Gaussian Two-way Relay Channel with Arbitrary Inputs

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***Abstract*—A two-way relay channel with independent parallel Gaussian channels between the relay and the two terminals is considered. Focusing on the decode-and-forward protocol, the second phase of the communication, in which the relay broadcasts the two messages to their respective receivers, is studied. Precisely, the problem of computing the power allocation among the parallel channels that maximizes the weighted sum rate assuming arbitrarily distributed channel inputs (such as m-QAM) is stated and shown to be convex. A numerical algorithm is provided to solve the problem for the general case and, for the particular cases of high and low power regimes, expressions for the optimal power allocation are derived in closed form.**

I. Introduction

We consider two-way communication between two terminals where the connection is established over a relay terminal. This can model a scenario such as wireless communication between two mobile user terminals which have no direct channel in between (see Fig. 1). In such a scenario, the relay is essential to achieve any nonzero transmission rate in either direction. The capacity region of this two-way relay channel is open; however, there has been a considerable recent interest in developing novel achievability techniques and tight outer bounds for this model due to its practical relevance. Several achievability schemes have been proposed in the literature based on the type of coding used and the operation at the relay terminal. The classical amplify-and-forward and decode-and-forward (DF) schemes have been extended to this scenario in [1]. The more advanced compress-and-forward (CF) scheme is considered in [2] and [3] while structured codes, rather than random coding, have been proposed in [4], [5]. Recently, the capacity of the Gaussian two-way relay channel has been computed to within 1/2 bit in [6].

While no single scheme dominates the others in all channel conditions, schemes based on CF relaying or structured codes are more complicated when it comes to practical applications as the former requires the use of joint source-channel coding for the broadcasting of the relay's received signal, while the latter requires perfect synchronization of the transmissions from the two users. Hence, we focus here on the DF strategy, in which both messages of the users are decoded at the relay terminal, before being forwarded to their respective recipients. Hence, the transmission can be divided into two phases, the first one is a multiple access channel from the users to the relay, and the second stage is a special broadcast channel, in which each receiver has access to the message that is to be decoded by the other receiver.

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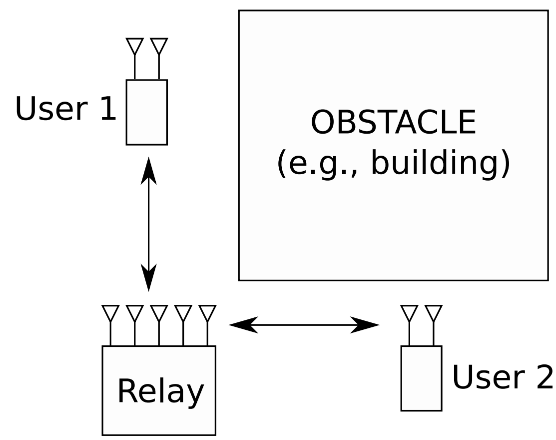


Fig. 1. Illustration of the two-way relay channel model in which a base station acts as a relay terminal helping two separate user terminals exchange messages simultaneously.

In this work, we explicitly focus on the second stage of the transmission, i.e., the broadcast phase. It can be seen, using the result of [7], that, in the case of Gaussian channels, the relay can transmit simultaneously to both users at the capacity of each channel, as if each user is the sole receiver in the system. In a way, the transmissions to the users do not interfere with each other. This is due to the fact that the interfering message is already known at the receiver, and each receiver receives at a rate as if she is the sole receiver. This can be achieved by transmitting a modulo sum of the coded bits from the relay terminal rather than superimposing the signals or time-sharing between the messages. This can be considered as an analog network coding strategy.

In this paper we focus on the achievable rate region over the two-way relay channel, i.e., the set of information rate pairs at which the two users can exchange information simultaneously. We study the problem of optimal power allocation to identify the region of achievable rate pairs in the case of multiple parallel channels from the relay to the users. Observe that in many wireless applications nodes are connected through a set of parallel channels over which the available power needs to be allocated. MIMO systems or multicarrier schemes such as OFDM or discrete multitone (DMT) are the most common examples for such systems.

In the studied case of independent parallel channels, the mutual information to each user is maximized by a Gaussian input; however, the optimal power allocation among the subchannels is not a direct extension of the classical waterfilling solution which achieves the highest mutual information for a given power constraint [8]. Here our focus is on practical applications, and hence, we allow arbitrary input constellations at the relay input rather than simply focusing on the case of Gaussian input distributions. Our goal is to find the optimal power allocation strategy that achieves a specific point on the boundary of the achievable rate region with DF relaying over independent parallel Gaussian channels with arbitrary input constellations.

The rest of the paper is organized as follows. We introduce the system model in Section II. The optimization problem that solves for the optimal power allocation strategy is presented in Section III and its necessary and sufficient conditions are characterized. Section IV and Section V focus on the low and high power regimes, respectively. Finally, in Section VI we present some simulation results and we close the paper with the conclusions.

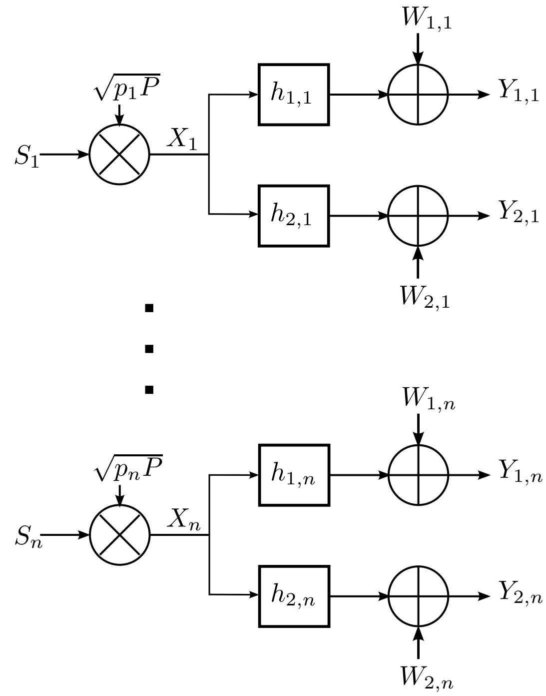


Fig. 2. Bidirectional broadcast channel with n parallel channels.

II. System model and problem statement

We consider n parallel broadcast channels from the relay to the two receivers (see Fig. 2). The input-output relationship of the ith channel from the relay to the kth user, $k = 1, 2$, is

$$Y_{k,i} = h_{k,i} X_i + W_{k,i}, \tag{1}$$

where X_i is the relay transmitted signal at the ith channel, $W_{k,i}$ is a zero-mean unit variance complex Gaussian random variable independent of the noise at the other user or at the other channels, and $h_{k,i}$ is the complex channel gain. We assume that the two receivers perfectly know their own channel gains, while the relay knows the magnitudes of the channel gains to both receivers.

The transmission power at the relay is constrained by

$$\sum_{i=1}^{n} E[|X_i|^2] \leq P. \tag{2}$$

Then, the transmitted signal X_i can be expressed as a scaled version of a unit-power zero-mean arbitrarily distributed input signal S_i according to

$$X_i = \sqrt{p_i P} S_i, \tag{3}$$

where p_i allocates the power among the channels satisfying

$$\sum_{i=1}^{n} p_i \leq 1. \tag{4}$$

Now, defining the normalized mutual information $\mathcal{I}_i(\rho)$ as

$$\mathcal{I}_i(\rho) = I(S_i; \sqrt{\rho} S_i + W), \tag{5}$$

the mutual information between the ith input signal, S_i, and the ith output of user k is simply given by

$$I(S_i; Y_{k,i}) = \mathcal{I}_i\big(|h_{k,i}|^2 p_i P\big). \tag{6}$$

Now the goal is to maximize a weighted average of the information flow to each receiver:

$$\sum_{k=1}^{2}\sum_{i=1}^{n} \alpha_k I(S_i; Y_{k,i}) = \sum_{k=1}^{2}\sum_{i=1}^{n} \alpha_k \mathcal{I}_i\big(|h_{k,i}|^2 p_i P\big), \tag{7}$$

where $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.

With this formulation, we can characterize the whole rate region by considering all possible weight pairs. For a fixed weight pair (α_1, α_2), the optimal power allocation problem can be expressed as follows

$$\max_{\{p_i\}} \quad \sum_{k=1}^{2}\sum_{i=1}^{n} \alpha_k \mathcal{I}_i(p_i \gamma_{k,i}) \tag{8a}$$

$$\text{s.t.} \quad p_1, \ldots, p_n \geq 0 \tag{8b}$$

$$\sum_{i=1}^{n} p_i = 1, \tag{8c}$$

where we defined the received signal-to-noise ratio of user k on channel i as $\gamma_{k,i}$, i.e., $\gamma_{k,i} \triangleq P|h_{k,i}|^2$ and where we have written the inequality constraint in (4) as an equality in (8c) as the optimal $\{p_i^\star\}$ will make use of all available power.

III. Optimal power allocation

A. Preliminaries

In order to study and solve the optimization problem in (8), we will make use of the results that stem from the relation between the minimum mean-square error (MMSE) and the mutual information, [9], [10]. Consequently, we will now briefly review them.

Consider the generic scalar Gaussian channel $Y = \sqrt{\rho}S_i + W$. Given an observation of the channel output y, the MMSE estimation of S_i is given by the conditional mean

$$\hat{S}_i(\rho, y) = E[S_i | y = \sqrt{\rho}S_i + W], \tag{9}$$

where we have explicitly indicated the dependence of $\hat{S}_i$ on the channel gain ρ and the channel output y. For the sake of notation, this dependence will be dropped in the following.

We now define the following two quantities

$$\Phi_i(\rho, y) = E\big[|S_i - \hat{S}_i|^2 \big| y\big], \tag{10}$$

$$\overline{\Phi}_i(\rho, y) = E\big[(S_i - \hat{S}_i)^2 \big| y\big]. \tag{11}$$

It is straightforward to check that the corresponding mean-square error of the estimation in (9) is given by

$$\mathrm{MMSE}_i(\rho) = E[\Phi_i(\rho, y)], \tag{12}$$

where the expectation is with respect to the statistics of the channel output y. Note that $\mathrm{MMSE}_i(\rho) \in [0, 1]$ due to the unit power assumption on S_i.

Now we can recall the result, which will be instrumental in solving the optimal power allocation problem for non-Gaussian channel input distributions.

Lemma 1: For any input distribution of S_i we have

$$\dot{\mathcal{I}}_i(\rho) = E[\Phi_i(\rho, y)] = \mathrm{MMSE}_i(\rho), \tag{13}$$

$$\ddot{\mathcal{I}}_i(\rho) = -E\big[(\Phi_i(\rho, y))^2 + |\overline{\Phi}_i(\rho, y)|^2\big], \tag{14}$$

where we have used $\dot{f}(\rho) = df(\rho)/d\rho$.

Proof: Equation (13) is derived in [9] and (14), in [10, Eq. (110), arXiv version]. ■

Corollary 2: For the particular case of low power ($\rho = 0$), the derivatives of the mutual information in Lemma 1 read as

$$\dot{\mathcal{I}}_i(0) = \dot{\mathcal{I}}(0) = 1, \tag{15}$$

$$\ddot{\mathcal{I}}_i(0) = -\big(E\big[|S_i|^2\big]\big)^2 - \big|E\big[S_i^2\big]\big|^2 \tag{16}$$

$$= -1 - \big|E\big[S_i^2\big]\big|^2. \tag{17}$$

Proof: The proof follows from Lemma 1 and by noting that $\hat{S}_i(0, y) = E[S_i] = 0, \forall y$. In (15) and (17) we have used the unit power assumption for S_i, $E\big[|S_i|^2\big] = 1$. ■

It must be highlighted that the first derivative in (15) is a constant value and that the value of the second derivative in (16), which can be easily computed, depends only on the distribution of the input S_i.

Proper complex constellations (i.e., those that are quadrature symmetric, such as QPSK, 8-PSK or 16-QAM) fulfill $E\big[S_i^2\big] = 0$, as given in [11], which, from Corollary 2 implies that $\ddot{\mathcal{I}}_i^{\text{proper}}(0) = -1$. Alternatively, for real valued constellations (for example BPSK and m-PAM) we have that $E\big[S_i^2\big] = E\big[|S_i|^2\big] = 1$ and, thus, $\ddot{\mathcal{I}}_i^{\text{real}}(0) = -2$. Observe that both cases agree with the results given in [12]. Since $E\big[|S_i|^2\big] \geq \big|E\big[S_i^2\big]\big|$, for general (not necessarily proper) complex input distributions we will thus have

$$-2 \leq \ddot{\mathcal{I}}_i(0) \leq -1. \tag{18}$$

B. Problem solution

We are now ready to solve the problem stated in (8). First, we equivalently put it into the standard minimization form

$$\min_{\{p_i\}} \quad -\sum_{k=1}^{2}\sum_{i=1}^{n} \alpha_k \mathcal{I}_i(p_i \gamma_{k,i}) \tag{19a}$$

$$\text{s.t.} \quad -p_i \leq 0, \quad i \in [1, n] \tag{19b}$$

$$\sum_{i=1}^{n} p_i - 1 = 0. \tag{19c}$$

From (14), it can be easily seen that the mutual information is a concave function, $\ddot{\mathcal{I}}_i(\rho) < 0, \forall \rho$, which, together with the fact that the constraints in (19b) and (19c) are linear, implies that (19) is a convex optimization problem [13].

Introducing the Lagrange multipliers λ_i for the inequality constraints on each power p_i in (19b) and η for the equality constraint on the sum power in (19c), the Karush-Kuhn-Tucker conditions give us necessary and sufficient conditions for the optimal power allocation, which is denoted by $\{p_i^\star\}$, as

$$-p_i^\star \leq 0, \; i \in [1, n], \tag{20a}$$

$$\sum_{i=1}^{n} p_i^\star - 1 = 0, \tag{20b}$$

$$\lambda_i \geq 0, \; i \in [1, n], \tag{20c}$$

$$-\lambda_i p_i^\star = 0, \; i \in [1, n], \tag{20d}$$

$$\sum_{k=1}^{2} \alpha_k \gamma_{k,i} \mathrm{MMSE}_i(p_i^\star \gamma_{k,i}) + \lambda_i - \eta = 0, \; i \in [1, n]. \tag{20e}$$

Here, λ_i's act as slack variables and, thus, can be eliminated. The remaining following conditions can be rewritten as

$$p_i^\star \left[\eta - \sum_{k=1}^{2} \alpha_k \gamma_{k,i} \mathrm{MMSE}_i(p_i^\star \gamma_{k,i})\right] = 0, \; i \in [1, n], \tag{21a}$$

$$\sum_{i=1}^{n} p_i^\star - 1 = 0, \tag{21b}$$

$$-p_i^\star \leq 0, \; i \in [1, n], \tag{21c}$$

$$\sum_{k=1}^{2} \alpha_k \gamma_{k,i} \mathrm{MMSE}_i(p_i^\star \gamma_{k,i}) - \eta \leq 0, \; i \in [1, n]. \tag{21d}$$

Unfortunately, the optimal power allocation policy for this problem cannot be expressed in a simpler closed-form using a threshold value as in the case of waterfilling or mercury/waterfilling [14]. This is true even in the case of Gaussian inputs. Hence, we are not able to provide an intuitive graphical interpretation for the optimal power allocation in the general case and, rather, provide numerical results (see Section VI).

Nonetheless, in the special cases of low and high power regimes, the MMSE function can be approximated using a series expansion and closed form solutions for the optimal power allocation exist as shown in the following two sections.

IV. Low power regime ($P \to 0$)

Using a first order Taylor expansion and applying Corollary 2, the low power behavior of the MMSE function is given by

$$\mathrm{MMSE}_i(\rho) = \dot{\mathcal{I}}_i(0) + \ddot{\mathcal{I}}_i(0)\rho + \mathcal{O}(\rho^2) \quad (22)$$
$$= 1 + \ddot{\mathcal{I}}_i(0)\rho + \mathcal{O}(\rho^2), \quad (23)$$

similarly as it was done in [10], [14], [15]. We recall that, $\ddot{\mathcal{I}}_i(0)$ is a constant value that depends only on the input distribution used in the ith channel (see (17)).

With the approximation in (23), the optimal power allocation $\{p_i^\star\}$ that solves (19) can be put into the simpler form

$$p_i^\star = 0, \quad \text{if } \gamma_i^w \le \eta, \quad (24)$$
$$\gamma_i^w + \ddot{\mathcal{I}}_i(0) p_i^\star (\alpha_1 \gamma_{1,i}^2 + \alpha_2 \gamma_{2,i}^2) = \eta, \quad \text{if } \gamma_i^w > \eta, \quad (25)$$

where γ_i^w denotes the weighted channel quality defined as

$$\gamma_i^w \triangleq \alpha_1 \gamma_{1,i} + \alpha_2 \gamma_{2,i} = P(\alpha_1 |h_{1,i}|^2 + \alpha_2 |h_{2,i}|^2). \quad (26)$$

In the case of a single receiver, as studied in [14], the optimal policy in the asymptotic low power regime is to allocate all the power to the channel with the highest gain. If there are multiple channels with the strongest channel gain, then the power is allocated equally among them in the case of Gaussian inputs (waterfilling), or inversely proportional to the second derivative of the mutual information at zero power in the case of constrained input constellations.

A similar argument applies to the two-way relay channel model studied here as well. However, in our case we compare the weighted averages of the channel strengths of the two users as in (26) to determine the channel (or channels) to which positive power is allocated. All the power is allocated to the channel with the highest value for $\alpha_1|h_{1,i}|^2 + \alpha_2|h_{2,i}|^2$. If multiple channels have the same maximum value, indexed by $i \in \mathcal{M}$, then the power is divided as

$$\begin{array}{lll} p_i^\star & = \frac{\theta}{\ddot{\mathcal{I}}_i(0)}, & i \in \mathcal{M}, \\ p_i^\star & = 0, & i \notin \mathcal{M}, \end{array} \quad (27)$$

with $\theta = \left(\sum_{i\in\mathcal{M}} (\ddot{\mathcal{I}}_i(0))^{-1}\right)^{-1}$ being the normalization factor such that $\sum_{i=1}^n p_i^\star = 1$.

Note that the channels to which non-zero power is allocated depend on the objective function, i.e., α_1 and α_2. Hence, different channels can be chosen for different operating points on the boundary of the rate region. Once the channels for which the power is to be allocated are chosen, the power allocation is the same as in the case of a single receiver, and depends only on the constellation used in those channels.

V. High power regime ($P \to \infty$)

In the limit of high power, $\mathrm{MMSE}_i(\rho)$, in the case of Gaussian inputs, expands as [14]

$$\mathrm{MMSE}_i^{\mathcal{G}}(\rho) = \frac{1}{\rho} + \mathcal{O}(1/\rho^2).$$

Plugging this expansion in the KKT conditions in (21), we can see that the optimal power allocation behaves as

$$p_i^{\mathcal{G}\star} = \frac{1}{n} + \mathcal{O}(1/P),$$

which is the same behavior as in the case of a single receiver: for Gaussian inputs, in the high power regime, power is allocated uniformly among the channels.

In the case of discrete m-ary constellations, the high power behavior of $\mathrm{MMSE}_i(\rho)$ depends on the minimum distance, d_i, between any two points of the constellation used in ith channel. Then $\mathrm{MMSE}_i(\rho)$ decays exponentially as [14]

$$\mathrm{MMSE}_i(\rho) = K_i(\rho) \exp\left(-\frac{d_i^2}{4}\rho\right), \quad (28)$$

with $\mathcal{O}(1/\sqrt{\rho}) \le K_i(\rho) \le C_i$, where C_i is a constant value.

It can be shown that, in the high power regime, power is allocated to all the channels, i.e., $p_i^\star > 0$, $\forall i$, which further implies that the condition in (21d) is fulfilled with equality. With this result and plugging (28) into (21d) we obtain

$$\sum_{k=1}^{2} \alpha_k \gamma_{k,i} K_i(p_i^\star \gamma_{k,i}) \exp\left(-\frac{d_i^2}{4} p_i^\star \gamma_{k,i}\right) = \eta, \quad \forall i. \quad (29)$$

It is now easy to see that, as $P \to \infty$ and with $\alpha_1, \alpha_2 > 0$, the term with the smallest $\gamma_{k,i}$ dominates the sum in (29) obtaining, thus, the new condition, $\forall i$,

$$\alpha_{k_{\min}^{(i)}} \gamma_{k_{\min}^{(i)},i} K_i(p_i^\star \gamma_{k_{\min}^{(i)},i}) \exp\left(-\frac{d_i^2}{4} p_i^\star \gamma_{k_{\min}^{(i)},i}\right) = \eta, \quad (30)$$

where we have used $k_{\min}^{(i)} = \arg\min_k \{\gamma_{k,i}\}$. Observe that, if either $\alpha_1 = 0$ or $\alpha_2 = 0$, one of the terms in the summation in (29) disappears obtaining a similar expression as in (30).

We now apply the similar operations as in [14] to the expression in (30) and we obtain the optimal power allocation in the high power regime with discrete constellations:

$$p_i^\star = \frac{\theta}{d_i^2 \min_k |h_{k,i}|^2} + \mathcal{O}\left(\frac{\log P}{P}\right), \quad (31)$$

with $\theta = \left(\sum_{i=1}^n (d_i^2 \min_k |h_{k,i}|^2)^{-1}\right)^{-1}$.

Observe that the expression in (31) has been obtained assuming that both α_1 and α_2 are different from 0. Interestingly, under this assumption, the expression for the optimal power allocation in the high power regime in (31) is independent of the actual values of α_1 and α_2.

However, for the case where one of the α_k is equal to zero, the expression in (31) is no longer valid and is replaced by

$$p_i^\star = \frac{\theta}{d_i^2 |h_{\bar{k},i}|^2} + \mathcal{O}\left(\frac{\log P}{P}\right), \quad (32)$$

where $\bar{k}$ is such that $\alpha_{\bar{k}} > 0$. Note that (32) agrees with [14].

VI. Simulations

A. Numerical algorithm

Since the optimization problem in (19) is convex, various standard algorithms can be used to compute its optimal solution [13]. However, we believe that it will be useful to provide the actual numerical algorithm used in our simulations.

We have numerically solved the non-linear system of equations in (21a) and (21b) using Newton's method and projecting the solution obtained at each iteration so that the inequalities in (21c) and (21d) were fulfilled. Precisely, we have defined the vector argument $\mathbf{z} \in \mathbb{R}^{n+1}$ such that its elements fulfill

$$z_i = p_i, \quad i \in [1, n], \tag{33}$$

$$z_{n+1} = \eta. \tag{34}$$

Similarly, we have defined the vector function $\mathbf{f}(\mathbf{z}) \in \mathbb{R}^{n+1}$:

$$f_i(\mathbf{z}) = p_i \left[\eta - \sum_{k=1}^{2} \alpha_k \gamma_{k,i} \mathrm{MMSE}_i(p_i \gamma_{k,i}) \right], i \in [1, n],$$

$$f_{n+1}(\mathbf{z}) = \sum_{i=1}^{n} p_i - 1.$$

Starting with a given $\mathbf{z}^{(0)}$ and, until convergence is met, the iteration update is given by

$$\mathbf{z}^{(l+1)} = \mathbf{z}^{(l)} - (\nabla_{\mathbf{z}} \mathbf{f}(\mathbf{z}^{(l)}))^{-1} \mathbf{f}(\mathbf{z}^{(l)}), \tag{35}$$

$$\mathbf{z}^{(l+1)} = \mathbf{Proj}(\mathbf{z}^{(l+1)}), \tag{36}$$

where $\mathbf{Proj}$ projects its argument into the feasible set defined by (21c) and (21d).

The computation of $\nabla_{\mathbf{z}} \mathbf{f}$ can be done applying the results in Lemma 1 and elementary differentiation techniques. Observe that, as opposed to other methods proposed in [13], the main advantage of using Newton's method is that convergence is very fast as we are able to directly compute the gradient $\nabla_{\mathbf{z}} \mathbf{f}$.

B. Results

Firstly, we consider the case with $n = 3$ parallel channels with the following channel gains

$$|h_{1,1}|^2 = 2, \qquad |h_{1,2}|^2 = 1.2, \qquad |h_{1,3}|^2 = 0.2, \tag{37}$$

$$|h_{2,1}|^2 = 0.8, \qquad |h_{2,2}|^2 = 3, \qquad |h_{2,3}|^2 = 1.2, \tag{38}$$

and we assume that the input to each channel is from a QPSK constellation. For this scenario, we study the evolution of $\{p_i^\star\}_{i=1}^{i=3}$ as a function of the available power, P, and for different choices of α_k. The corresponding $\{p_i^\star\}_{i=1}^{i=3}$ are plotted in Figs. 3(a), 3(b), and 3(c) together with the limiting distribution as $P \to \infty$, given by (31). It can be seen that the limiting distribution for the power allocation does not depend on the specific values of α_k as long as both of them are nonzero (see Figs. 3(a) and 3(b)). It can also be observed that, whenever one $\alpha_k = 0$, the limiting distribution changes as it is now given by (32). Finally, from Figs. 3(a), 3(b), and 3(c) it is also clear that, as $P \to 0$ all the power is allocated to the channel with highest $\gamma_i^w = P(\alpha_1 |h_{1,i}|^2 + \alpha_2 |h_{2,i}|^2)$, which, for this particular case, corresponds to channel 2 in Figs. 3(a) and 3(a) and to channel 1 in Fig. 3(c).

Secondly, we consider the following two configurations:

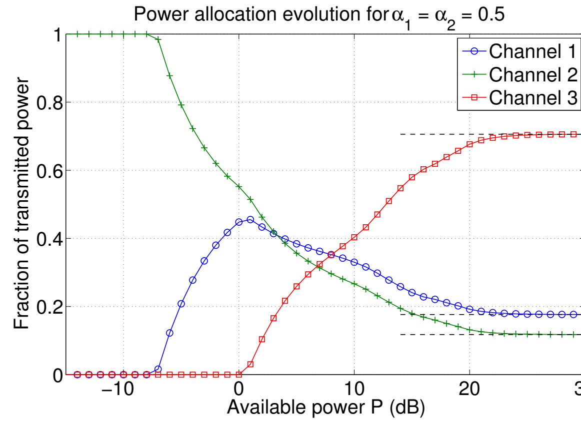
$$|h_{1,1}|^2 = 4, \qquad |h_{1,2}|^2 = 0.1, \qquad |h_{1,3}|^2 = 0.1, \tag{39}$$

$$|h_{2,1}|^2 = 2.1, \qquad |h_{2,2}|^2 = 1.9, \qquad |h_{2,3}|^2 = 0.1, \tag{40}$$

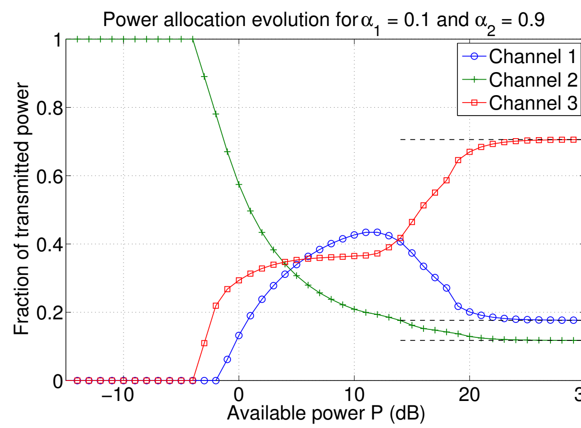
which will be referred to as *balanced* configuration because the best channel for the two users is the same (channel 1), and

$$|h_{1,1}|^2 = 0.1, \qquad |h_{1,2}|^2 = 0.1, \qquad |h_{1,3}|^2 = 4, \tag{41}$$

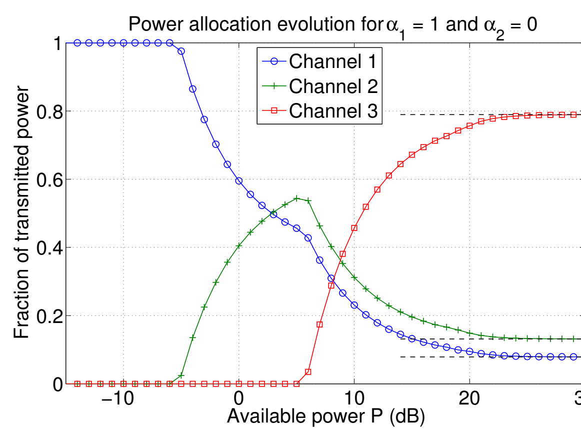
$$|h_{2,1}|^2 = 2.1, \qquad |h_{2,2}|^2 = 1.9, \qquad |h_{2,3}|^2 = 0.1, \tag{42}$$



(a) Case where $\alpha_1 = \alpha_2 = 0.5$. This implies that $\gamma_1^w = 1.4P$, $\gamma_2^w = 2.1P$, and $\gamma_3^w = 0.7P$, which means that, in the low power regime, all the power is allocated to channel 2. The limiting distribution as $P \to \infty$ is given by $p_1^\star \approx 0.18$, $p_2^\star \approx 0.12$, and $p_3^\star \approx 0.7$.



(b) Case where $\alpha_1 = 0.1$ and $\alpha_2 = 0.9$. This implies that $\gamma_1^w = 0.9P$, $\gamma_2^w = 2.8P$, and $\gamma_3^w = 1.1P$, which means that, in the low power regime, all power is allocated to channel 2. Observe that the limiting power allocation is the same as in Fig. 3(a).



(c) Case where $\alpha_1 = 1$ and $\alpha_2 = 0$. This implies that $\gamma_1^w = 2P$, $\gamma_2^w = 1.2P$, and $\gamma_3^w = 0.2P$, which means that, in the low power regime, all the power is allocated to channel 1. Observe that the limiting power allocation ($p_1^\star \approx 0.08$, $p_2^\star \approx 0.12$, and $p_3^\star \approx 0.8$) is different than in Figs. 3(a) and 3(b) as $\alpha_2 = 0$.

Fig. 3. Evolution of the optimal fraction of power assigned to each channel, $p_i^\star$, as a function of the transmission available power, P.

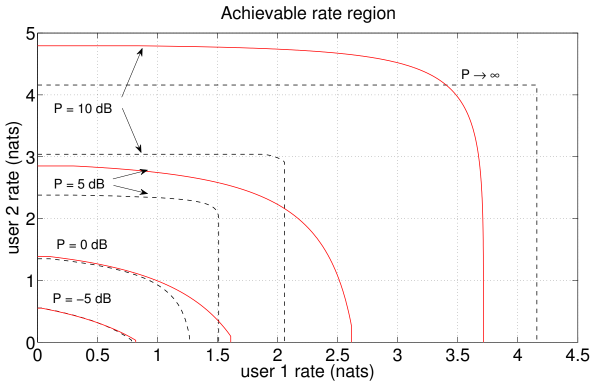


Fig. 4. Achievable rate regions for different power constraints for the cases of Gaussian (red, solid) and QPSK (black, dashed) input distributions for a fixed channel with $n = 3$.

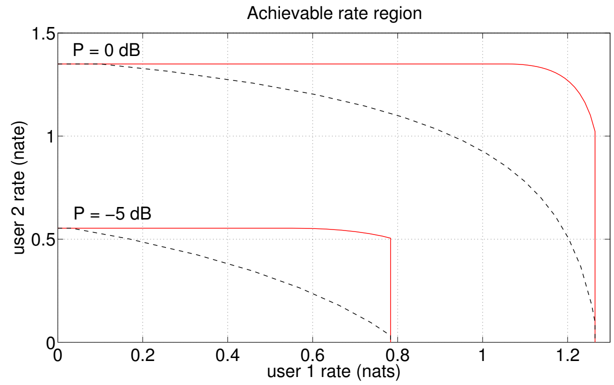


Fig. 5. Achievable rate regions for different power constraints for QPSK input distribution and two different channel configurations: balanced (red, solid) and unbalanced (black, dashed).

which will be referred to as *unbalanced* configuration because the best channels for the two users are different (channel 3 for user 1 and channel 1 for user 2). Observe that the only difference between the two channels is that, in the unbalanced case, the values of $|h_{1,1}|^2$ and $|h_{1,3}|^2$ are swapped.

For the unbalanced configuration, we plot, in Fig. 4, the achievable rate regions for the case of Gaussian and QPSK inputs and for different P values. It can be observed that, as P increases, the Gaussian rate region is always larger and grows faster than the QPSK region. Moreover, it can be observed that as $P \to \infty$ the QPSK region saturates as the mutual information with an input with 3 QPSK constellations is upper bounded by $\log 2^6 \approx 4.16$ nat. At low power, both regions coincide providing yet another evidence of the excellent performance of QPSK in the low power regime.

In Fig. 5, we compare the achievable rate regions for the balanced and unbalanced configurations. It can be seen that the unbalanced region is always included inside the balanced region. Moreover, it must be highlighted that the borders of both regions coincide near the axes because this situation corresponds to the case where the information flow to only one of the users is maximized, thus, the achievable rate is a function of the channel gains of that user, independently of the order of these channels (since the input is QPSK for all the channels).

VII. Conclusions

We have considered DF relaying in a two-way relay channel with independent parallel Gaussian links from the relay to the users. Focusing on the second phase of transmission, we have studied the optimal power allocation policy at the relay terminal to achieve the boundary points of the rate region by considering arbitrary input constellations. We have identified the necessary and sufficient conditions for the optimal power allocation policy and provided numerical solutions. Unfortunately, the well-known waterfilling or mercury/waterfilling interpretations do not apply in this setup due to the weighting in the objective function. We have also considered the low and the high power regimes. In the low power regime, all the power is allocated to the channels with the highest weighted channel quality, whereas in the high power regime, the power is allocated uniformly in the case of Gaussian inputs, and according to the minimum distances and the minimum of the channel qualities in the case of discrete constellations.

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