

Uncoded Transmission of Correlated Gaussian Sources Over Broadcast Channels With Feedback

Yonathan Murin

Ben-Gurion University, Israel
moriny@ee.bgu.ac.il

Yonatan Kaspi

UCSD, San Diego, USA
ykaspi@eng.ucsd.edu

Ron Dabora

Ben-Gurion University, Israel
ron@ee.bgu.ac.il

Deniz Gündüz

Imperial College London, UK
d.gunduz@imperial.ac.uk

Abstract—Motivated by the practical requirement for delay and complexity constrained broadcasting, we study uncoded transmission of a pair of correlated Gaussian sources over a two-user Gaussian broadcast channel with unit-delay noiseless feedback links (GBCF). Differently from previous works, in the present work we focus on the finite horizon regime. We present two joint source-channel coding schemes, one is based on the Ozarow-Leung (OL) coding scheme for the GBCF and the other is based on the linear quadratic Gaussian (LQG) code by Ardestanizadeh et al. Our LQG-oriented code uses an improved decoder which outperforms the original decoder of Ardestanizadeh et al. in the finite horizon regime. We further derive lower and upper bounds on the minimal number of channel uses needed to achieve a specified pair of distortion levels for each scheme, and using these bounds, we explicitly characterize a range of transmit powers in which the OL code outperforms the LQG-oriented code.

I. INTRODUCTION

We study the transmission of a pair of correlated Gaussian sources over a two-user Gaussian broadcast channel (GBC) with correlated noises at the receivers in the presence of noiseless causal feedback (FB) from both receivers to the transmitter. Motivated by practical broadcast scenarios with strict delay constraints, e.g., live multimedia broadcast, we focus on linear uncoded transmission, namely, we do not consider source coding over sequences of source symbol pairs. We further assume that each pair of source symbols is transmitted using a finite number of channel symbols. Our objective is to characterize the minimal number of channel symbol transmissions needed to achieve a target non-zero distortion pair.

Previous works on GBCs with noiseless FB (GBCF) links mainly considered independent messages at the transmitter, i.e., the channel coding problem, [2]–[8]: In [2] the Schalkwijk-Kailath (SK) scheme of [1] was extended to the two-user GBCF with two independent messages, and achieved reliable communications at rates which are outside the capacity region of the GBC without FB. The scheme of [2] was later extended to GBCs with more than two users as well as to interference channels (ICs) in [3].

Another approach to the channel coding problem for the GBCF is based on control theory. Elia [4] derived a (linear) code for the two-user GBCF with *independent noises* at the receivers which outperforms the scheme of [2]. Then, using tools from the theory of linear quadratic Gaussian (LQG)

control, the work [5] removed the restriction of independent noises in [4], and studied the case of more than two users. For independent noises, the scheme of [5] was shown to outperform the schemes of [2] and [3] in terms of achievable rates. Furthermore, in [6] it was shown that for independent and equal noise variances, the scheme of [5] achieves the largest sum-rate among all linear-FB schemes.

In [7], GBCs and Gaussian ICs with noiseless FB were considered and a scheme that achieves a sum-rate which approaches the full-cooperation bound, as the signal-to-noise ratio (SNR) increases to infinity, was presented. Finally, the work [8] presented a non-linear FB scheme which is optimal for the two-user GBCF and only a common message. In this work, however, we focus on linear schemes.

While the works [1]–[8] focused on channel coding problems, the work [9] studied the transmission of correlated Gaussian sources over the two-user multiple-access channel (MAC) with noiseless FB, via the uncoded scheme developed in [10]. The work [9] also established an upper bound on the energy-distortion tradeoff for the symmetric scenario.

All previous studies on GBCFs applied an infinite horizon analysis, i.e., the results are valid as the number of channel uses increases to infinity. In the present work we study lossy joint source-channel coding (JSCC) for GBCFs, thus, our focus is on the finite horizon regime. Specifically, we implement uncoded transmission based on two coding schemes: The scheme developed in [2], which is referred to as OL (Ozarow-Leung), and the scheme derived in [5], which is referred to as LQG. We apply both schemes for the transmission of correlated Gaussian sources over the GBCF and derive bounds on the number of channel symbol transmissions needed to achieve a target non-zero distortion pair. Our main contributions are as follows:

Main Contributions: We derive a new decoder for the LQG-oriented scheme based on the minimum mean-square error (MMSE) criterion, which, in the finite horizon regime outperforms the LQG decoder presented in [5]. These two LQG-decoders are shown to be equivalent in the infinite horizon regime. We then derive lower and upper bounds on the minimal number of channel uses needed to achieve a specified pair of distortion levels for the OL-based scheme and the LQG-oriented scheme (for the LQG we use our new decoder). Finally, for the symmetric case with independent noises we explicitly characterize a range of transmit powers for which OL outperforms LQG. This result is in contrast to the infinite horizon regime in which LQG strictly outperforms OL.

This work was partially supported by the European Commission's Marie Curie IRG Fellowship PIRG05-GA-2009-246657 under the Seventh Framework Programme, and by the Israel Science Foundation under grant 396/11.

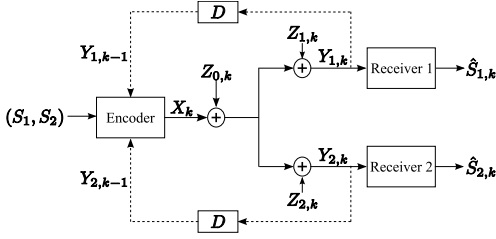


Fig. 1: Gaussian broadcast channel with correlated sources and feedback links. $\hat{S}_{1,k}, \hat{S}_{2,k}$ are the reconstructions of S_1, S_2 , respectively, after the k 'th transmission, and D represents a unit delay.

The rest of this paper is organized as follows: The problem formulation is introduced in Section II. The OL and LQG schemes, along with the improved LQG decoder, are presented in Section III in the context of JSCC, and the finite horizon analysis is presented in Section IV. A discussion and a numerical example are given in Section V and conclusions appear in Section VI.

Notations: We use capital letters to denote random variables (RVs), e.g., X , boldface letters to denote column vectors, e.g., \mathbf{X} and sans-serif font to denote matrices, e.g., \mathbf{M} . $[\mathbf{M}]_{m,n}$ denotes the entry at the m 'th row and n 'th column of a matrix \mathbf{M} and $\det(\mathbf{M})$ denotes its determinant. We use $\mathbb{E}\{\cdot\}$, $(\cdot)^T$, $\log(\cdot)$ and \Re to denote expectation, transpose, natural basis logarithm, and the set of real numbers, respectively. Lastly, define $[x]^+ \triangleq \max\{x, 0\}$, let $\text{sgn}(x)$ to be the sign of x , where $\text{sgn}(0) \triangleq 1$, and denote the ceiling function of x by $\lceil x \rceil$.

II. PROBLEM FORMULATION

The two-user GBCF is depicted in Fig. 1. The encoder observes a realization of a jointly Gaussian correlated pair of sources denoted by $\mathbf{S} = (S_1, S_2)$, and is required to send the source $S_i, i = 1, 2$, to the i 'th receiver. Let $\mathbf{S} \sim \mathcal{N}(0, \mathbf{Q}_s)$, where $\mathbf{Q}_s = \begin{bmatrix} \sigma_1^2 & \rho_s \sigma_1 \sigma_2 \\ \rho_s \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$, $\rho_s = \frac{\mathbb{E}\{S_1 S_2\}}{\sigma_1 \sigma_2}$, $|\rho_s| < 1$. Each pair of source symbols is transmitted using K channel uses indexed by $k = 1, 2, \dots, K$. The channel outputs at the decoders are given by $Y_{i,k} = X_k + Z_{0,k} + Z_{i,k}, i = 1, 2$, where $Z_{j,k} \sim \mathcal{N}(0, \sigma_{z,j}^2), j = 0, 1, 2$, are i.i.d for $k = 1, 2, \dots, K$, and mutually independent of each other. Let $\mathbf{B} \triangleq [1, 1]^T$, $\mathbf{Y}_k \triangleq [Y_{1,k}, Y_{2,k}]^T$ and $\mathbf{Z}_k \triangleq [Z_{0,k} + Z_{1,k}, Z_{0,k} + Z_{2,k}]^T$. Note that $\mathbf{Z}_k \sim \mathcal{N}(0, \mathbf{Q}_z)$, where $\mathbf{Q}_z = \mathbb{E}\{\mathbf{Z}_k \mathbf{Z}_k^T\}$. Thus, we have the following vector representation for the signal model:

$$\mathbf{Y}_k = \mathbf{B}X_k + \mathbf{Z}_k, \quad k = 1, 2, \dots, K. \quad (1)$$

The i 'th receiver, $i = 1, 2$, uses the first k channel outputs, $\mathbf{Y}_{i,1}^k$, to estimate S_i : $\hat{S}_{i,k} = \phi_{i,k}(\mathbf{Y}_{i,1}^k), \phi_{i,k} : \Re^k \rightarrow \Re, k = 1, 2, \dots, K$. The encoder has access to a noiseless causal channel output feedback from both receivers. At time k , the encoder maps the observed pair of sources and the received noiseless FB into a channel input via $X_k = f_k(S_1, S_2, \mathbf{Y}_{1,1}^{k-1}, \mathbf{Y}_{2,1}^{k-1}), f_k : \Re^{2k} \rightarrow \Re$. The transmitted signal is subject to a per-symbol average power constraint defined as:

$$\mathbb{E}\{X_k^2\} \leq P, \quad \forall k = 1, 2, \dots, K. \quad (2)$$

For a specific set of parameters $(P, \sigma_{z,0}^2, \sigma_{z,1}^2, \sigma_{z,2}^2, \sigma_1^2, \sigma_2^2, \rho_s)$, we define a (D_1, D_2, K) code to be a collection of K encoding

functions that satisfy (2), and two decoders such that the MSEs of the received source symbols satisfy:

$$\mathbb{E}\{(S_i - \hat{S}_{i,K})^2\} \leq D_i, \quad 0 < D_i \leq \sigma_i^2, \quad i = 1, 2. \quad (3)$$

Our objective is to characterize the minimal number of channel uses K for a given distortion pair (D_1, D_2) such that a (D_1, D_2, K) code exists.

Next, we recall some results and definitions from [2] and [5], and present an improved decoder for the LQG scheme.

III. THE OL AND THE LQG SCHEMES

A. The OL Scheme

In the OL scheme, prior to the transmission of a channel symbol, the transmitter calculates the estimations at the receivers using the causal noiseless FB, from which it computes the estimation errors. The transmitter then sends a linear combination of these estimation errors. Thus, each receiver obtains its estimation error corrupted by a correlated noise term, consisting of the other receiver's error, and additive noise. Each receiver then updates its estimation accordingly, thereby, decreasing the variance of its estimation error. The scheme is terminated after K channel uses, where K is chosen such that the target MSE for each source is achieved at the corresponding receiver.

Definitions: Define $\hat{S}_{i,k}$ to be the estimate of S_i at the i 'th receiver after the reception of the k 'th channel output $Y_{i,k}$. Let $\epsilon_{i,k} \triangleq \hat{S}_{i,k} - S_i$ be the estimation error after k transmissions, and define $\hat{\epsilon}_{i,k-1} \triangleq \hat{S}_{i,k-1} - \hat{S}_{i,k}$, which implies that $\epsilon_{i,k} = \epsilon_{i,k-1} - \hat{\epsilon}_{i,k-1}$. Further define $\alpha_{i,k} \triangleq \mathbb{E}\{\epsilon_{i,k}^2\}$ to be the MSEs after k transmissions, $\rho_k \triangleq \frac{\mathbb{E}\{\epsilon_{1,k} \epsilon_{2,k}\}}{\sqrt{\alpha_{1,k} \alpha_{2,k}}}$ to be the correlation between the estimation errors, and $\Psi_{k-1} \triangleq \sqrt{\frac{P}{1+g^2+2g|\rho_k|}}$.

Encoding: Set $\epsilon_{i,0} = S_i, \alpha_{i,0} = \mathbb{E}\{\epsilon_{i,0}^2\} = \sigma_i^2$ and $\hat{S}_{i,0} = 0$. It follows that $\rho_0 = \rho_s$. Next, let $g > 0$ be a constant which facilitates a tradeoff between the information rates to receivers 1 and 2. At the k 'th iteration, $1 \leq k \leq K$, the transmitter sends $X_k = \Psi_{k-1} \left(\frac{\epsilon_{1,k-1}}{\sqrt{\alpha_{1,k-1}}} + \frac{\epsilon_{2,k-1}}{\sqrt{\alpha_{2,k-1}}} \cdot g \cdot \text{sgn}(\rho_{k-1}) \right)$, and the corresponding channel outputs are given by (1).

Decoding: The receivers estimate $\epsilon_{i,k-1}, i = 1, 2$, based only on $Y_{i,k}$: $\hat{\epsilon}_{i,k-1} = \frac{\mathbb{E}\{\epsilon_{i,k-1} Y_{i,k}\}}{\mathbb{E}\{Y_{i,k}^2\}} Y_{i,k}$. Let $\pi_i \triangleq P + \sigma_{z,0}^2 + \sigma_{z,i}^2$. The variances of $\epsilon_{i,k}$ are given by the recursive expressions [2, Eqs. (5)–(6)]:

$$\alpha_{1,k} = \alpha_{1,k-1} \frac{\sigma_{z,0}^2 + \sigma_{z,1}^2 + \Psi_{k-1}^2 g^2 (1 - \rho_{k-1}^2)}{\pi_1} \quad (4a)$$

$$\alpha_{2,k} = \alpha_{2,k-1} \frac{\sigma_{z,0}^2 + \sigma_{z,2}^2 + \Psi_{k-1}^2 (1 - \rho_{k-1}^2)}{\pi_2}, \quad (4b)$$

with a recursive expression for ρ_k given in [2, Eqn. (7)].

Remark 1. The OL scheme described above does not apply the initialization procedure described in [2, pg. 669], since when only a finite number of channel uses is allowed it might result in higher MSE. Instead we set $\epsilon_{i,0} = S_i$ and $\rho_0 = \rho_s$ in order to take advantage of the correlation among the sources.

B. The LQG scheme

The LQG scheme of [5] is based on a mapping from the noiseless FB control problem to a linear code for the

GBCF. The asymptotic performance of this scheme is determined by the eigenvalues of the open-loop matrix of a linear system. These eigenvalues are determined by the minimal power required to stabilize the system using the noiseless FB. Similarity to the OL scheme, the considered LQG scheme is also terminated after K channel uses.

Consider a two-dimensional unstable dynamical system which is stabilized by a controller which observes the entire system state vector, $\mathbf{U}_k = [U_{1,k}, U_{2,k}]^T$. The controller outputs a scalar signal, $X_k = f_k(\mathbf{U}_k^T)$, $f_k : \mathfrak{R}^{2k} \rightarrow \mathfrak{R}$. The control signal is received at the system via a noisy channel. Let \mathbf{Y}_k denote the noisy observed control signal at the output of the channel at time k , given in (1). Let $\mathbf{A} = \text{diag}(a_1, a_2)$, where $a_i \in \mathfrak{R}$, $a_1 \neq a_2$, $|a_i| > 1$, and let $\mathbf{S} = [S_1, S_2]^T$. The system state vector at time k , \mathbf{U}_k , is recursively given by:

$$\mathbf{U}_1 = \mathbf{S}, \quad \mathbf{U}_k = \mathbf{A}\mathbf{U}_{k-1} + \mathbf{Y}_{k-1}, \quad k = 2, 3, \dots, K. \quad (5)$$

In the corresponding communication problem the encoder consists of the system given in (5) and of the controller. At each time index the encoder recursively computes \mathbf{U}_k and transmits X_k . In this work we use the controller that minimizes the asymptotic average power which is presented in [5, Lemma 4]. This is motivated by: 1) The approximate optimality of this controller for large K 's (alternatively low D_i 's), and 2) This choice results in a linear time-invariant scheme. The output of the linear controller is given by $X_k = -\mathbf{C}^T \mathbf{U}_k$, where $\mathbf{C} = [c_1, c_2]^T$ is given by $\mathbf{C} = (\mathbf{B}^T \mathbf{G} \mathbf{B} + 1)^{-1} \mathbf{A} \mathbf{G}^T \mathbf{B}$, and \mathbf{G} is the unique positive-definite solution of the discrete algebraic Riccati equation $\mathbf{G} = \mathbf{A}^T \mathbf{G} \mathbf{A} - \mathbf{A}^T \mathbf{G} \mathbf{B} (\mathbf{B}^T \mathbf{G} \mathbf{B} + 1)^{-1} \mathbf{B}^T \mathbf{G} \mathbf{A}$, such that all the eigenvalues of the matrix $\mathbf{A} - \mathbf{B} \mathbf{C}^T$ have magnitudes smaller than 1. From [5, Lemma 4] it follows that for this controller, the covariance matrix of \mathbf{U}_k , $\mathbf{Q}_{u,k}$, converges as $k \rightarrow \infty$ to \mathbf{Q}_u , which is the solution of $\mathbf{Q}_u = (\mathbf{A} - \mathbf{B} \mathbf{C}^T) \mathbf{Q}_u (\mathbf{A} - \mathbf{B} \mathbf{C}^T)^T + \mathbf{Q}_z$ within the class of positive semidefinite matrices. Finally, from [5, Lemma 4] it follows that the minimum asymptotic average power of this controller is given by $P(\mathbf{A}, \mathbf{Q}_z) = \mathbf{C}^T \mathbf{Q}_u \mathbf{C} = \text{trace}(\mathbf{G} \mathbf{Q}_z)$.

The work [5] used a decoder which follows the zero trajectory. First it estimates $U_{i,k}$ via:

$$\hat{U}_{i,1} = 0, \quad \hat{U}_{i,k} = a_i \hat{U}_{i,k-1} + Y_{i,k-1}, \quad k = 2, 3, \dots, K. \quad (6)$$

Then it estimates S_i from $\hat{U}_{i,k+1}$ via:

$$\hat{S}_{i,k} = -a_i^{-k} \hat{U}_{i,k+1}, \quad (7)$$

thus, resulting in the MSE: $\mathbb{E}\{(S_i - \hat{S}_{i,k})^2\} = a_i^{-2k} \mathbb{E}\{U_{i,k+1}^2\}$.

C. An Improved LQG Decoder

The estimation (7) is clearly suboptimal in the MMSE sense. The optimal estimator of S_i , based on the observation $\hat{U}_{i,k+1}$, obtained as in (6), is stated in the following theorem:

Theorem 1. Let $\mathbf{M} \triangleq \mathbf{A} - \mathbf{B} \mathbf{C}^T$ and $\mathbf{Q}_{u,k} \triangleq \mathbb{E}\{\mathbf{U}_k \mathbf{U}_k^T\}$ with $\mathbf{Q}_{u,1} = \mathbf{Q}_s$. The optimal estimator of S_i , $i = 1, 2$, at time k , based on the observation $\hat{U}_{i,k+1}$, given in (6), is:

$$\hat{S}_{i,k} = \frac{[\mathbf{M}^k \mathbf{Q}_s]_{i,i} - \sigma_i^2 a_i^k}{[\mathbf{Q}_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k \mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}} \hat{U}_{i,k+1}. \quad (8)$$

Furthermore, the MSE of $\hat{S}_{i,k}$ is given by:

$$\mathbb{E}\{(S_i - \hat{S}_{i,k})^2\} = \frac{\sigma_i^2 [\mathbf{Q}_{u,k+1}]_{i,i} - ([\mathbf{M}^k \mathbf{Q}_s]_{i,i})^2}{[\mathbf{Q}_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k \mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}}, \quad (9)$$

and as $k \rightarrow \infty$ the MSE expression (9) converges to the MSE of the decoder in (7).

Proof: The proof is detailed in Appendix A. \blacksquare

Remark 2. Since the estimator (8) is the optimal estimator of S_i , based on the observation $\hat{U}_{i,k+1}$, it clearly achieves an MSE smaller than or equal to the MSE achieved by the estimator (7). In particular, (8) outperforms (7) in the finite horizon regime, i.e., for large distortions, see Section V.

IV. FINITE HORIZON ANALYSIS

The considered OL and LQG-oriented schemes terminate after K channel uses, where K is chosen such that the target distortions $0 < D_i$, $i = 1, 2$, are achieved for both sources. We now analyze the finite horizon performance of these schemes.

A. Finite Horizon Analysis of LQG

In contrast to the OL scheme which has a constant average instantaneous transmission power, the average instantaneous transmission power in the LQG scheme, P_k , changes over time and converges to P as $k \rightarrow \infty$. In general P_k may be larger than P and (2) may not be satisfied. This implies that for a specific value of P and for specific noise variances there are pairs of sources which cannot be transmitted using the LQG-oriented scheme of Subsection III-B. Let K_{LQG} denote the minimal number of channel uses for achieving a target distortion pair (D_1, D_2) by the LQG-oriented scheme with the decoder (8) (regardless whether (2) is satisfied or not). Thm. 2 below provides an upper bound on K_{LQG} , denoted by $K_{\text{LQG}}^{\text{ub}}$. It follows that (2) needs to be verified only for $k \leq K_{\text{LQG}}^{\text{ub}}$. This can be done numerically using a finite number of calculations.

Let $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2)$ be a diagonal matrix of the eigenvalues of \mathbf{M} , and $\mathbf{V} \triangleq \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ be a 2×2 matrix whose columns are the corresponding eigenvectors of \mathbf{M} . We now have the following theorem:

Theorem 2. Let $\eta_1 \triangleq \frac{\sigma_1 (|v_1 v_4 \lambda_1| + |v_2 v_3 \lambda_2|) + |\rho_s \sigma_2 v_1 v_2| (|\lambda_2| + |\lambda_1|)}{|\det(\mathbf{V})|}$, $\eta_2 \triangleq \frac{|\sigma_1 v_3 v_4| (|\lambda_1| + |\lambda_2|) + |\rho_s \sigma_2| (|v_1 v_4 \lambda_2| + |v_2 v_3 \lambda_1|)}{|\det(\mathbf{V})|}$, $\eta_3 \triangleq \frac{\sigma_2 \sqrt{1 - \rho_s^2} (|v_1 v_2| (|\lambda_2| + |\lambda_1|))}{|\det(\mathbf{V})|}$, and $\eta_4 \triangleq \frac{\sigma_2 \sqrt{1 - \rho_s^2} (|v_1 v_4 \lambda_2| + |v_2 v_3 \lambda_1|)}{|\det(\mathbf{V})|}$. Furthermore, define $\vartheta_1 \triangleq \eta_1^2 + \eta_3^2 + [\mathbf{Q}_u]_{1,1}$ and $\vartheta_2 \triangleq \eta_2^2 + \eta_4^2 + [\mathbf{Q}_u]_{2,2}$. Then,

$$K_{\text{LQG}}^{\text{ub}} \triangleq \left\lceil \max \left\{ \frac{\left[\log \left(\frac{\vartheta_1}{D_1} \right) \right]^+}{2 \log |a_1|}, \frac{\left[\log \left(\frac{\vartheta_2}{D_2} \right) \right]^+}{2 \log |a_2|} \right\} \right\rceil, \quad (10)$$

is an upper bound on K_{LQG} .

Proof: The proof is detailed in Appendix B. \blacksquare

Next, we present a lower bound on K_{LQG} . This is summarized in the following theorem:

Theorem 3. Let $\beta_1 \triangleq \frac{\sigma_1^2 (|v_1 v_4 \lambda_1| + |v_2 v_3 \lambda_2|) + |\rho_s \sigma_1 \sigma_2 v_1 v_2| (|\lambda_2| + |\lambda_1|)}{|\det(\mathbf{V})|}$, $\beta_2 \triangleq \frac{\sigma_2^2 (|v_1 v_4 \lambda_2| + |v_2 v_3 \lambda_1|) + |\rho_s \sigma_1 \sigma_2 v_3 v_4| (|\lambda_2| + |\lambda_1|)}{|\det(\mathbf{V})|}$, $\mu_1 \triangleq \frac{[\sigma_1^2 [\mathbf{Q}_z]_{1,1} - \beta_1^2 - D_1 [\mathbf{Q}_z]_{1,1}]^+}{(2\beta_1 + \sigma_1^2) D_1}$ and $\mu_2 \triangleq \frac{[\sigma_2^2 [\mathbf{Q}_z]_{2,2} - \beta_2^2 - D_2 [\mathbf{Q}_z]_{2,2}]^+}{(2\beta_2 + \sigma_2^2) D_2}$. Then,

$$K_{\text{LQG}}^{\text{lb}} \triangleq \left\lceil \max \left\{ \frac{[\log(\mu_1)]^+}{2 \log |a_1|}, \frac{[\log(\mu_2)]^+}{2 \log |a_2|} \right\} \right\rceil, \quad (11)$$

is a lower bound on K_{LQG} .

Proof: The proof is detailed in Appendix C. ■

B. Finite Horizon Analysis of OL

The following theorem presents upper and lower bounds on K_{OL} , the minimal number of channel uses required to achieve distortion pair (D_1, D_2) with the OL scheme:

Theorem 4. Define

$$K_{\text{OL}}^{\text{ub}} = \left\lceil \frac{(1+g^2)}{P} \max \left\{ \pi_1 \log \left(\frac{\sigma_1^2}{D_1} \right), \frac{\pi_2}{g^2} \log \left(\frac{\sigma_2^2}{D_2} \right) \right\} \right\rceil, \quad (12a)$$

$$K_{\text{OL}}^{\text{lb}} = \left\lceil \max \left\{ \frac{\pi_1 - P}{P} \log \left(\frac{\sigma_1^2}{D_1} \right), \frac{\pi_2 - P}{P} \log \left(\frac{\sigma_2^2}{D_2} \right) \right\} \right\rceil. \quad (12b)$$

Then $K_{\text{OL}}^{\text{lb}} \leq K_{\text{OL}} \leq K_{\text{OL}}^{\text{ub}}$.

Proof: The proof is detailed in Appendix D. ■

V. A DISCUSSION AND A NUMERICAL EXAMPLE

The bounds presented in Thm. 3 and Thm. 4 help in answering the question whether the considered schemes can be used to communicate a given pair of sources over a specific GBCF within strict delay constraints.

Remark 3. Recall that in the infinite horizon regime LQG outperforms OL, see [5, Subsection V.A]. However, since the LQG scheme is time invariant while the OL scheme is time varying, one may expect that in the finite horizon regime OL may outperform LQG. To check this conjecture, we sought for power constraint value P for which $K_{\text{OL}}^{\text{ub}} < K_{\text{LQG}}^{\text{lb}}$ which would imply that $K_{\text{OL}} < K_{\text{LQG}}$, i.e., OL would outperform LQG. Using Thm. 3 and Thm. 4, the following corollary characterizes a region of power constraint values P for which OL outperforms LQG, for the symmetric case with independent noises:

Corollary 1. Consider a symmetric case: $D_1 = D_2, \sigma_1^2 = \sigma_2^2, \sigma_{z,1}^2 = \sigma_{z,2}^2 = \sigma_z^2, \sigma_{z,0}^2 = 0$, and $g = 1$. Let $0 < \delta \leq 4$, and define $\varphi_0(P, \delta) \triangleq \frac{4P^2}{(4+\delta)\sigma_z^4 \sqrt{1 + \frac{2P}{(4+\delta)\sigma_z^2}}}$, $\varphi_1(P, \delta) \triangleq \frac{(1 + \frac{P}{2\sigma_z^2})^{-1}}{2(1 + \frac{2P}{(4+\delta)\sigma_z^2})^{\frac{3}{2}}}$, $\varphi_2(P) \triangleq \frac{P^2}{8\sigma_z^4 (1 + \frac{2P}{(4+\delta)\sigma_z^2})}$ and $\tau(P, \delta) \triangleq \frac{\sigma_1^2 \left(\left(1 + \frac{P}{2\sigma_z^2}\right) \varphi_1^2(P, \delta) + \varphi_2^2(P, \delta) + 2|\rho_s| \varphi_1(P, \delta) \varphi_2(P, \delta) \sqrt{1 + \frac{P}{2\sigma_z^2}} \right)}{\varphi_0(P, \delta)}$.

If the following conditions hold:

$$\frac{P}{\sigma_z^2} \log \left(\frac{\sigma_1^2}{D_1} \right) < \left\lceil \log \left(\frac{[\sigma_z^2(\sigma_1^2 - D_1) - \tau^2(P, \delta)]^+}{\sigma_1^2(\sigma_1^2 + 2\tau(P, \delta))} \right) \right\rceil, \quad (13a)$$

$$P < \frac{1}{4} \delta (4 + \delta) \sigma_z^2 + \sqrt{\frac{1}{4} \delta ((4 + \delta) \sigma_z^2)^2 \left(1 + \frac{\delta}{4}\right)}, \quad (13b)$$

then $K_{\text{OL}} < K_{\text{LQG}}$.

Proof: The proof is detailed in Appendix E. ■

Remark 4. Corollary 1 implies that for the symmetric case with independent noises, if the target distortion D_1 is large enough and if P is sufficiently small, then the OL scheme can outperform the LQG scheme.

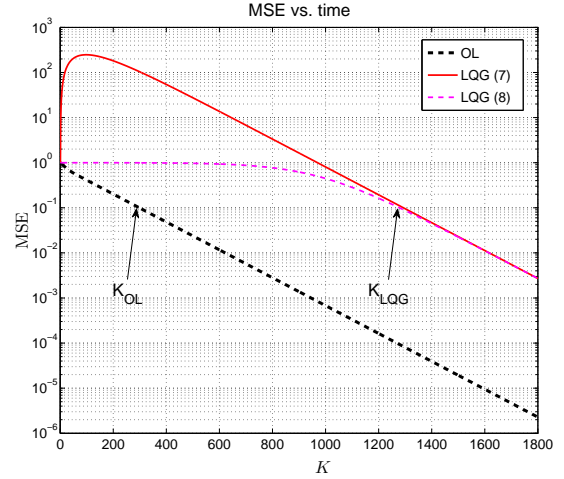


Fig. 2: MSE vs. k for $\sigma_1^2 = \sigma_2^2 = 1, \rho_s = 0.9, \sigma_{z,0}^2 = 0, \sigma_{z,1}^2 = \sigma_{z,2}^2 = 7, g = 1$ and $P = 0.1$.

Lastly, we demonstrate our results via a numerical example: Consider the transmission of a pair of Gaussian sources with $\sigma_1^2 = \sigma_2^2 = 1, \rho_s = 0.9$, over a GBCF with $\sigma_{z,0}^2 = 0, \sigma_{z,1}^2 = \sigma_{z,2}^2 = 7$ and $g = 1$. Furthermore, let $P = 0.1$ and $D_1 = D_2 = 0.1$. For these parameters (10) is evaluated to be $K_{\text{LQG}}^{\text{ub}} = 1293$, (11) is evaluated to be $K_{\text{LQG}}^{\text{lb}} = 404$, (12a) is evaluated to be $K_{\text{OL}}^{\text{ub}} = 327$, and (12b) is evaluated to be $K_{\text{OL}}^{\text{lb}} = 162$. Therefore, for the considered scenario we have that OL outperforms LQG. Furthermore, using numerical methods it can be shown that for $\delta = 0.011$ condition (13a) holds for $P < 1.5516$, while condition (13b) also requires that $P < 1.5516$.

Fig. 2 depicts the MSEs (4a) and (9), and the MSE of the estimator in (7), for the scenario considered above. It can be observed that $K_{\text{OL}} = 298, K_{\text{LQG}} = 1278$, and the LQG scheme using the estimator (7) achieves $D_1 = 0.1$ at $K = 1293$. Thus, here the upper bound (10) is very close. Finally, it can be observed that Fig. 2 supports the result of Thm. 1: at large values of k , the lines corresponding to the decoders (7) and (8) are nearly the same. Therefore, (8) is superior compared to (7) for large distortions. Based on Fig. 2, “large” for the present scenario corresponds to approximately $D_1 > 0.05$.

VI. CONCLUSIONS

In this work we studied uncoded transmission of two correlated Gaussian sources over the two-user GBCF. We first derived a decoder for the LQG approach based on the MMSE criterion. Then, we presented bounds on the minimal number of channel uses required to achieve a target distortion pair for the OL scheme and LQG scheme with the new decoder. For the symmetric case with independent noises we presented an explicit characterization of a range of transmit power constraint for which the OL scheme outperforms the LQG scheme. This is in contrast to the situation at the infinite horizon scenario. These results were also demonstrated via a numerical example. These results are a step towards identifying efficient and simple coding schemes for the transmission of correlated sources over multi-user channels with noiseless FB subject to delay constraints.

APPENDIX A

PROOF OF THEOREM 1

First, recall that the MMSE estimator of S_i based on $\hat{U}_{i,k}$ is the conditional expectation $\mathbb{E}\{S_i|\hat{U}_{i,k}\}$, [11, Eqn. (11.10)]. Furthermore, from (5), from the fact that the optimal control is linear and from the fact that $\hat{U}_{i,k} = a_i\hat{U}_{i,k-1} + Y_{i,k-1}$ it follows that for $i = 1, 2$, $\hat{U}_{i,k}$ and S_i are jointly Gaussian, both with zero mean. From [11, Eqn. (10.16)] it follows that $\mathbb{E}\{S_i|\hat{U}_{i,k}\} = \frac{\mathbb{E}\{S_i\hat{U}_{i,k}\}}{\mathbb{E}\{\hat{U}_{i,k}^2\}}\hat{U}_{i,k}$. Now, from (5) we have:

$$\begin{aligned} \mathbf{U}_k &= \mathbf{A}\mathbf{U}_{k-1} + \mathbf{Y}_{k-1} \\ &= \mathbf{A}^2\mathbf{U}_{k-2} + \mathbf{A}\mathbf{Y}_{k-2} + \mathbf{Y}_{k-1} \\ &= \mathbf{A}^{k-1}\mathbf{U}_1 + \sum_{m=1}^{k-2} \mathbf{A}^{k-m-1}\mathbf{Y}_m \\ &= \mathbf{A}^{k-1}\mathbf{S} + \sum_{m=1}^{k-2} \mathbf{A}^{k-m-1}\mathbf{Y}_m. \end{aligned} \quad (\text{A.1})$$

From (6) we have:

$$\begin{aligned} \hat{\mathbf{U}}_k &= \mathbf{A}\hat{\mathbf{U}}_{k-1} + \mathbf{Y}_{k-1} \\ &= \mathbf{A}^{k-1}\hat{\mathbf{U}}_1 + \sum_{m=1}^{k-2} \mathbf{A}^{k-m-1}\mathbf{Y}_m \\ &= \mathbf{0} + \sum_{m=1}^{k-2} \mathbf{A}^{k-m-1}\mathbf{Y}_m. \end{aligned} \quad (\text{A.2})$$

Therefore, combining (A.1) and (A.2) we have that $\mathbf{U}_{k+1} - \hat{\mathbf{U}}_{k+1} = \mathbf{A}^k\mathbf{S} \Rightarrow \hat{\mathbf{U}}_{k+1} = \mathbf{U}_{k+1} - \mathbf{A}^k\mathbf{S}$, and since \mathbf{A} is a diagonal matrix it follow that $\hat{U}_{i,k+1} = U_{i,k+1} - a_i^k S_i$. Thus, we have:

$$\begin{aligned} \hat{S}_{i,k} &= \frac{\mathbb{E}\{S_i(U_{i,k+1} - a_i^k S_i)\}}{\mathbb{E}\{(U_{i,k+1} - a_i^k S_i)^2\}}\hat{U}_{i,k+1} \\ &= \frac{\mathbb{E}\{S_i U_{i,k+1}\} - a_i^k \sigma_i^2}{\mathbb{E}\{U_{i,k+1}^2\} - 2a_i^k \mathbb{E}\{S_i U_{i,k+1}\} + a_i^{2k} \sigma_i^2} \hat{U}_{i,k+1}. \end{aligned} \quad (\text{A.3})$$

Next, from (5) we have:

$$\begin{aligned} \mathbf{U}_k &= \mathbf{A}\mathbf{U}_{k-1} + \mathbf{Y}_{k-1} \\ &= \mathbf{A}\mathbf{U}_{k-1} - \mathbf{B}\mathbf{C}^T\mathbf{U}_{k-1} + \mathbf{Z}_{k-1} \\ &= (\mathbf{A} - \mathbf{B}\mathbf{C}^T)\mathbf{U}_{k-1} + \mathbf{Z}_{k-1}. \end{aligned} \quad (\text{A.4})$$

From the independence of \mathbf{S} and \mathbf{Z}_k we have that $\mathbb{E}\{\mathbf{U}_{k+1}\mathbf{S}^T\} = (\mathbf{A} - \mathbf{B}\mathbf{C}^T)\mathbb{E}\{\mathbf{U}_k\mathbf{S}^T\}$, and since $\mathbf{U}_1 = \mathbf{S}$ it follows that $\mathbb{E}\{\mathbf{U}_{k+1}\mathbf{S}^T\} = (\mathbf{A} - \mathbf{B}\mathbf{C}^T)^k \mathbf{Q}_s$. Recalling that $\mathbf{M} = \mathbf{A} - \mathbf{B}\mathbf{C}^T$ we conclude that:

$$\mathbb{E}\{S_i U_{i,k+1}\} = [\mathbf{M}^k \mathbf{Q}_s]_{i,i}. \quad (\text{A.5})$$

Plugging (A.5) and denoting $\mathbb{E}\{U_{i,k+1}^2\} = [\mathbf{Q}_{u,k+1}]_{i,i}$ into (A.3) we obtain (8). Next, we use (8) to write the MSE. By plugging the expression for $\hat{S}_{i,k}$ in (A.3) into $\mathbb{E}\{(S_i - \hat{S}_{i,k})^2\}$ we obtain that:

$$\begin{aligned} \mathbb{E}\{(S_i - \hat{S}_{i,k})^2\} &= \sigma_i^2 - \frac{([\mathbf{M}^k \mathbf{Q}_s]_{i,i} - \sigma_i^2 a_i^k)^2}{[\mathbf{Q}_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k \mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}} \\ &= \frac{\sigma_i^2 [\mathbf{Q}_{u,k+1}]_{i,i} - ([\mathbf{M}^k \mathbf{Q}_s]_{i,i})^2}{[\mathbf{Q}_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k \mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}} \end{aligned} \quad (\text{A.6})$$

which is Eqn. (9). Finally, we consider (A.6) for $k \rightarrow \infty$. As the magnitudes of eigenvalues of the matrix \mathbf{M} are smaller than unity it follows that $\lim_{k \rightarrow \infty} ([\mathbf{M}^k \mathbf{Q}_s]_{i,i})^2 = 0$ and $\lim_{k \rightarrow \infty} [\mathbf{M}^k \mathbf{Q}_s]_{i,i} = 0$. Furthermore, since $|a_i| > 1$ and since $\lim_{k \rightarrow \infty} \mathbf{Q}_{u,k} = \mathbf{Q}_u$ it follows that for k large enough $([\mathbf{Q}_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k \mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}) \approx \sigma_i^2 a_i^{2k}$.¹ Therefore, for k large enough we have:

$$\begin{aligned} \frac{\sigma_i^2 [\mathbf{Q}_{u,k+1}]_{i,i} - ([\mathbf{M}^k \mathbf{Q}_s]_{i,i})^2}{[\mathbf{Q}_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k \mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}} &\approx a_i^{-2k} [\mathbf{Q}_{u,k+1}]_{i,i} \\ &= a_i^{-2k} \mathbb{E}\{U_{i,k+1}^2\}. \end{aligned}$$

APPENDIX B

PROOF OF THEOREM 2

From Remark 2 it follows that the MSE in (9) is upper bounded by the MSE of the decoder (7). Recall that the MSE of the decoder (7) is given by $\mathbb{E}\{(S_i - \hat{S}_{i,k})^2\} = |a_i|^{-2k} \mathbb{E}\{U_{i,k+1}^2\}$. Thus, we upper bound the MSE in (9) via upper bounding $\mathbb{E}\{U_{i,k+1}^2\}$ which leads to an upper bound on K .

From (A.4) and from the fact that \mathbf{U}_k and \mathbf{Z}_k are independent we have:

$$\begin{aligned} \mathbb{E}\{\mathbf{U}_{k+1} \mathbf{U}_{k+1}^T\} &= \mathbf{M} \mathbb{E}\{\mathbf{U}_k \mathbf{U}_k^T\} \mathbf{M}^T + \mathbf{Q}_z, \\ &= \mathbf{M}^k \mathbb{E}\{\mathbf{U}_1 \mathbf{U}_1^T\} (\mathbf{M}^T)^k + \sum_{l=0}^{k-1} \mathbf{M}^l \mathbf{Q}_z (\mathbf{M}^T)^l \\ &= \mathbf{M}^k \mathbf{Q}_s (\mathbf{M}^T)^k + \sum_{l=0}^{k-1} \mathbf{M}^l \mathbf{Q}_z (\mathbf{M}^T)^l. \end{aligned} \quad (\text{B.1})$$

Since the eigenvalues of \mathbf{M} are inside the unit circle (the controller stabilizes the system), it follows that $[\mathbf{M}^k \mathbf{Q}_s (\mathbf{M}^T)^k]_{i,i} \rightarrow 0$ as $k \rightarrow \infty$, and therefore from the fact that $\mathbf{Q}_{u,k} \rightarrow \mathbf{Q}_u$ for $k \rightarrow \infty$ we have from (B.1) that $[\sum_{l=0}^{k-1} \mathbf{M}^l \mathbf{Q}_z (\mathbf{M}^T)^l]_{i,i} \rightarrow [\mathbf{Q}_u]_{i,i}$ as $k \rightarrow \infty$.² Thus, we have:

$$\mathbb{E}\{U_{i,k+1}^2\} \leq [\mathbf{M}^k \mathbf{Q}_s (\mathbf{M}^T)^k]_{i,i} + [\mathbf{Q}_u]_{i,i}.$$

Next, we upper bound $[\mathbf{M}^k \mathbf{Q}_s (\mathbf{M}^T)^k]_{1,1}$. The bound for $i = 2$ is obtained by following similar steps. First, we note that $(\mathbf{M}^T)^k = (\mathbf{M}^k)^T$. Furthermore, we can apply Cholesky decomposition [12, Subsection 19.2.1.2] to \mathbf{Q}_s .³

$$\mathbf{M}^k \mathbf{Q}_s (\mathbf{M}^k)^T = \mathbf{M}^k \mathbf{L} \mathbf{L}^T (\mathbf{M}^k)^T, \quad \mathbf{L} = \begin{bmatrix} \sigma_1 & 0 \\ \rho_s \sigma_2 & \sigma_2 \sqrt{1 - \rho_s^2} \end{bmatrix}.$$

Now, we write \mathbf{M}^k in terms of the eigenvalues and eigenvectors of \mathbf{M} , see [12, Subsection 4.5.2.2]. Let $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2)$ be a diagonal matrix of the eigenvalues of \mathbf{M} , and let $\mathbf{V} = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ be a 2×2 matrix whose columns are the corresponding eigenvectors of \mathbf{M} . We have:

$$\mathbf{M} = \mathbf{V} \mathbf{D} \mathbf{V}^{-1} \Rightarrow \mathbf{M}^k = \mathbf{V} \mathbf{D}^k \mathbf{V}^{-1}.$$

¹From the fact that $\lim_{k \rightarrow \infty} ([\mathbf{M}^k \mathbf{Q}_s]_{i,i})^2 = 0$ it follows that $a_i^k [\mathbf{M}^k \mathbf{Q}_s]_{i,i}$ increases to infinity slower than $\sigma_i^2 a_i^{2k}$.

²Note that $[\sum_{l=0}^{k-1} \mathbf{M}^l \mathbf{Q}_z (\mathbf{M}^T)^l]_{i,i} \geq 0, k = 1, 2, \dots, i = 1, 2$, since the diagonal elements are sums of variances of the noise.

³Since \mathbf{Q}_s is a correlation matrix, and $\rho_s \neq 1$ we have that \mathbf{Q}_s is positive-definite, hence, Cholesky decomposition exists.

Next, we define $\mathbf{R} \triangleq \mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}\mathbf{L} = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$. This implies that:

$$[\mathbf{M}^k\mathbf{Q}_s(\mathbf{M}^T)^k]_{1,1} = [\mathbf{R}\mathbf{R}^T]_{1,1} = r_1^2 + r_2^2. \quad (\text{B.2})$$

The next step is to upper bound r_1 and r_2 . Writing $\mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ explicitly we have:

$$\begin{aligned} \mathbf{V}\mathbf{D}^k\mathbf{V}^{-1} &= \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} v_1\lambda_1^k & v_2\lambda_2^k \\ v_3\lambda_1^k & v_4\lambda_2^k \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}^{-1} \\ &= \frac{1}{\det(\mathbf{V})} \begin{bmatrix} v_1\lambda_1^k & v_2\lambda_2^k \\ v_3\lambda_1^k & v_4\lambda_2^k \end{bmatrix} \begin{bmatrix} v_4 & -v_2 \\ -v_3 & v_1 \end{bmatrix} \\ &= \frac{1}{\det(\mathbf{V})} \begin{bmatrix} v_1v_4\lambda_1^k - v_2v_3\lambda_2^k & v_1v_2(\lambda_2^k - \lambda_1^k) \\ v_3v_4(\lambda_1^k - \lambda_2^k) & v_1v_4\lambda_2^k - v_2v_3\lambda_1^k \end{bmatrix}. \end{aligned} \quad (\text{B.3})$$

Therefore, we have:

$$\mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}\mathbf{L} = \begin{bmatrix} v_1v_4\lambda_1^k - v_2v_3\lambda_2^k & v_1v_2(\lambda_2^k - \lambda_1^k) \\ v_3v_4(\lambda_1^k - \lambda_2^k) & v_1v_4\lambda_2^k - v_2v_3\lambda_1^k \end{bmatrix} \times \begin{bmatrix} \sigma_1 & 0 \\ \rho_s\sigma_2 & \sigma_2\sqrt{1-\rho_s^2} \end{bmatrix} \cdot \frac{1}{\det(\mathbf{V})},$$

which implies that

$$r_1 = \frac{1}{\det(\mathbf{V})} \left(\sigma_1 (v_1v_4\lambda_1^k - v_2v_3\lambda_2^k) + \rho_s\sigma_2 (v_1v_2(\lambda_2^k - \lambda_1^k)) \right) \quad (\text{B.4a})$$

$$r_2 = \frac{1}{\det(\mathbf{V})} \sigma_2 \sqrt{1-\rho_s^2} (v_1v_2(\lambda_2^k - \lambda_1^k)). \quad (\text{B.4b})$$

Now, we upper bound $|r_1|$ as follows:

$$\begin{aligned} |r_1| &\leq \frac{1}{|\det(\mathbf{V})|} \left(\sigma_1 (|v_1v_4||\lambda_1|^k + |v_2v_3||\lambda_2|^k) \right. \\ &\quad \left. + |\rho_s|\sigma_2 (|v_1v_2|(|\lambda_2|^k + |\lambda_1|^k)) \right) \\ &\stackrel{(a)}{\leq} \frac{\sigma_1 (|v_1v_4\lambda_1| + |v_2v_3\lambda_2|) + |\rho_s|\sigma_2 v_1v_2 (|\lambda_2| + |\lambda_1|)}{|\det(\mathbf{V})|} \\ &\triangleq \eta_1 \end{aligned}$$

where (a) follows from the fact that $|\lambda_i| < 1, i = 1, 2$. Following similar arguments we bound $|r_2|$ as follows:

$$|r_2| \leq \frac{\sigma_2 \sqrt{1-\rho_s^2} (|v_1v_2|(|\lambda_2| + |\lambda_1|))}{|\det(\mathbf{V})|} \triangleq \eta_2.$$

Hence, we have that $[\mathbf{M}^k\mathbf{Q}_s(\mathbf{M}^T)^k]_{1,1} \leq \eta_1^2 + \eta_2^2$, and this implies that:

$$\mathbb{E} \{U_{1,k+1}^2\} \leq \eta_1^2 + \eta_2^2 + [Q_u]_{1,1} \triangleq \vartheta_1.$$

Following similar arguments we have that $[\mathbf{M}^k\mathbf{Q}_s(\mathbf{M}^T)^k]_{2,2} \leq \eta_3^2 + \eta_4^2$, where:

$$\eta_3 \triangleq \frac{|\sigma_1 v_3 v_4| (|\lambda_1| + |\lambda_2|) + |\rho_s \sigma_2| (|v_1 v_4 \lambda_2| + |v_2 v_3 \lambda_1|)}{|\det(\mathbf{V})|}$$

$$\eta_4 \triangleq \frac{\sigma_2 \sqrt{1-\rho_s^2} (|v_1 v_4 \lambda_2| + |v_2 v_3 \lambda_1|)}{|\det(\mathbf{V})|},$$

and therefore:

$$\mathbb{E} \{U_{2,k+1}^2\} \leq \eta_3^2 + \eta_4^2 + [Q_u]_{2,2} \triangleq \vartheta_2.$$

Finally, we Let $\mathbb{E}\{(S_i - \hat{S}_{i,k})^2\} \triangleq D_{i,k}$ be the achieved distortion at time instance k . Therefore we have $D_{i,k} = a_i^{-2k} \mathbb{E}\{U_{i,k+1}^2\}$. Thus, we have:

$$K \leq \frac{\log\left(\frac{\vartheta_i}{D_{i,k}}\right)}{2 \log|a_i|}, \quad i = 1, 2,$$

and by setting $D_{i,k} = D_i$ we obtain the bound (10).

APPENDIX C PROOF OF THEOREM 3

In order to lower bound K_{LQG} , we lower bound the MSE in (9) as follows:

$$\begin{aligned} &\mathbb{E} \left\{ (S_i - \hat{S}_{i,k})^2 \right\} \\ &= \sigma_i^2 \frac{[Q_{u,k+1}]_{i,i} - \frac{1}{\sigma_i^2} ([\mathbf{M}^k\mathbf{Q}_s]_{i,i})^2}{[Q_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k\mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}} \\ &\stackrel{(a)}{\geq} \sigma_i^2 \frac{[Q_z]_{i,i} - \frac{1}{\sigma_i^2} ([\mathbf{M}^k\mathbf{Q}_s]_{i,i})^2}{[Q_z]_{i,i} - 2a_i^k [\mathbf{M}^k\mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}}, \end{aligned} \quad (\text{C.1})$$

where (a) follows from the fact that $[Q_{u,k+1}]_{i,i} \geq [Q_z]_{i,i}$, see (B.1), and from the fact that $\frac{[Q_{u,k+1}]_{i,i} - \frac{1}{\sigma_i^2} ([\mathbf{M}^k\mathbf{Q}_s]_{i,i})^2}{[Q_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k\mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}} \leq 1$. To see this recall that $\mathbb{E}\{(S_i - \hat{S}_{i,k})^2\} = \sigma_i^2 - \mathbb{E}\{\hat{S}_{i,k}^2\}$, and therefore we have:

$$\sigma_i^2 \left(1 - \frac{[Q_{u,k+1}]_{i,i} - \frac{1}{\sigma_i^2} ([\mathbf{M}^k\mathbf{Q}_s]_{i,i})^2}{[Q_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k\mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}} \right) = \mathbb{E} \left\{ \hat{S}_{i,k}^2 \right\} \geq 0,$$

which implies that $\frac{[Q_{u,k+1}]_{i,i} - \frac{1}{\sigma_i^2} ([\mathbf{M}^k\mathbf{Q}_s]_{i,i})^2}{[Q_{u,k+1}]_{i,i} - 2a_i^k [\mathbf{M}^k\mathbf{Q}_s]_{i,i} + \sigma_i^2 a_i^{2k}} \leq 1$. Next, we lower bound the numerator of (C.1) and upper bound the denominator of (C.1).

Recall that $\mathbf{M}^k = (\mathbf{A} - \mathbf{B}\mathbf{C}^T)^k$ and consider upper bounding $[\mathbf{M}^k\mathbf{Q}_s]_{1,1}$. Similarly to Appendix B writing \mathbf{M}^k in terms of its eigenvalues matrix \mathbf{D} and eigenvectors matrix \mathbf{V} we obtain $\mathbf{M}^k = \mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}$. Using (B.3) we obtain:

$$\mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}\mathbf{Q}_s = \begin{bmatrix} v_1v_4\lambda_1^k - v_2v_3\lambda_2^k & v_1v_2(\lambda_2^k - \lambda_1^k) \\ v_3v_4(\lambda_1^k - \lambda_2^k) & v_1v_4\lambda_2^k - v_2v_3\lambda_1^k \end{bmatrix} \times \begin{bmatrix} \sigma_1^2 & \rho_s\sigma_1\sigma_2 \\ \rho_s\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \cdot \frac{1}{\det(\mathbf{V})},$$

from which we compute:

$$[\mathbf{M}^k\mathbf{Q}_s]_{1,1} = \frac{\sigma_1^2 (v_1v_4\lambda_1^k - v_2v_3\lambda_2^k) + \rho_s\sigma_1\sigma_2 v_1v_2 (\lambda_2^k - \lambda_1^k)}{\det(\mathbf{V})}. \quad (\text{C.2})$$

Next, using the fact that the eigenvalues of \mathbf{M} are inside the unit circle we obtain the following upper bound on $[\mathbf{M}^k\mathbf{Q}_s]_{1,1}$, for $k = 1, 2, \dots$:

$$[M^k \mathbf{Q}_s]_{1,1} \leq \frac{1}{|\det(\mathbf{V})|} \left(\sigma_1^2 (|v_1 v_4 \lambda_1| + |v_2 v_3 \lambda_2|) + |\rho_s \sigma_1 \sigma_2 v_1 v_2| (|\lambda_2| + |\lambda_1|) \right) \triangleq \beta_1.$$

Following similar arguments we also bound:

$$[M^k \mathbf{Q}_s]_{2,2} \leq \frac{1}{|\det(\mathbf{V})|} \left(\sigma_2^2 (|v_1 v_4 \lambda_2| + |v_2 v_3 \lambda_1|) + |\rho_s \sigma_1 \sigma_2 v_3 v_4| (|\lambda_2| + |\lambda_1|) \right) \triangleq \beta_2.$$

Now, plugging into (C.1) and recalling that $\mathbb{E}\{(S_i - \hat{S}_{i,k})^2\} = D_{i,k}$ we have:

$$D_{i,k} \geq \frac{\sigma_i^2 [\mathbf{Q}_z]_{i,i} - \beta_i^2}{[\mathbf{Q}_z]_{i,i} + 2|a_i|^k \beta_i + \sigma_i^2 |a_i|^{2k}},$$

which can also be written as

$$\begin{aligned} D_{i,k} [\mathbf{Q}_z]_{i,i} - \sigma_i^2 [\mathbf{Q}_z]_{i,i} + \beta_i^2 &\geq -D_{i,k} (2|a_i|^k \beta_i + \sigma_i^2 |a_i|^{2k}) \\ \Rightarrow \frac{\sigma_i^2 [\mathbf{Q}_z]_{i,i} - \beta_i^2 - D_{i,k} [\mathbf{Q}_z]_{i,i}}{D_{i,k}} &\leq 2|a_i|^k \beta_i + \sigma_i^2 |a_i|^{2k}. \end{aligned}$$

Next, we recall that $|a_i| > 1$ and obtain the following bound:

$$\frac{\sigma_i^2 [\mathbf{Q}_z]_{i,i} - \beta_i^2 - D_{i,k} [\mathbf{Q}_z]_{i,i}}{D_{i,k}} \leq (2\beta_i + \sigma_i^2) |a_i|^{2k}.$$

Taking the log from both sides we have

$$\begin{aligned} \log \left(\frac{\sigma_i^2 [\mathbf{Q}_z]_{i,i} - \beta_i^2 - D_{i,k} [\mathbf{Q}_z]_{i,i}}{D_{i,k}} \right) \\ \leq \log \left((2\beta_i + \sigma_i^2) |a_i|^{2k} \right), \end{aligned}$$

which can be written as:

$$\log \left(\frac{\sigma_i^2 [\mathbf{Q}_z]_{i,i} - \beta_i^2 - D_{i,k} [\mathbf{Q}_z]_{i,i}}{(2\beta_i + \sigma_i^2) D_{i,k}} \right) \leq 2k \log |a_i|,$$

and therefore we have

$$\frac{\log \left(\frac{\sigma_i^2 [\mathbf{Q}_z]_{i,i} - \beta_i^2 - D_{i,k} [\mathbf{Q}_z]_{i,i}}{(2\beta_i + \sigma_i^2) D_{i,k}} \right)}{2 \log |a_i|} \leq k,$$

which is stated in Eqn. (11).

APPENDIX D

PROOF OF THEOREM 4

Recall that $\alpha_{i,0} = \sigma_i^2$. From (4a) we have:

$$\log \left(\frac{\alpha_{1,K}}{\sigma_1^2} \right) = \sum_{k=1}^K \log \left(\frac{\sigma_{z,0}^2 + \sigma_{z,1}^2 + \Psi_{k-1}^2 g^2 (1 - \rho_{k-1}^2)}{\pi_1} \right).$$

As $|\rho_k| \in [0, 1]$, it follows that:

$$\Psi_{k-1}^2 g^2 (1 - \rho_{k-1}^2) = \frac{P g^2 (1 - \rho_{k-1}^2)}{1 + g^2 + 2g|\rho_k - 1|} \leq \frac{P g^2}{1 + g^2}.$$

Thus, we obtain the following upper bound:

$$\frac{\sigma_{z,0}^2 + \sigma_{z,1}^2 + \Psi_{k-1}^2 g^2 (1 - \rho_{k-1}^2)}{\pi_1} \leq \frac{\sigma_{z,0}^2 + \sigma_{z,1}^2 + \pi_1 g^2}{\pi_1 + \pi_1 g^2}.$$

Next, we use the fact that $\log(x) \leq x - 1$ to obtain

$$\begin{aligned} \log \left(\frac{\sigma_{z,0}^2 + \sigma_{z,1}^2 + \pi_1 g^2}{\pi_1 + \pi_1 g^2} \right) &\leq \frac{\sigma_{z,0}^2 + \sigma_{z,1}^2 + \pi_1 g^2}{\pi_1 + \pi_1 g^2} - 1 \\ &= -\frac{P}{\pi_1 + \pi_1 g^2}, \end{aligned}$$

and therefore it follows that

$$\log \left(\frac{\alpha_{1,K}}{\sigma_1^2} \right) = \log \left(\frac{D_1}{\sigma_1^2} \right) \leq -\frac{KP}{\pi_1 + \pi_1 g^2},$$

which implies that

$$K_{\text{OL}}^{\text{ub}} = \left\lceil \frac{(1 + g^2)}{P} \max \left\{ \pi_1 \log \left(\frac{\sigma_1^2}{D_1} \right), \frac{\pi_2}{g^2} \log \left(\frac{\sigma_2^2}{D_2} \right) \right\} \right\rceil.$$

To obtain $K_{\text{OL}}^{\text{lb}}$ we note that $0 \leq \Psi_{k-1}^2 g^2 (1 - \rho_{k-1}^2)$ where equality is obtained by setting $\rho_{k-1} = 1$. Then, we use the inequality $1 - \frac{1}{x} \leq \log x$ to obtain:

$$\begin{aligned} \log \left(\frac{\sigma_{z,0}^2 + \sigma_{z,1}^2}{\sigma_{z,0}^2 + \sigma_{z,1}^2 + P} \right) &\geq 1 - \frac{\sigma_{z,0}^2 + \sigma_{z,1}^2 + P}{\sigma_{z,0}^2 + \sigma_{z,1}^2} \\ &= -\frac{P}{\sigma_{z,0}^2 + \sigma_{z,1}^2}. \end{aligned}$$

Thus, we have:

$$\log \left(\frac{D_1}{\sigma_1^2} \right) \geq -\frac{KP}{\sigma_{z,0}^2 + \sigma_{z,1}^2},$$

which results in the following lower bound:

$$K_{\text{OL}}^{\text{lb}} = \left\lceil \max \left\{ \frac{\sigma_{z,0}^2 + \sigma_{z,1}^2}{P} \log \left(\frac{\sigma_1^2}{D_1} \right), \frac{\sigma_{z,0}^2 + \sigma_{z,2}^2}{P} \log \left(\frac{\sigma_2^2}{D_2} \right) \right\} \right\rceil.$$

APPENDIX E

PROOF OF COROLLARY 1

We begin with explicitly writing the condition $K_{\text{OL}}^{\text{ub}} < K_{\text{LQG}}^{\text{lb}}$, for the symmetric case:

$$\frac{2(P + \sigma_z^2)}{P} \log \left(\frac{\sigma_1^2}{D_1} \right) < \frac{\log \left(\frac{\sigma_1^2 \sigma_z^2 - \beta_1^2 - D_1 \sigma_z^2}{(2\beta_1 + \sigma_1^2) D_1} \right)}{2 \log |a_1|}. \quad (\text{E.1})$$

As we aim to provide a characterization of P 's for which OL outperforms LQG, we lower bound the right-hand-side (RHS) of (E.1) by upper bounding β_1 and $2 \log |a_1|$ in terms of P and of the system parameters (noise and source parameters). We begin with an upper bound on $2 \log |a_1|$.

A. An Upper Bound on $2 \log |a_1|$

We follow steps similar to the steps presented in [5, Section IV.C], for the symmetric case without common noise, and obtain that $a_1^2 = x_0$, where x_0 is the unique real positive root⁴ of the equation:

$$\sigma_z^2 x^3 + \sigma_z^2 x^2 - (\sigma_z^2 + 2P)x - \sigma_z^2 = 0.$$

This equation can be written as:

$$x^3 + x^2 - \left(1 + \frac{2P}{\sigma_z^2} \right) x - 1 = 0. \quad (\text{E.2})$$

Next, we use Budan's theorem [13] to upper bound x_0 :

Theorem. (Budan's theorem) Let $p(x) = a_0 + a_1 x + \dots + a_n x^n$ be a polynomial of degree n , and let $p^{(j)}(x)$ be its

⁴The uniqueness of a real positive root follows from Descartes rule, [12, Subsection 1.6.3.2]

$$\beta_1 = \frac{f_0^2(a_1, c_1) \left(\frac{\sigma_1^2}{a_1} (a_1^2 c_1^2 + (1 - (a_1 - c_1)a_1)^2) + \frac{2|\rho_s \sigma_1 \sigma_2|}{a_1} |a_1 c_1 (1 - (a_1 - c_1)a_1)| \right)}{f_0^2(a_1, c_1) (|a_1^2 c_1^2 - (1 - (a_1 - c_1)a_1)^2|)} \quad (\text{E.6})$$

$$\stackrel{(a)}{=} \frac{\sigma_1^2 (a_1^2 c_1^2 + (1 - (a_1 - c_1)a_1)^2) + 2|\rho_s \sigma_1 \sigma_2| |a_1 c_1 (1 - (a_1 - c_1)a_1)|}{|a_1 (a_1^2 c_1^2 - (1 - (a_1 - c_1)a_1)^2)|},$$

j 'th derivative. Define the function $V(\alpha)$ to be the number of sign variations in the sequence $p(\alpha), p^{(1)}(\alpha), \dots, p^{(n)}(\alpha)$. Then, the number of roots of the polynomial $p(x)$ in the open interval (a, b) is either equal to $V(a) - V(b)$ or less by an even number.

Let $p(x)$ be the polynomial in (E.2). Then we have:

$$p^{(0)}(x) = x^3 + x^2 - \left(1 + \frac{2P}{\sigma_z^2}\right)x - 1, \quad (\text{E.3a})$$

$$p^{(1)}(x) = 3x^2 + 2x - \left(1 + \frac{2P}{\sigma_z^2}\right), \quad (\text{E.3b})$$

$$p^{(2)}(x) = 6x + 2, \quad (\text{E.3c})$$

$$p^{(3)}(x) = 6. \quad (\text{E.3d})$$

For $x = 1$ we have $V(1) = 1$. Note that $\text{sgn}(p^{(1)}(1))$ depend on the term $\frac{2P}{\sigma_z^2}$, however, since $\text{sgn}(p^{(0)}(1)) = -1$ and $\text{sgn}(p^{(2)}(1)) = 1$, in both cases we have $V(1) = 1$. Next, we let $\chi = \frac{P}{2\sigma_z^2}$ and set $x = 1 + \chi$ to obtain:

$$\begin{aligned} p^{(0)}(1 + \chi) &= \chi^3 > 0, \\ p^{(1)}(1 + \chi) &= 3\chi^2 + 4\chi + 4 > 0, \\ p^{(2)}(1 + \chi) &= 6\chi + 8 > 0, \\ p^{(3)}(1 + \chi) &= 6 > 0. \end{aligned}$$

Therefore, $V(1 + \chi) = 0$. Thus, Budan's theorem implies that the number of roots of (E.2) in the interval $(1, 1 + \chi)$ is 1. From Descartes rule we know that there is a unique positive root, therefore $1 + \chi$ is an upper bound on x_0 : $x_0 < 1 + \frac{P}{2\sigma_z^2}$.

Next, recall that $a_1^2 = x_0$, which implies that $2 \log(|a_1|) = \log(x_0) \leq \log\left(1 + \frac{P}{2\sigma_z^2}\right)$. Using the fact that $\log(x) \leq x - 1$ we have the following bound on $2 \log(|a_1|)$:

$$2 \log(|a_1|) \leq \frac{P}{2\sigma_z^2}. \quad (\text{E.4})$$

Next, we upper bound β_1 .

B. An Upper Bound on β_1

Recall the definition of β_1 :

$$\beta_1 = \frac{\sigma_1^2 (|v_1 v_4 \lambda_1| + |v_2 v_3 \lambda_2|) + |\rho_s \sigma_1 \sigma_2 v_1 v_2| (|\lambda_2| + |\lambda_1|)}{|\det(\mathbf{V})|}.$$

For the symmetric case the diagonal elements of \mathbf{A} can be selected as in [5, Subsection IV.C]: $\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ 0 & -a_1 \end{bmatrix}$, where $a_1 > 1$.⁵ In the following we will show that due to symmetry we can equivalently interchange a_1 with $-a_1$ on the main

diagonal of \mathbf{A} , which implies that $\lambda_1 = -\lambda_2$, $v_1 = v_4$, $v_2 = v_3$, and that $c_2 = -c_1$. Therefore, we write β_1 as follows:

$$\beta_1 = \frac{\sigma_1^2 |\lambda_1| (v_1^2 + v_2^2) + 2|\lambda_1| |\rho_s \sigma_1 \sigma_2 v_1 v_2|}{|v_1^2 - v_2^2|}.$$

Let \mathbf{I} denote the identity matrix. Next, we explicitly express λ_1, v_1 , and v_2 . From [14, Lemma 2.4] we know that $\lambda_1 = \frac{1}{a_1}$.

Furthermore, recall that $\mathbf{M} = \begin{bmatrix} a_1 - c_1 & c_1 \\ -c_1 & -(a_1 - c_1) \end{bmatrix}$. Now, an eigenvector \mathbf{v} of \mathbf{M} , corresponding to the eigenvalue λ_1 , obeys $\mathbf{M}\mathbf{v} = \lambda_1 \mathbf{v}$. This equation can also be written using a matrix form:

$$(\mathbf{M} - \lambda_1 \mathbf{I})\mathbf{v} = \begin{bmatrix} a_1 - c_1 - \lambda_1 & c_1 \\ -c_1 & -(a_1 - c_1) - \lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}.$$

Therefore, recalling that eigenvectors has unity norm, we obtain an explicit expression for \mathbf{v} :

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{\sqrt{c_1^2 + (a_1 - c_1 - \lambda_1)^2}} \\ -\frac{a_1 - c_1 - \lambda_1}{\sqrt{c_1^2 + (a_1 - c_1 - \lambda_1)^2}} \end{bmatrix}.$$

Now, plugging the expression for λ we write:

$$\begin{aligned} v_1 &= \frac{c_1}{\sqrt{c_1^2 + (a_1 - c_1 - \frac{1}{a_1})^2}} \\ &= \frac{a_1 c_1}{\sqrt{a_1^2 c_1^2 + ((a_1 - c_1)a_1 - 1)^2}} \end{aligned} \quad (\text{E.5a})$$

$$\begin{aligned} v_2 &= -\frac{a_1 - c_1 - \frac{1}{a_1}}{\sqrt{c_1^2 + (a_1 - c_1 - \frac{1}{a_1})^2}} \\ &= \frac{1 - (a_1 - c_1)a_1}{\sqrt{a_1^2 c_1^2 + ((a_1 - c_1)a_1 - 1)^2}}. \end{aligned} \quad (\text{E.5b})$$

Let $f_0(a_1, c_1) = \frac{1}{\sqrt{a_1^2 c_1^2 + ((a_1 - c_1)a_1 - 1)^2}}$. Using (E.5) we write β_1 as in (E.6) at the top of this page, where (a) follows from the fact that $a_1 > 1$. Since we are interested in an upper bound on β_1 , we lower bound the denominator of (E.6) and upper bound the numerator. For this purpose, we first explicitly write c_1 in terms of a_1 . From the definition of the vector \mathbf{C} , see Subsection III-B we have:

$$\mathbf{C} = (\mathbf{B}^T \mathbf{G} \mathbf{B} + \mathbf{1})^{-1} \mathbf{A} \mathbf{G}^T \mathbf{B},$$

where \mathbf{G} is the unique positive-definite solution of the discrete algebraic Riccati equation $\mathbf{G} = \mathbf{A}^T \mathbf{G} \mathbf{A} - \mathbf{A}^T \mathbf{G} \mathbf{B} (\mathbf{B}^T \mathbf{G} \mathbf{B} + \mathbf{1})^{-1} \mathbf{B}^T \mathbf{G} \mathbf{A}$, such that all the eigenvalues of the matrix $\mathbf{A} - \mathbf{B} \mathbf{C}^T$ have magnitudes smaller than 1. Let $\mathbf{G} = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix}$.

⁵Recall that $a_1 \neq a_2$, see Subsection III-B.

From [15, Prop. 1] we have that for the symmetric case and the considered A, the elements of G are given by:

$$g_1 = g_4 = \frac{(a_1^2 - 1)(1 + a_1^2)^2}{4a_1^2},$$

$$g_2 = g_3 = \frac{(1 - a_1^2)^2(1 + a_1^2)}{4a_1^2}.$$

Writing $AG^T\mathbf{B} = \text{AGB}$ explicitly we have:

$$\begin{aligned} \text{AGB} &= \begin{bmatrix} a_1 & 0 \\ 0 & -a_1 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a_1 g_1 & a_1 g_2 \\ -a_1 g_3 & -a_1 g_4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a_1(g_1 + g_2) \\ -a_1(g_3 + g_4) \end{bmatrix}. \end{aligned}$$

Writing $\mathbf{B}^T\text{GB} + 1$ explicitly we have:

$$\begin{aligned} \mathbf{B}^T\text{GB} + 1 &= 2(g_1 + g_2) + 1 \\ &= 2 \left(\frac{(a_1^2 - 1)(1 + a_1^2)^2}{4a_1^2} + \frac{(1 - a_1^2)^2(1 + a_1^2)}{4a_1^2} \right) + 1 \\ &= 2 \left(\frac{(a_1^2 - 1)(1 + a_1^2)^2 + (1 - a_1^2)^2(1 + a_1^2)}{4a_1^2} \right) + 1 \\ &= 2 \left(\frac{(1 + a_1^2)(a_1^2 - 1)(1 + a_1^2 - 1 + a_1^2)}{4a_1^2} \right) + 1 \\ &= (1 + a_1^2)(a_1^2 - 1) + 1. \end{aligned}$$

Therefore, since $g_1 + g_2 = \frac{(1+a_1^2)(a_1^2-1)}{2}$, we have:

$$\begin{aligned} c_1 &= \frac{a_1(g_1 + g_2)}{2(g_1 + g_2) + 1} \\ &= \frac{a_1(1 + a_1^2)(a_1^2 - 1)}{2((1 + a_1^2)(a_1^2 - 1) + 1)} \\ &= \frac{a_1^4 - 1}{2a_1^3} \end{aligned} \quad (\text{E.7})$$

Computing c_2 via following similar arguments it follows that $c_2 = -c_1$. Note that (E.7) implies that $0 \leq c_1 \leq a_1$. Next, we lower bound the denominator of β_1 in (E.6). We write:

$$\begin{aligned} &a_1(a_1^2 c_1^2 - (1 - (a_1 - c_1)a_1)^2) \\ &= a_1(a_1^2 c_1^2 - (1 - 2(a_1 - c_1)a_1 + (a_1 - c_1)^2 a_1^2)) \\ &= a_1(a_1^2 c_1^2 - (1 - 2a_1^2 + 2a_1 c_1 + a_1^4 - 2a_1^3 c_1 + a_1^2 c_1^2)) \\ &= a_1(a_1^2 c_1^2 - 1 + 2a_1^2 - 2a_1 c_1 - a_1^4 + 2a_1^3 c_1 - a_1^2 c_1^2) \\ &= 2a_1^3 + 2a_1^4 c_1 - 2a_1^2 c_1 - a_1^5 - a_1 \\ &= 2a_1^3 + 2a_1^2 c_1 (a_1^2 - 1) - a_1^5 - a_1 \\ &= 2a_1^2 c_1 (a_1^2 - 1) - a_1(1 - a_1^2)^2 \\ &\triangleq f_1(a_1, c_1). \end{aligned} \quad (\text{E.8})$$

In order to lower bound $f_1(a_1, c_1)$ we present the following lemma:

Lemma 1. For $a_1 > 1$, the function $f_1(a_1, c_1)$ in (E.8) is positive and monotonic increasing in a_1 .

Proof: First, we write $f_1(a_1, c_1)$ only in terms of a_1 . By plugging (E.7) into (E.8) we obtain:

$$\begin{aligned} f_1(a_1, c_1) &= \frac{2a_1^3(a_1^2 - 1)^2(1 + a_1^2)}{2((1 + a_1^2)(a_1^2 - 1) + 1)} - a_1(1 - a_1^2)^2 \\ &= a_1(1 - a_1^2)^2 \left(\frac{2a_1^2(1 + a_1^2)}{2((1 + a_1^2)(a_1^2 - 1) + 1)} - 1 \right) \\ &= a_1(1 - a_1^2)^2 \left(\frac{a_1^4 + a_1^2}{a_1^4} - 1 \right) \\ &= \frac{a_1^3(1 - a_1^2)^2}{a_1^4} \\ &= \frac{(a_1^2 - 1)^2}{a_1} \\ &= \frac{a_1^4 - 2a_1^2 + 1}{a_1} \\ &\triangleq f_2(a_1). \end{aligned} \quad (\text{E.9})$$

As $a_1 > 1$, we have that $f_2(a_1) > 0$. Furthermore, we have:

$$f_2^{(1)}(a_1) = \frac{3a_1^4 - 2a_1^2 - 1}{a_1^2} > 0, \quad \forall a_1 > 1. \quad (\text{E.10})$$

Therefore, as the derivative of $f_2(a_1)$ is positive for $a_1 > 1$ we conclude that in this regime $f_2(a_1)$ is monotonically increasing. \blacksquare

Lemma 1 implies that if $1 \leq a_1^{\text{lb}} < a_1$ then $f_2(a_1^{\text{lb}}) < f_2(a_1)$. Therefore, in order to lower bound the denominator of β_1 in (E.6), i.e., lower bound $f_1(a_1, c_1)$ in (E.8) we derive a lower bound on a_1 . The following lemma presents a class of lower bounds on a_1 .

Lemma 2. Let $0 < \delta \leq 4$. If $P < \frac{1}{4}\delta(4 + \delta)\sigma_z^2 + \sqrt{\frac{1}{4}\delta((4 + \delta)\sigma_z^2)^2(1 + \frac{\delta}{4})}$ then $\sqrt{1 + \frac{2P}{(4 + \delta)\sigma_z^2}} < a_1$.

Proof: From Budan's theorem, see Appendix E-A, it follows that if $V(\tilde{x}) = V(1)$, then \tilde{x} is a lower bound on x_0 , where x_0 is the unique root of (E.2) which is real and larger than 1 (see Appendix E-A for the steps leading to an upper bound on x_0). Now, let $\chi = \frac{2P}{(4 + \delta)\sigma_z^2}$, for some $0 < \delta \leq 4$, and evaluate (E.3) at $\tilde{x} = 1 + \chi$:

$$\begin{aligned} p^{(0)}(1 + \chi) &= (1 + \chi)^3 + (1 + \chi)^2 \\ &\quad - (1 + \chi)(1 + (4 + \delta)\chi) - 1 \\ &= \chi^3 - \delta\chi^2 - \delta\chi, \end{aligned} \quad (\text{E.11a})$$

$$\begin{aligned} p^{(1)}(1 + \chi) &= 3(1 + \chi)^2 + 2(1 + \chi) - (1 + (4 + \delta)\chi) \\ &= 3\chi^2 + (4 - \delta)\chi + 4 > 0, \end{aligned} \quad (\text{E.11b})$$

$$p^{(2)}(1 + \chi) = 6(1 + \chi) + 2 = 6\chi + 8 > 0, \quad (\text{E.11c})$$

$$p^{(3)}(1 + \chi) = 6 > 0. \quad (\text{E.11d})$$

Note that (E.11b) holds since $\delta \leq 4$. Thus, in order to have $V(1 + \chi) = 1$ we must have $\chi^3 - \delta\chi^2 - \delta\chi < 0$, or equivalently $\chi^2 - \delta\chi - \delta < 0$. Plugging the value of χ we obtain the following polynomial inequality in terms of P :

$$q(P) = P^2 - \frac{1}{2}\delta(4 + \delta)\sigma_z^2 P - \frac{1}{4}\delta((4 + \delta)\sigma_z^2)^2 < 0.$$

$$\begin{aligned} & \sigma_1^2 (a_1^2 c_1^2 + ((a_1 - c_1)a_1 - 1)^2) + 2|\rho_s \sigma_1 \sigma_2| a_1 c_1 ((a_1 - c_1)a_1 - 1) \\ & \leq \sigma_1^2 \left(\left(1 + \frac{P}{2\sigma_z^2}\right) \varphi_1^2(P, \delta) + \varphi_2^2(P, \delta) \right) + 2|\rho_s \sigma_1 \sigma_2| \varphi_1(P, \delta) \varphi_2(P, \delta) \sqrt{1 + \frac{P}{2\sigma_z^2}}. \end{aligned} \quad (\text{E.17})$$

$$\beta_1 \leq \frac{\sigma_1^2 \left(\left(1 + \frac{P}{2\sigma_z^2}\right) \varphi_1^2(P, \delta) + \varphi_2^2(P, \delta) \right) + 2|\rho_s \sigma_1 \sigma_2| \varphi_1(P, \delta) \varphi_2(P, \delta) \sqrt{1 + \frac{P}{2\sigma_z^2}}}{\varphi_0(P, \delta)} \triangleq \tau(P, \delta). \quad (\text{E.18})$$

The roots of $q(P)$ are given by:

$$P_0 = \frac{1}{4} \delta (4 + \delta) \sigma_z^2 \pm \sqrt{\frac{1}{4} \delta ((4 + \delta) \sigma_z^2)^2 \left(1 + \frac{\delta}{4}\right)},$$

which implies that one of the roots is positive while the other is negative. Therefore, we have that $V(1 + \chi) = 1$ for

$$P < \frac{1}{4} \delta (4 + \delta) \sigma_z^2 + \sqrt{\frac{1}{4} \delta ((4 + \delta) \sigma_z^2)^2 \left(1 + \frac{\delta}{4}\right)}. \quad (\text{E.12})$$

If (E.12) holds then $1 + \frac{2P}{(4+\delta)\sigma_z^2}$ is a lower bound on x_0 . The bound on a_1 directly follows. ■

Recalling the definition of $f_2(a_1)$ in (E.9), Lemma 1 and Lemma 2 imply that if (E.12) holds then a lower bound on the denominator of (E.6) is given by:

$$\begin{aligned} \varphi_0(P, \delta) & \triangleq f_2 \left(\sqrt{1 + \frac{2P}{(4 + \delta) \sigma_z^2}} \right) \\ & \leq |a_1 (a_1^2 c_1^2 - (1 - (a_1 - c_1) a_1)^2)|, \end{aligned} \quad (\text{E.13})$$

Next, we address the upper bound on the numerator of (E.6):

$$\begin{aligned} & \sigma_1^2 (a_1^2 c_1^2 + (1 - (a_1 - c_1) a_1)^2) \\ & + 2|\rho_s \sigma_1 \sigma_2| |a_1 c_1 (1 - (a_1 - c_1) a_1)|. \end{aligned} \quad (\text{E.14})$$

First, we write:

$$\begin{aligned} a_1(a_1 - c_1) - 1 & = a_1 \left(a_1 - \frac{(a_1^4 - 1)}{2a_1^3} \right) - 1 \\ & = \frac{a_1^5 + a_1}{2a_1^3} - 1 \\ & = \frac{a_1^4 + 1}{2a_1^2} - 1 \\ & = \frac{(a_1^2 - 1)^2}{2a_1^2}, \end{aligned}$$

which is clearly positive. Hence, we write (E.14) explicitly without the absolute value sign as follows:

$$\begin{aligned} & \sigma_1^2 (a_1^2 c_1^2 + ((a_1 - c_1)a_1 - 1)^2) \\ & + 2|\rho_s \sigma_1 \sigma_2| a_1 c_1 ((a_1 - c_1)a_1 - 1). \end{aligned} \quad (\text{E.15})$$

Next, we upper bound c_1 . Recall from Subsection E-A that $x_0 < 1 + \frac{P}{2\sigma_z^2}$ which implies that $a_1 < \sqrt{1 + \frac{P}{2\sigma_z^2}}$. Furthermore, Lemma 2 implies that if $P < \frac{1}{4} \delta (4 + \delta) \sigma_z^2 +$

$\sqrt{\frac{1}{4} \delta ((4 + \delta) \sigma_z^2)^2 \left(1 + \frac{\delta}{4}\right)}$ then $\sqrt{1 + \frac{2P}{(4+\delta)\sigma_z^2}} < a_1$. Thus, we write:

$$c_1 = \frac{a_1^4 - 1}{2a_1^3} \leq \frac{\left(1 + \frac{P}{2\sigma_z^2}\right)^2 - 1}{2 \left(1 + \frac{2P}{(4+\delta)\sigma_z^2}\right)^{\frac{3}{2}}} \triangleq \varphi_1(P, \delta). \quad (\text{E.16})$$

Similarly, we also have:

$$\begin{aligned} a_1(a_1 - c_1) - 1 & = \frac{(a_1^2 - 1)^2}{2a_1^2} \\ & \leq \frac{P^2}{8\sigma_z^4 \left(1 + \frac{2P}{(4+\delta)\sigma_z^2}\right)} \\ & \triangleq \varphi_2(P, \delta). \end{aligned}$$

Hence, we upper bound (E.15) with (E.17) which appears at the top of the page. Finally, combining (E.13) and (E.17) we have that if $P < \frac{1}{4} \delta (4 + \delta) \sigma_z^2 + \sqrt{\frac{1}{4} \delta ((4 + \delta) \sigma_z^2)^2 \left(1 + \frac{\delta}{4}\right)}$ then (E.18) at the top of this page constitutes an upper bound on β_1 .

C. Lower bound the RHS of (E.1)

Next, we use the upper bounds on β_1 and on $2 \log |a_1|$ to lower bound the RHS of (E.1):

$$\begin{aligned} & \frac{\log \left(\frac{\sigma_1^2 \sigma_z^2 - \beta_1^2 - D_1 \sigma_z^2}{(2\beta_1 + \sigma_1^2) D_1} \right)}{2 \log |a_1|} \\ & \geq \frac{2\sigma_z^2 \log \left(\frac{\sigma_1^2 \sigma_z^2 - \tau^2(P, \delta) - D_1 \sigma_z^2}{(2\tau(P, \delta) + \sigma_1^2) D_1} \right)}{P}. \end{aligned}$$

Thus, using this bound explicitly in (E.1) we obtain a sufficient condition for $K_{\text{OL}}^{\text{ub}} < K_{\text{LQG}}^{\text{lb}}$:

$$\begin{aligned} & \frac{2\sigma_z^2 \log \left(\frac{\sigma_1^2 \sigma_z^2 - \tau^2(P, \delta) - D_1 \sigma_z^2}{(2\tau(P, \delta) + \sigma_1^2) D_1} \right)}{P} \\ & - \frac{2(P + \sigma_z^2)}{P} \log \left(\frac{\sigma_1^2}{D_1} \right) > 0, \end{aligned}$$

which can also be written as:

$$\begin{aligned} & \sigma_z^2 \log \left(\frac{\sigma_1^2 \sigma_z^2 - \tau^2(P, \delta) - D_1 \sigma_z^2}{(2\tau(P, \delta) + \sigma_1^2) D_1} \right) \\ & - (P + \sigma_z^2) \log \left(\frac{\sigma_1^2}{D_1} \right) > 0. \end{aligned}$$

Therefore, we obtain (13):

$$\sigma_z^2 \log \left(\frac{\sigma_1^2 \sigma_{z,1}^2 - \tau^2(P, \delta) - D_1 \sigma_{z,1}^2}{(2\tau(P, \delta) + \sigma_1^2) \sigma_1^2} \right) - P \log \left(\frac{\sigma_1^2}{D_1} \right) > 0,$$

which concludes the proof of Corollary 1.

REFERENCES

- [1] J. P. M. Schalkwijk and T. Kailath, "Coding scheme for additive noise channels with feedback-Part I: No bandwidth constraint.", *IEEE Trans. Inf. Theory*, vol. 12, no. 2, pp. 172–182, Apr. 1966.
- [2] L. H. Ozarow and S. K. Leung-Yan-Cheong, "An achievable region and outer bound for the Gaussian broadcast channel with feedback", *IEEE Trans. Inf. Theory*, vol. 30, no. 4, pp. 667–671, Jul. 1984.
- [3] G. Kramer, "Feedback strategies for white Gaussian interference networks", *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1423–1438, Jun. 2002.
- [4] N. Elia, "When Bode meets Shannon: Control oriented feedback communication schemes", *IEEE Trans. Automat. Control*, vol. 49, no. 9, pp. 1477–1488, Sep. 2004.
- [5] E. Ardestanizadeh, P. Minero and M. Franceschetti, "LQG control approach to Gaussian broadcast channels with feedback", *IEEE Trans. Inf. Theory*, vol. 58, no. 8, pp. 5267–5278, Aug. 2012.
- [6] S. B. Amor, Y. Steinberg and M. Wigger, "Duality with linear-feedback schemes for the scalar Gaussian MAC and BC", *Proc. Int. Zurich Seminar Commun.*, Zurich, Switzerland, Feb. 2014, pp. 25–28.
- [7] M. Gastpar, A. Lapidoth, Y. Steinberg, and M. Wigger, "Coding schemes and asymptotic capacity for the Gaussian broadcast and interference channels with feedback", *IEEE Trans. Inf. Theory*, vol. 60, no. 1, pp. 54–71, Jan. 2014.
- [8] Y. Wu, P. Minero, and M. Wigger, "Insufficiency of linear-feedback schemes in Gaussian broadcast channels with common message," *IEEE Trans. Inf. Theory*, vol. 60, no. 8, pp. 4553–4566, Aug. 2014.
- [9] A. Jain, D. Gündüz, S. R. Kulkarni, V. Poor and S. Verdú, "Energy-distortion tradeoffs in Gaussian joint source-channel coding problems", *IEEE Trans. Inf. Theory*, vol. 58, no. 5, pp. 3153–3168, May 2012.
- [10] L. H. Ozarow, "The capacity of the white Gaussian multiple access channel with feedback", *IEEE Trans. Inf. Theory*, vol. 30, no. 4, pp. 623–629, Jul. 1984.
- [11] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Englewood Cliffs, NJ: Prentice Hall, 1993.
- [12] I. N. Bronshtein, K. A. Semendyayev, G. Musiol and H. Muehlig *Handbook of Mathematics*. 5th ed. Springer Berlin Heidelberg New York, 2007.
- [13] H. S. Wilf, "Budán's theorem for a class of entire functions", *Proc. Amer. Math. Soc.*, vol. 13, no. 1, pp. 122–125, Feb. 1962.
- [14] Y. H. Kim, "Feedback capacity of stationary Gaussian channels", *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 57–85, Jan. 2010.
- [15] A. J. Rojas, "Explicit solution for a class of discrete-time algebraic Riccati equations", *Asian Jour. Cont.*, vol. 15, no. 1, pp. 132–141, Jan. 2013.