 u(\tau_{\max})}{\partial b_k \partial b_j} = \boldsymbol{H}_{-}^{T}(\tau_{\max})\frac{\partial^2 \boldsymbol{U}}{\partial b_k \partial b_j} + \dot{\boldsymbol{H}}_{-}^{T}(\tau_{\max})\left(\frac{\partial \boldsymbol{U}_j}{\partial b_j}\frac{\partial \tau_{\max}}{\partial b_k} + \frac{\partial \boldsymbol{U}_j}{\partial b_k}\frac{\partial \tau_{\max}}{\partial b_j}\right) + \ddot{u}(\tau_{\max})\frac{\partial \tau_{\max}}{\partial b_j}\frac{\partial \tau_{\max}}{\partial b_k} \tag{23}$$

where $\partial^2 \boldsymbol{U}/\partial b_k \partial b_j$ is a vector of the second-order sensitivity for harmonic coefficients selected from vector $\partial^2 \boldsymbol{Q}/\partial b_k \partial b_j$. The derivative of the time, $\tau_{\max}$, when the coordinate displacement reaches its maximum value is determined by differentiation of Eq. (21), which gives the expression:

$$\frac{\partial \tau_{\max}}{\partial b_j} = -\frac{(\dot{\boldsymbol{H}}_{-}^{T}(\tau_{\max})(\partial \boldsymbol{U}/\partial b_j))}{\ddot{u}(\tau_{\max})}. \tag{24}$$

2.5. Dimensionless sensitivity coefficients

Design parameters can be measured in different units and their magnitudes can differ significantly. Moreover, the forced response level can vary by several orders of magnitudes, for example for resonance peak values and for vibration level excited far from resonance peaks. The sensitivity coefficients calculated for response levels can vary accordingly. Hence, there is some difficulty in comparing values of the sensitivity coefficients, with respect to different design parameters and for different frequency ranges. Therefore, in order to characterise sensitivity of the forced response, it is often more convenient to calculate dimensionless relative sensitivity parameters which indicate for how many percents the forced response level can change when each of design parameters changes by 1%.

Let us assume that γ is some forced response characteristic, for example one of those considered in the previous sections, i.e. value of a harmonic coefficient or a value of the maximum displacement, and its sensitivity coefficients $s_j = \partial\gamma/\partial b_j$ and $s_{ij} = \partial^2\gamma/\partial b_j \partial b_j$ are calculated using the expressions derived above. Therefore, the dimensionless first- and second-order sensitivity coefficients, $s_j^{\%} = \partial\gamma^{\%}/\partial b_j^{\%}$ and $s_{ij}^{\%} = \partial^2\gamma^{\%}/\partial b_i^{\%}\partial b_i^{\%}$ are expressed in the form:

$$s_j^{\%} = \left(\frac{b_j^{(0)}}{\gamma_0}\right)s_j \quad \text{and} \quad s_{ij}^{\%} = \left(\frac{10^{-2} b_i^{(0)} b_j^{(0)}}{\gamma_0}\right)s_{ij} \tag{25}$$

where superscript '%' indicates dimensionless values that are measured in percents of their nominal values, and γ_0 is the scaling coefficient for the forced response characteristic analysed. Choice of the scaling coefficient value is determined by a designer and is dependent on practical requirements. In many cases, it is convenient to choose for the scaling coefficient the maximum forced response level calculated for a current frequency, ω, i.e. $\gamma_0 = \gamma(\omega)$, in other cases, the maximum value of $\gamma(\omega)$ found over the whole analysed frequency range, i.e. $\gamma_0 = \max\limits_{\omega} \gamma(\omega)$ can be used.

3. Analytical derivation of expressions for contact interfaces with friction and gaps

3.1. Nonlinear contact forces and tangent stiffness matrix

As seen in Section 2.2, calculation of the nonlinear forced response requires calculation of the multiharmonic nonlinear contact interaction forces, $\boldsymbol{F}(\boldsymbol{Q})$, together with their tangent stiffness matrix, $\boldsymbol{K}_t = \partial \boldsymbol{F}(\boldsymbol{Q}, \boldsymbol{b})/\partial \boldsymbol{Q}$.

Expressions for these vectors and matrices have been derived analytically for each friction contact element introduced in Ref. [18] to describe interaction between nodes of pairing contact surfaces. Clearances and interferences are included in the friction contact models developed together with the effects of variable normal stresses on friction forces and on time instants of stick–slip transitions. Unilateral character of the interaction forces acting along a direction normal to a contact surface is allowed for, i.e. compression stresses can be transferred from one of the pairing contact surfaces to another one but tensile stresses are not allowed. Parameters of the friction contact interfaces considered in those papers includes: (i) a clearance/interference value, g; (ii) the normal load/stress, N_0; (iii) the friction coefficient, μ; (iv) normal stiffness of the contact surface, k_n; and (v) tangential stiffness of the contact surface, k_t.

Time domain expressions for tangential and normal components of the interaction forces through interface parameters and relative motion of the contact nodes along tangential and normal directions, $x(\tau)$ and $y(\tau)$, are shown in Table 1 for all possible contact conditions.

Where τ_j are times of beginning of the current contact state (e.g. stick, slip or separation) and $\xi = \pm 1$ is a sign function. Equations that are used to determine time instants when contact state changes are:

$$\begin{aligned} &\text{(a) stick to slip}: \quad f_x(\tau) = \pm\mu f_y(\tau); \\ &\text{(b) slip to stick}: \quad \xi k_x \dot{x}(\tau) = \mu k_y \dot{y}(\tau); \\ &\text{(c) contact}-\text{separation}: \quad N_0 + k_y y(\tau) = 0 \end{aligned} \tag{26}$$

Vectors of multiharmonic components for tangential, $\boldsymbol{F}_x$, and normal, $\boldsymbol{F}_y$, forces can be expressed in the form:

$$\boldsymbol{F}_e = \begin{Bmatrix} \boldsymbol{F}_x \\ \boldsymbol{F}_y \end{Bmatrix} = \frac{1}{\pi}\sum_{j=1}^{n_\tau}\int_{\tau_j}^{\tau_{j+1}} \begin{Bmatrix} \boldsymbol{H}_+(\tau) f_x(x(\tau), y(\tau)) \\ \boldsymbol{H}_+(\tau) f_y(x(\tau), y(\tau)) \end{Bmatrix} \mathrm{d}\tau \tag{27}$$

where $\boldsymbol{H}_+(\tau) = \{0.5, \cos l_1\tau, \sin l_1\tau, \dots, \cos l_n\tau, \sin l_n\tau\}^T$; τ_j are time instants when contact state changes, which are determined from Eq. (26), and n_τ is the total number of these changes over the period of vibrations. Variation of displacements in time domain, $x(\tau)$ and $y(\tau)$, is expressed through multiharmonic coefficients of expansions $\boldsymbol{X}$ and $\boldsymbol{Y}$ for the corresponding DOFs (see Eq. (20)) selected from vector $\boldsymbol{Q}$. Integrals included in right-hand side of Eq. (27) are evaluated analytically, and then expressions for interaction forces as functions of harmonic coefficients of relative displacements of the contacting surfaces are obtained in an explicit analytical form: $\boldsymbol{F}_e = \boldsymbol{F}_e(\boldsymbol{U}_e)$ (where $\boldsymbol{U}_e = \{\boldsymbol{X}, \boldsymbol{Y}\}^T$) together with the tangent stiffness matrix, $\partial \boldsymbol{F}_e/\partial \boldsymbol{U}_e$. Details of the derivation of these analytical expressions are described in Ref. [18].

The vector of contact interaction forces, $\boldsymbol{F}(\boldsymbol{Q})$, and the tangent stiffness matrix of the friction contact interfaces, $\boldsymbol{K}_t$, are formed for the whole structure by summing up vectors of nonlinear forces and stiffness matrices at all friction contact interfaces, i.e.:

$$\boldsymbol{F}(\boldsymbol{Q}) = \bigcup_{e=1}^{N_e} \boldsymbol{F}_e(\boldsymbol{Q}), \qquad \boldsymbol{K}_t(\boldsymbol{Q}) = \bigcup_{e=1}^{N_e} \frac{\partial \boldsymbol{F}_e}{\partial \boldsymbol{U}_e} \tag{28}$$

where N_e is the total number of contact elements. Since an analytical formulation is derived for the friction contact elements these matrices are calculated very fast and accurately.

3.2. First-order sensitivities for nonlinear contact forces

In order to determine the first-order sensitivity of the forced response sensitivity of the nonlinear contact interaction forces to variation of the friction contact parameters, $\partial \boldsymbol{F}(\boldsymbol{Q}, \boldsymbol{b})/\partial b_j$, have to be additionally calculated (see Section 2.3). For contact interfaces elements analytical expressions for sensitivity of the nonlinear friction and gap interfaces to contact

Table 1
Forces of the friction contact interaction

Status	Tangential force, f_x	Normal force, f_y
Contact		
Stick	$K_t(x(\tau)-x(\tau_j))-\xi\mu f_y(\tau_j)$	$N_0+k_n y(\tau)$
Slip	$\xi\mu f_y$	
Separation	0	0

parameters, $\partial \boldsymbol{F}_e(\boldsymbol{Q},\boldsymbol{b})/\partial b_j$, are derived analytically in Ref. [23] and the vector of sensitivities of nonlinear forces for the whole structure, $\partial \boldsymbol{F}(\boldsymbol{Q},\boldsymbol{b})/\partial b_j$, is formed similar to the case of the vector of nonlinear forces, given by Eq. (28).

3.3. Second-order sensitivities for nonlinear contact forces

In addition to the case of first-order sensitivity calculation, for calculation of the second-order sensitivity coefficients the following expressions have to be evaluated: (i) second-order derivatives of the contact interface forces with respect to interface parameters, $\partial^2 \boldsymbol{F}/\partial b_k \partial b_j$, (ii) a sensitivity of the tangent stiffness matrix to parameter variation, $\partial \boldsymbol{K}_t/\partial b_j$, and (iii) a tangent stiffness matrix of contact sensitivities which is obtained by differentiation of the product of the tangent stiffness matrix and first-order sensitivity of the harmonic coefficients of displacements, $\tilde{\boldsymbol{K}} = \partial(\boldsymbol{K}_t \partial \boldsymbol{Q}/\partial b_j)/\partial \boldsymbol{Q}$ with respect to the harmonic coefficients of displacements. Expressions for these matrices are derived analytically in Ref. [7].

3.4. Efficiency of sensitivity calculations

It should be noted that the calculation of first-order sensitivity coefficients, and, for many cases, even calculation of second-order sensitivity coefficients, do not increase noticeably the time of calculation which is required for forced response analysis. This is achieved by using the analytically derived expressions for when sensitivity of the contact forces are determined, when the customary need to calculate finite-difference approximations can be fully excluded. These analytical expressions provide exact and very fast calculation of the contact characteristics necessary for the first- and second-order sensitivity coefficient calculations. Efficiency of the calculation allows the sensitivity analysis to be performed simultaneously for different contact interfaces and with respect to the different contact interface parameters.

4. Assessment of robustness for nonlinear forced response

4.1. Sensitivity-based explicit expressions for forced response

For many practical cases, it is possible to construct sensitivity-based approximations for the forced response expressing forced response level explicitly as functions of the design parameters of interest. One of the simplest and efficient approaches is using Taylor series approximation for the forced response constructed in vicinity of the forced response calculated for a vector of some nominal design parameters, $\boldsymbol{b}_0$.

The first-order approximation for the forced response takes the following form:

$$\gamma_I(\boldsymbol{b}) = \gamma(\boldsymbol{b}_0) + \boldsymbol{s}^T(\boldsymbol{b}_0)(\boldsymbol{b} - \boldsymbol{b}_0) \tag{29}$$

and the second-order approximation takes the form:

$$\gamma_{II}(\boldsymbol{b}) = \gamma(\boldsymbol{b}_0) + \boldsymbol{s}^T(\boldsymbol{b}_0)(\boldsymbol{b} - \boldsymbol{b}_0) + \tfrac{1}{2}(\boldsymbol{b} - \boldsymbol{b}_0)^T \boldsymbol{S}(\boldsymbol{b}_0)(\boldsymbol{b} - \boldsymbol{b}_0) \tag{30}$$

where $\boldsymbol{s}(\boldsymbol{b}_0) = \{\partial\gamma/\partial b_1, \partial\gamma/\partial b_2, \ldots, \partial\gamma/\partial b_n\}^T$ is vector of first-order sensitivity coefficients calculated for a vector of design parameters, $\boldsymbol{b}_0$; $\boldsymbol{S}(\boldsymbol{b}_0) = [\partial^2\gamma/\partial b_i \partial b_j]$ is a matrix of second-order sensitivity coefficients, n is the number of design parameters which can affect forced response levels, and subscripts 'I' and 'II', indicate quantities related to first- and second-order approximations, respectively.

It should be noted that these approximations can be sufficiently accurate when applied in relatively small parameter variation ranges resulting from the technological imperfections and operational uncertainties.

The dimensionless first and second-order sensitivity coefficients, $\boldsymbol{s}^{\%}(\boldsymbol{b}_0)$ and $\boldsymbol{s}^{\%}(\boldsymbol{b}_0)$ allow constructing explicit expressions for relative variation of the forced response, $\Delta\gamma^{\%}(\boldsymbol{b}) = 100[(\gamma(\boldsymbol{b}) - \gamma(\boldsymbol{b}_0))/\gamma(\boldsymbol{b}_0)]$ through a vector of relative variation of design parameters: $\Delta b^{\%} = 100\left\{\left(b_1 - b_1^{(0)}\right)/b_1^{(0)}, \left(b_2 - b_2^{(0)}\right)/b_2^{(0)}, \ldots, \left(b_n - b_n^{(0)}\right)/b_n^{(0)}\right\}^T$. For cases of first- and second-order approximations, the relative variation of the forced response takes the form:

$$\Delta\gamma_I^{\%}(\boldsymbol{b}) = (\boldsymbol{s}^{\%})^T \Delta\boldsymbol{b}^{\%} \quad \text{and} \quad \Delta\gamma_{II}^{\%}(\boldsymbol{b}) = (\boldsymbol{s}^{\%})^T \Delta\boldsymbol{b}^{\%} + \tfrac{1}{2}(\Delta\boldsymbol{b}^{\%})^T \boldsymbol{S}\, \Delta\boldsymbol{b}^{\%}. \tag{31}$$

4.2. Robustness measure for the forced response levels

Although sensitivity coefficients can be used directly for assessment of the forced response robustness, in strict terms, they can provide estimates for the forced response robustness only in infinitesimal vicinity of the parameter values for which these sensitivities are calculated.

The robustness measure, R, can introduced, as a function of an absolute value of the maximum relative deviation of the response level from its nominal value which can be achieved for all possible variations of design parameter values and operating conditions, $|\Delta\gamma^{\%}_{\max}|$, in the following form:

$$R = 100 - |\Delta\gamma^{\%}_{\max}| \tag{32}$$

The robustness measure is dependent on two major factors: (i) the sensitivity coefficients of the forced response to design parameters and (ii) uncertainty ranges of the design parameters. Decreasing each of these can decrease uncertainty and increase robustness of the forced response. Assessment of forced response robustness can be made for: (i) individual effects of the parameter uncertainties and (ii) the total robustness determined by uncertainty of a large number of parameters.

4.2.1. Effects of individual design parameter uncertainty on the robustness

In order to compare effects of uncertainties of each design parameter, b_j, on the robustness measure, a vector of robustness measures, $\boldsymbol{R}$, determined by independent variation of each design parameter is calculated:

$$\boldsymbol{R} = \left\{100 - \Delta\hat{\gamma}_1^{\%}, 100 - \Delta\hat{\gamma}_2^{\%}, \ldots, 100 - \Delta\hat{\gamma}_n^{\%}\right\}^T \tag{33}$$

where $\Delta\hat{\gamma}_j^{\%} = \max_{b_j^- \leqslant b_j \leqslant b_j^+}(|\Delta\gamma^{\%}(\boldsymbol{b}_0)|)$ is the maximum possible deviation of forced response level resulting from prescribed uncertainty of the jth design parameter. For calculation of each jth component of this vector, it is assumed that jth design parameter varies within its uncertainty range $b_j^- \leqslant b_j \leqslant b_j^+$, while all the rest parameters keep their nominal values, i.e. $b_i = b_i^0$ for $i \neq j$.

Use of the sensitivity-based approximation allows the following explicit expressions for $\Delta\hat{\gamma}_j^{\%}$:

- for the first-order approximation

$$\Delta\hat{\gamma}_j^{\%} = \max\left(\left|s_j^{\%}\Delta b_j^{\%-}\right|, \left|s_j^{\%}\Delta b_j^{\%+}\right|\right) \tag{34}$$

- for the second-order approximation

$$\Delta\hat{\gamma}_j^{\%} = \max\left(\left|\Delta\hat{\gamma}_{II}^{\%}\left(\Delta b_j^{\%-}\right)\right|, \left|\Delta\hat{\gamma}_{II}^{\%}\left(\Delta b_j^{\%+}\right)\right|, \left|\Delta\hat{\gamma}_{II}^{\%}\left(\Delta b_j^{\%*}\right)\right|\right) \tag{35}$$

where $\Delta\gamma_{II}^{\%}\left(\Delta b_j^{\%}\right) = s_j^{\%}\Delta b_j^{\%} + 0.5 s_{jj}^{\%}\left(\Delta b_j^{\%}\right)^2$, and $\Delta b_j^{\%*} = -s_j/s_{jj}$ when $\Delta b_j^{\%-} \leqslant \Delta b_j^{\%*} \leqslant \Delta b_j^{\%+}$, otherwise $\Delta b_j^{\%*} = 0$.

4.2.2. Assessment of overall robustness under uncertainty of all parameters

The maximum relative deviation of forced response is caused by all uncertainties in design parameters, excitation and operating conditions and characterises the forced response uncertainty level. This robustness measure includes in consideration the forced response levels for all possible sets of design parameters within prescribed uncertainty ranges and is often more useful for practical quantitative estimate of the robustness than simply sensitivity coefficients. When the forced response is not affected by design parameters, the robustness measure takes its highest value 100%.

The absolute value of the maximum relative forced response, $|\Delta\gamma_{\max}^{\%}|$, is calculated by solving the following optimisation problem:

$$\left|\Delta\gamma_{\max}^{\%}\right| = |\Delta\gamma^{\%}(\boldsymbol{b})| \to \max \tag{36}$$

with the constraints applied to the design parameters in the form

$$\boldsymbol{b}^- \leqslant \boldsymbol{b} \leqslant \boldsymbol{b}^+ \quad \text{or} \quad \Delta\boldsymbol{b}^{\%-} \leqslant \Delta\boldsymbol{b}^{\%} \leqslant \Delta\boldsymbol{b}^{\%+}, \tag{37}$$

where $\boldsymbol{b}^-$ and $\boldsymbol{b}^+$ are minimum and maximum possible values of the design parameter vector, respectively.

When the sensitivity-based explicit approximations given by Eq. (31) are used for relative forced response level variation, $\Delta\gamma^{\%}(\boldsymbol{b})$, the optimisation problem (Eqs. (36) and (37)) becomes: (i) the linear programming problem—for the linear approximations or (ii) the quadratic programming problem—for quadratic approximations. These optimisation problems have simple bound constraints described by Eq. (37), and both these problems can be solved very efficiently by readily available optimisation methods and software (see Refs. [27,28]), which provide the global maximum in a finite number of iterations. This global maximum gives the maximum value for the objective function, $|\Delta\gamma_{\max}^{\%}|$, and also a vector of design parameters at which this maximum can be achieved.

Owing to the very fast sensitivity calculation method developed in Section 2 and fast solution of the optimisation problem given by Eqs. (36) and (37), the robustness measure, $R = 100 - |\Delta\gamma_{\max}^{\%}|$, can be calculated simultaneously with forced response analysis and without significant computation expense.

5. Numerical results

5.1. A cantilever beam with friction and gap contacts

A three-dimensional solid FE model of 37,600 DOFs for a cantilever beam (Fig. 1) with sides $1000 \times 200 \times 100$ and with the following material properties: elasticity modulus $E = 10^5$; density $\rho = 4.43 \times 10^{-9}$ and damping loss factor 0.003 is considered as an example. Vibrations are excited by a force $p_x = 100\cos\omega t$ applied at the beam free end along axis X, as shown in Fig. 1. Numerical properties of the developed approach and characteristic features of the dynamic behaviour of

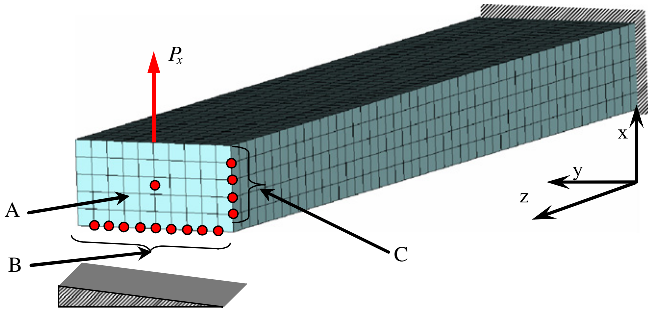

Fig. 1. Finite element model of the cantilever beam analysed and nodes where friction and gap contact elements are applied.

the structure are studied for two types of nonlinear contacts: (1) a case of a gap contact interface and (2) a case of friction interface.

For the case of gap contact interface, it is assumed that the cantilever beam can contact at the free end of the beam with support when vibration amplitudes of the beam close an initial gap. The gap value is varied linearly from 0.01 to 0.09 over the beam width (see Fig. 1) and nine gap contact interface elements are applied at nodes shown in red and marked by letter 'B' in Fig. 1. Stiffness coefficient of each gap contact element of this support is assumed to be $k_y = 10^3$.

For a case of friction contact, four friction contact elements are applied at one side of the beam. Each friction contact element has the following characteristics: $\mu = 0.3$; $k_t = k_n = 10^4$ and is located at node marked by letter 'C' in Fig. 1.

For forced response and sensitivity calculations, all harmonics from 1 to 10 are taken into account in the multiharmonic representation of the displacements. Forced response is determined for a middle node of the free end beam surface (marked by letter 'A' in Fig. 1).

Forced response level and its sensitivity to variation of gap value are shown in Fig. 2. The sensitivity coefficients are determined for three cases: (i) a case when gap value is simultaneously varied for all nine gap contact elements; (ii) a case when gap can vary only for first gap contact element (i.e. for the element which has gap value 0.01); and (iii) a case when gap can vary only for ninth gap contact element. The frequency range considered includes a major resonance at 100 Hz and several superharmonic resonances in frequency range 1–30 Hz. The sensitivity coefficient values vary significantly over this frequency range and, to demonstrate their peak values and their variation for wide frequency ranges where sensitivity values are much smaller than these peak values, they are plotted in two different scales in Fig. 2b and c. One can observe sharp peaks of the sensitivity coefficients at frequencies where there is a vertical tangent to the FRF curves and there are segments of the FRF corresponding to unstable solutions. This is due to high sensitivity of the forced response at such points where even very small variations of a design parameter can cause a jump of the forced response level to another branch of the FRF curve. Such sensitivity value peaks are observed in vicinity of the major resonance and in vicinity of the superharmonic resonances for sensitivity coefficient with respect to variation of gap value for first and for a case when sensitivities are calculated with respect to gap values of all nine gap contact elements considered. The sensitivity coefficient with respect to gap value of the ninth contact element, which has the largest gap value, is zero for most of the frequency range analysed and becomes significant only in the vicinity of the major resonance peak, i.e. when the gap corresponding to this contact element can be closed. Forced response sensitivity for nineth contact element is much higher than that for first contact element. As expected, forced response sensitivity with respect to simultaneous variation of gap values for all nine contact elements is much higher than sensitivity coefficients obtained with respect to any individual contact element parameter.

Forced response and first- and second-order sensitivity coefficients calculated for the beam with friction dampers are shown in Fig. 3. Forced response sensitivity is calculated here: (i) with respect to friction coefficient value of all four dampers, and (ii) with respect to friction coefficient value of the first damper. It should be noted that sensitivities with respect to friction coefficient of each individual damper are similar, and the plot corresponding to sensitivity to friction coefficient value of the first damper rather accurately describe also forced response sensitivities with respect to friction coefficients of second, third and fourth dampers. Similar to the case of gap contact elements, the forced response sensitivity has highly localised peak values at frequencies where a tangent to the forced response curve is vertical.

5.2. A high-pressure turbine bladed disc with friction contact of shrouds

As an example of a practical application, a turbine high-pressure bladed disc shown in Fig. 4 is considered. The bladed disc comprises 92 shrouded blades and a part of this bladed disc comprising 10 blades is shown in Fig. 4a. The damping loss factor is set to 0.003 and excitation by fourth engine order is considered in the analysis. Ten harmonics which are multipliers of fourth engine order are used in the multiharmonic representation, i.e. 4, 8, 12, …, 40. Nonlinear forces occur as a result of contact interactions between blade shrouds when vibration levels are large enough to cause closing/opening gaps and interferences or to cause slip–stick transition at any part of the friction contact interface. These nonlinear shroud

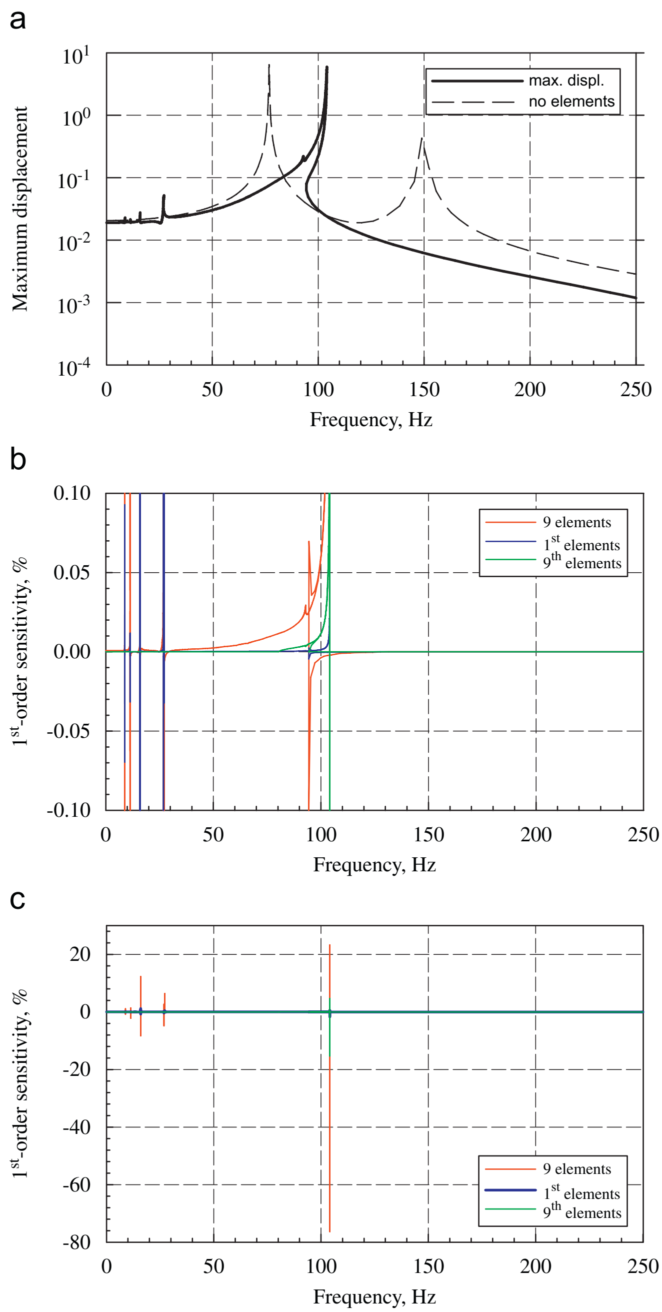

Fig. 2. Forced response (a) and its sensitivity to gap values (b, c).

interactions are modelled using friction contact interface elements developed in Ref. [18]. These contact interface elements allow accurate modelling of forces normal to the contact surface and tangential friction forces. Unilateral character of the normal forces is accounted for, i.e. at a contact interface compression normal stresses are allowed while tensile stresses cannot occur. Moreover, effects of the normal load variation on friction forces are allowed for, including their influence on stick–slip transitions. Cyclic symmetry of the bladed disc is accounted for by a method developed in Ref. [19], which allows using only one sector of the bladed disc with special boundary conditions to analyse accurately the whole assembly. Nine friction interface elements are distributed over each of two shroud surfaces at which shroud of the considered bladed disc sector contacts with adjacent blades of the bladed disc (see Fig. 4c). The interference value at both shroud contact interfaces used in calculations is 10^{-3} mm for all contact nodes.

Forced response level calculated at the blade tip and its sensitivity to variation of the interference value and the friction coefficient of the contacting surfaces is plotted in Fig. 5. A forced response of this bladed disc is also plotted for a case when

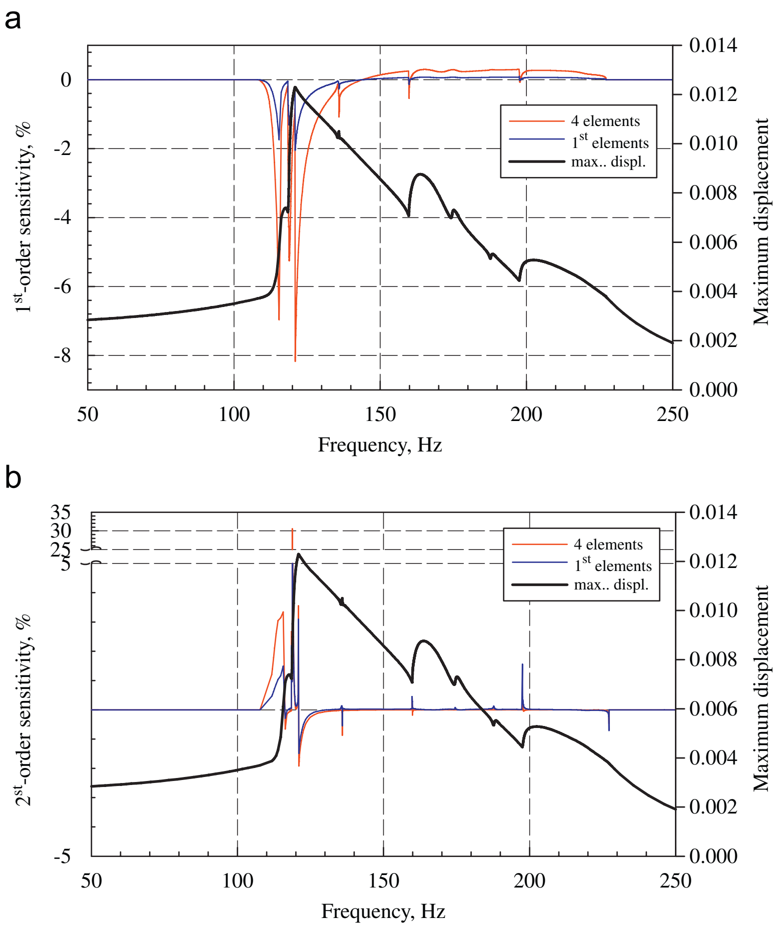

Fig. 3. Forced response and its first-order (a) and second-order (b) sensitivities to the friction coefficient value.

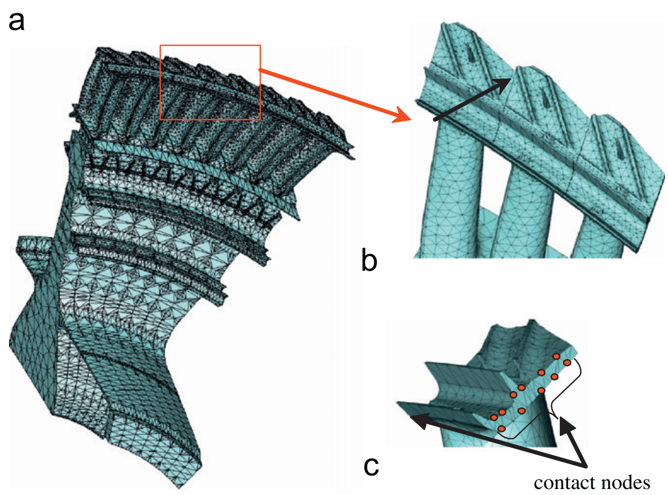

Fig. 4. Finite element model of a high-pressure turbine bladed disc: (a) a part of the bladed disc, (b) contacting shrouds, and (c) contact interface nodes.

interference value is so large that any contact–separation and slip–stick transitions are eliminated over all contact nodes. For this case, all nodes are always struck and all interaction forces are linear. Comparison of this linearised case with the case when contact interactions are realistically modelled by friction contact elements demonstrates importance of allowing for unilateral interactions and friction forces at shroud contact interfaces. It is evident that the resonance peak of the

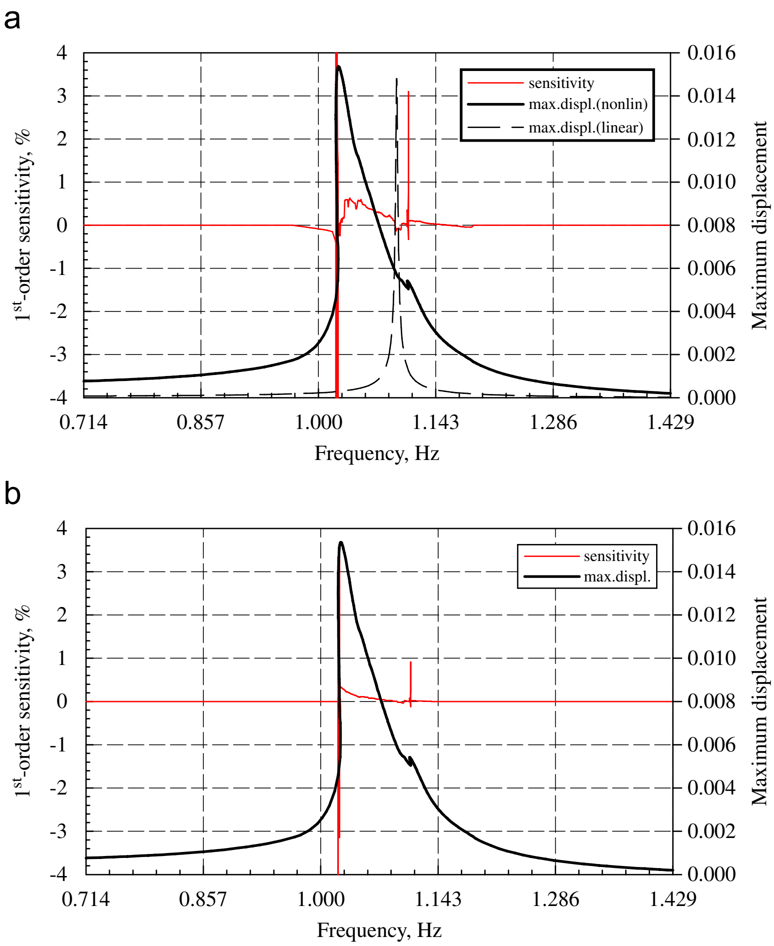

Fig. 5. Forced response and its sensitivity to contact interface parameters with respect to: (a) the interference value and (b) friction coefficient.

bladed disc with realistically modelled contact forces is much wider, due to higher damping caused by friction forces and also due to gradual variation of stiffness properties of the contact interface with increase of the forced response level, when number of contact nodes exhibiting contact–separation transitions during vibration period is increased gradually. The resonance frequency value for a structure with realistic contact modelling is lower than that of the structure with fully stuck contact nodes. Higher level of the resonance peak forced response for a structure with realistic contact interfaces is observed, which is owing to change of the resonating mode shape, which is not even compensated by the higher vibration energy dissipation at friction contact interfaces in the considered case.

One can see that the forced response becomes sensitive to variation of the both contact interface parameters considered in the vicinity of the resonance peak, i.e. where the forced response level become high enough to cause contact–separation and slip–stick transitions. Sensitivity coefficients with respect to interference value are significantly higher than those with respect to the friction coefficient. At frequencies where the forced response curve has vertical slope, the sensitivity coefficients increase sharply.

Robustness of the forced response calculated for cases: (i) when friction coefficient of the shroud contact interface is 5% uncertain, (ii) when the interference value is 5% uncertain, and (iii) when they both are uncertain is shown in Fig. 6. One can see that, for the considered case, the interference value is much more significant than friction coefficient.

5.3. AdTurbII test rig bladed disc with underplatform dampers

Another example of a structure with friction contact interfaces is a blisc (i.e. a jointless bladed disc for which bladed and disc are manufactured from one piece of metal), which was made for a test rig of an EU-funded ADTurbII project (see Ref. [29]). The blisc consists of 24 blades, and a finite element model of its one sector contains 21,555 DOFs (see Fig. 7a). Since there are no joints in this structure, the background damping in the blisc is mostly determined by damping in material of blades and disc and the damping loss factor is relatively low: $\eta = 7.5\text{e}{-}05$. In order to introduce more damping in the

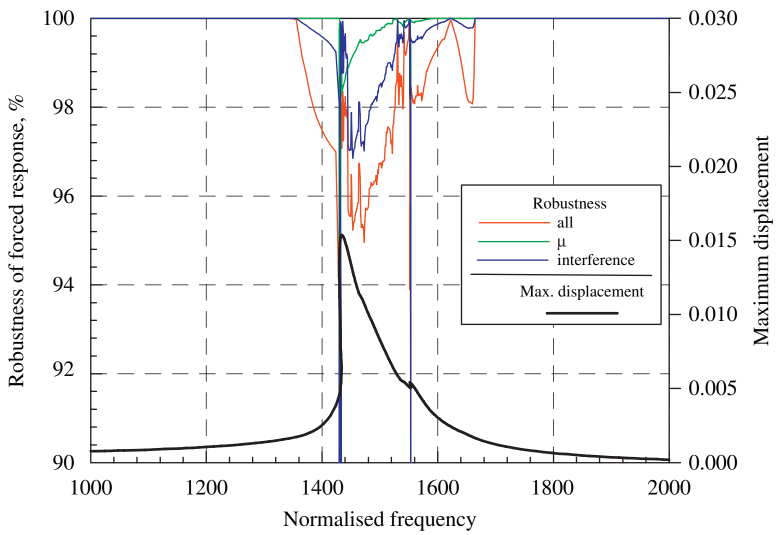

Fig. 6. Robustness of forced response with interference and friction coefficient uncertainties.

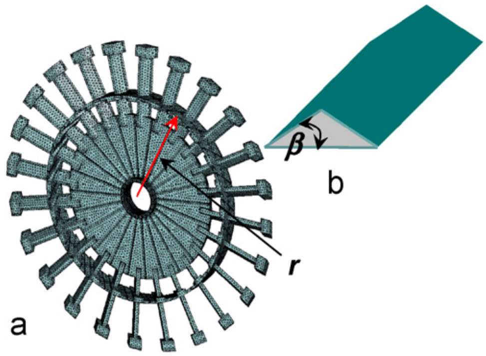

Fig. 7. FE model of a blisc with a cottage-roof damper.

structure and reduce resonance response levels special damping devices are applied: so-called cottage-roof (CR) underplatform dampers (UPDs). This damper is generally a prismatic piece of metal which is inserted under blade platforms and which has friction contact interfaces with each pair of adjacent blades in the bladed disc (see Fig. 7b). In operating conditions, the bladed disc is rotating and the UPDs are pressed by centrifugal forces to blade platforms. Blade vibration causes relative motion of the neighbouring blades and when such motion is large enough friction forces occurring at blade–UPD contact interfaces dissipate vibrational energy and moreover FE blisc model is shown in Fig. 7 and UPDs are modelled by a damper model developed in Ref. [30]. In the numerical studies presented here, a tuned cyclically symmetric bladed disc is considered.

The UPD model is formulated for a general case of multiharmonic forced response analysis. It allows for inertia forces occurring due to damper vibration and effects of normal load variation on stick–slip transitions at the contact interfaces, including contact–separation. The UPD damper is characterised by following parameters: (i) damper mass, $m_{\mathrm{UPD}} = 2.6\,\mathrm{g}$; (ii) friction coefficient, $\mu = 0.3$; (iii) distance of the UPD inertia centre from rotation axis, $r = 137\,\mathrm{mm}$; and (iv) damper angle, $\beta = 30^{\circ}$ (see Fig. 7a and b). These UPD parameters define inertial forces: static centrifugal and dynamic forces, and, moreover, friction contact interaction forces at damper contact surfaces. Effect of variation of the centrifugal forces with variation of the rotor rotation speed is also included in the analysis.

Forced response characteristics level calculated for a node located at blade tip the blisc is plotted in Fig. 8 for cases of three different UPD mass values. The damper mass of 2.6 g is referred as 100% value. For comparison forced response of two limiting cases are also plotted in this figure here: (i) forced response of a blisc without dampers and (ii) forced response of a blisc with fully stuck dampers. Large effect of the damper parameters on resonance and out of resonance response levels can be observed: due to energy dissipation and due to change in the stiffness properties.

Dimensionless first-order sensitivity coefficients of forced response levels with respect to all major UPD parameters are shown in Fig. 9, for a case of 100% damper mass. One can see that, for the case considered, the forced response is the most sensitive to variation of all parameters at the resonance peak. The most influential parameter is a value of the damper mass: its 1% increase can result in 2.5% reduction of the resonance response levels. One percent increase of friction coefficient can reduce response level by 2.2%, while for damper radius and angle change the response level by 1% and 0.14%, accordingly.

Accuracy of the sensitivity-based approximations (SBA), of the forced response has been investigated. Examples illustrating high accuracy of these approximations are shown in Fig. 10. Forced response curves obtained by two ways are compared here: (i) forced response and sensitivity coefficients are calculated for a set of nominal parameter values and then first-order SBA given by Eq. (29) are used to obtain forced response curves for new parameters values; (ii) forced response is calculated directly, for new parameter values. Cases of the most influential UPD parameters are considered here: (i) damper mass (Fig. 10a) and (ii) friction coefficient (Fig. 10b). A range of parameter variation is chosen in both cases 10%, which is expected to cover possible scatter of damper parameters due to manufacturing tolerances for most cases. One can see that results obtained by first-order SBA are practically indistinguishable from accurate results obtained by explicit calculation for new parameter values over the whole frequency range. Naturally, for larger ranges (e.g. $\pm 50\%$) the one-point approximations based on the sensitivity coefficients calculated can loose their accuracy and therefore two- or multiple-point approximations may need to be constructed. It should be noted that the method developed in papers [23,24] for direct parametric analysis allows direct and exact calculation of the forced response as a function of design parameters without restrictions on the design parameter variation.

The methodology developed in Section 4 for calculation of the forced response robustness for forced response caused by uncertainty of parameters of structure has been applied to calculate the robustness measure (see Eq. (32)) for the forced

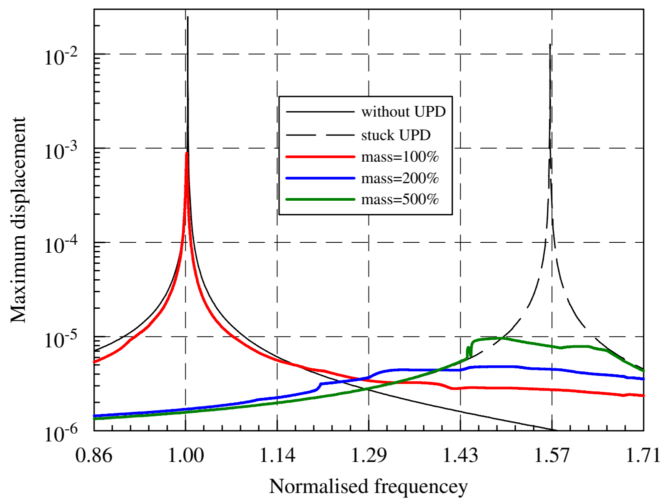

Fig. 8. Forced response of the blisc with underplatform dampers.

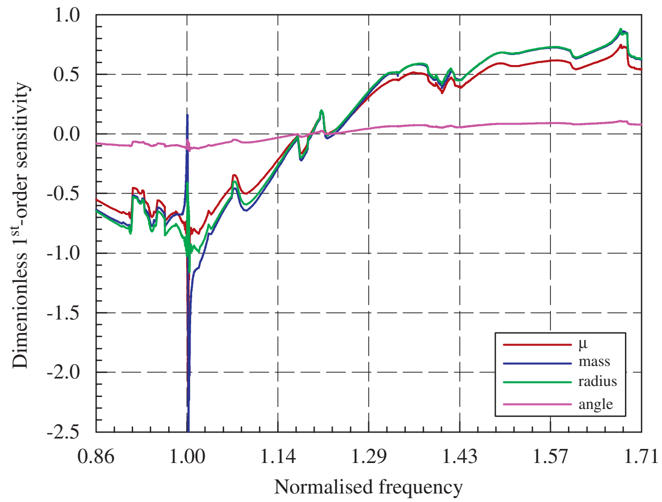

Fig. 9. Dimensionless sensitivity of the forced response to underplatform damper parameters.

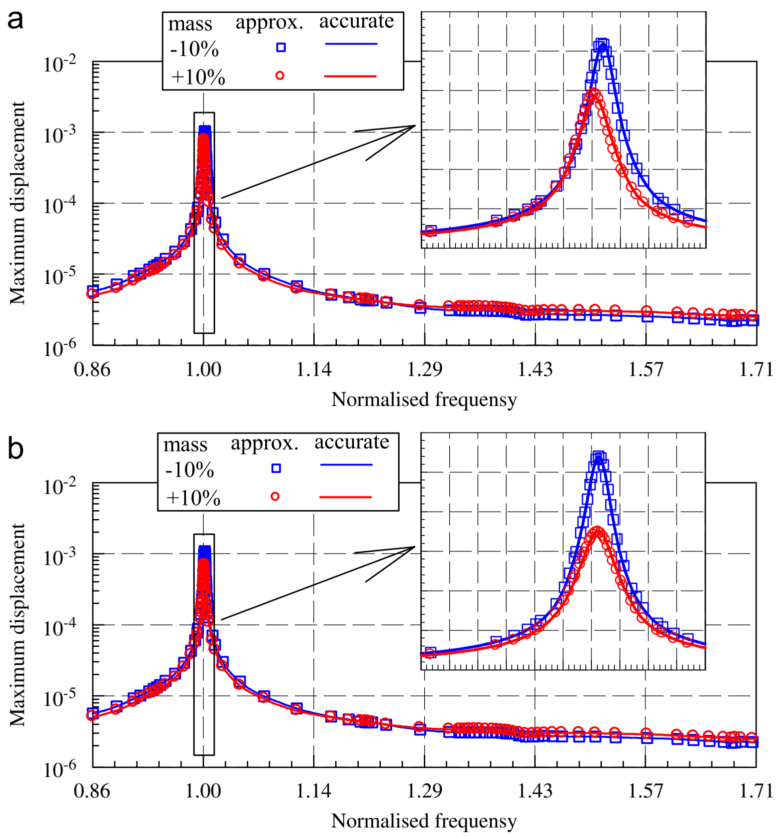

Fig. 10. Comparison of forced responses obtained by sensitivity-based approximations with accurate ones a case of: (a) damper mass value variation and (b) friction coefficient variation.

response when there are the following ranges of the UPD parameters: (i) damper mass is $\pm 6\%$, (ii) friction coefficient: $\pm 5\%$, (iii) distance from rotation axis: $\pm 2\%$, and (iv) damper angle: $\pm 3\%$. Robustness of forced response caused by each of these parameters individually and forced response uncertainty when all these parameters are uncertain simultaneously are shown in Fig. 11 together with forced response plots for all three values of the damper mass analysed. As expected, the forced response robustness when several all damper parameters are uncertain is much lower than that for cases when only one of four parameters is uncertain. It is interesting to note that for the case of 100% damper mass, the robustness measure calculated for a case of all uncertain parameters has its minimum value 74% at the resonance frequency, while for cases of heavier dampers, with 200% and 500% damper mass, the robustness measure takes values around 90% at resonance peaks. For 100% damper mass, the effect of damper mass variation and friction coefficient on robustness measure is almost identical and the most influential while for cases of 200% and 500% damper mass values they are clearly distinctive. For all cases, the uncertainty of the damper angle value was not significant.

6. Conclusions

A method is proposed to calculate, for a strongly nonlinear structure with friction contact interfaces, sensitivity of nonlinear forced response levels to variation of parameters of the friction contact interfaces, excitation forces and design parameters affecting dynamic properties of linear components of the assembled structure.

The effectiveness of the method allows the first- and second-order sensitivity coefficients to be calculated simultaneously with the calculation of the forced response without a significant increase of the computation effort.

The method is based on analytical derivation of equations for determination the sensitivity coefficients including sensitivity of the contact interaction forces for interfaces with gaps and frictions.

An approach for assessment of forced response robustness for nonlinear structures with uncertain design parameters is proposed and a measure of the robustness is introduced.

Numerical investigations of the sensitivity and robustness of the multiharmonic steady-state forced response of structures with friction and gaps have been performed showing the capabilities and efficiency of the methodology proposed.

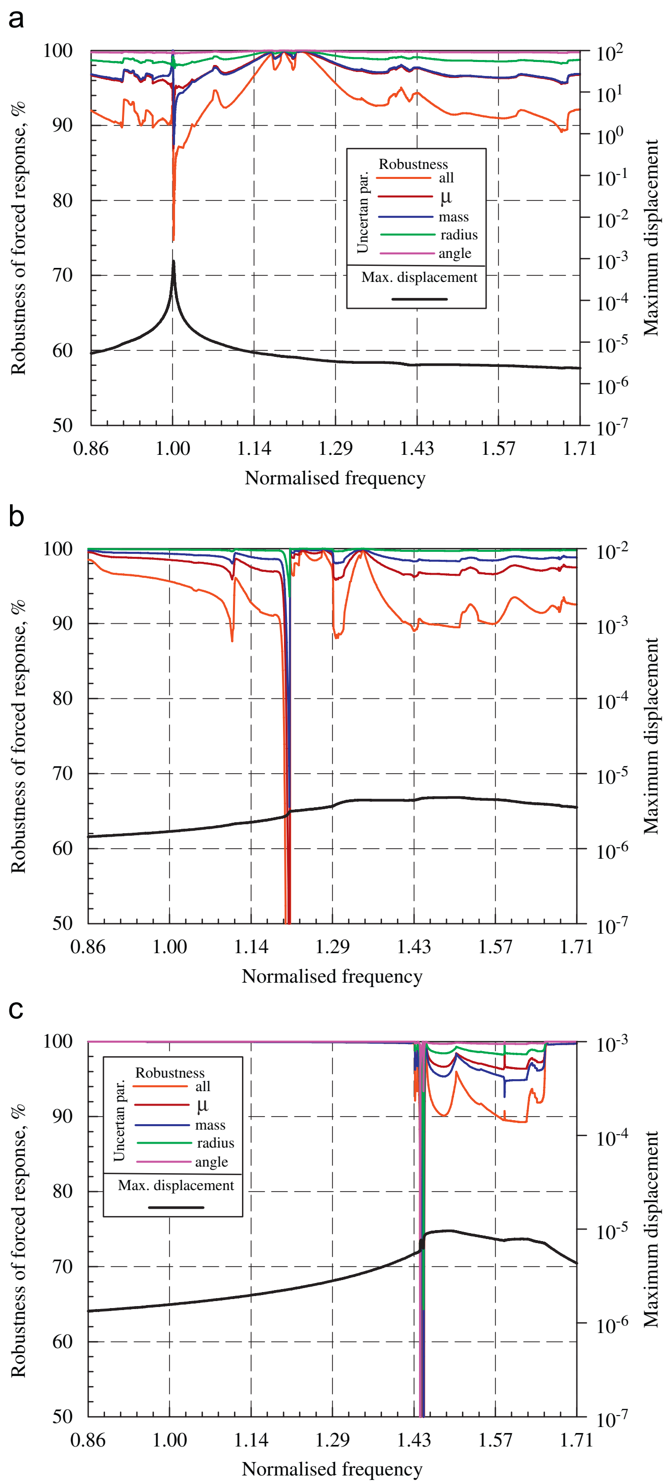

Fig. 11. Robustness of forced response a case of: (a) 100% damper mass, (b) 200% damper mass, and (c) 500% damper mass.

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