INTEGRABILITY ESTIMATES FOR GAUSSIAN ROUGH DIFFERENTIAL EQUATIONS

THOMAS CASS, CHRISTIAN LITTERER, AND TERRY LYONS

Abstract. We derive explicit tail-estimates for the Jacobian of the solution flow for stochastic differential equations driven by Gaussian rough paths. In particular, we deduce that the Jacobian has finite moments of all order for a wide class of Gaussian process including fractional Brownian motion with Hurst parameter $H > 1/4$. We remark on the relevance of such estimates to a number of significant open problems.

1. Introduction

Gaussian processes that are not necessarily semimartingales arise in modeling a large variety of natural phenomena. The range of their applications reaches from fluid dynamics (e.g. randomly forced Navier-Stokes systems [17]) via the modeling of financial markets under transaction costs ([15]) to the study of internet traffic through queueing models based on fractional Brownian motion (fBm) [16]. These applications motivate the study of stochastic differential equations of the form

$$dY_t = V(Y_t)dX_t, \quad Y(0) = y_0 \tag{1.1}$$

driven by Gaussian signals. Over the past decade extensive progress has been made understanding the behaviour of solutions to such equations. In particular, for the case of fBm with Hurst parameter $H > 1/4$ the work of Cass and Friz [3] shows the existence of the density for (1.1) under Hörmander's condition; Hairer et al. [1], [18] have shown the smoothness of this density and established ergodicity under the regime $H > 1/2$.

Various recent works (Coutin-Qian [5], Ledoux-Qian-Zhang [24], Friz-Victoir [10], Lyons-Hambly [20]) have explored the use of rough paths to understand differential equations driven by non-semimartingale noise processes. Within this framework we can make sense of the solutions to (1.1) driven by a broader class of Gaussian noises (which includes fBm $H > 1/4$) than classical analysis based on Young integration. Thus, if we consider the flow $U^{\mathbf{X}}_{t\leftarrow 0}(y_0) \equiv Y_t$ of the RDE (1.1) then under sufficient regularity on V, the map $U^{\mathbf{X}}_{t\leftarrow 0}(\cdot)$ is a differentiable function (see, for example, [10]) and its derivative ("the Jacobian"):

$$J^{\mathbf{X}}_{t\leftarrow 0}(y_0) \equiv DU^{\mathbf{X}}_{t\leftarrow 0}(\cdot)\,|_{\cdot=y_0}$$

The research of Thomas Cass is supported by EPSRC grant EP/F029578/1 and a Career Development Fellowship at Christ Church, Oxford.

The research of Christian Litterer is supported by a grant of the European Research Council (ERC grant nr. 258237).

Part of this research was carried out during a stay of the first two authors at the Hausdorff Institute, Bonn.

The research of Terry Lyons is supported by EPSRC grant EP/H000100/1.

satisfies path-by-path an RDE of linear growth driven by $\mathbf{X}$.

A careful reading of the diverse applications in [1], [18] reveals a surprisingly generic common obstacle to the extensions of such results to the rough path regime. This obstacle eventually boils down to the need for sharp estimates on the integrability of the Jacobian of the flow $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ of an RDE. Cass, Lyons [2] and Inahama [22] establish such integrability for the Brownian rough path but only by using the independence of the increments; for more general Gaussian processes a more careful analysis is needed. To understand the difficulty of this problem, we note from [10] that the standard deterministic estimate on $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ gives

$$|J^{\mathbf{x}}_{t\leftarrow 0}(y_0)| \leq C \exp\left(C \left|\left|\mathbf{x}\right|\right|^p_{p\text{-var};[0,T]}\right). \tag{1.2}$$

But in the case where $\mathbf{X}$ is a Gaussian rough path and $p > 2$ (i.e. Brownian-type paths or rougher) the Fernique-type estimates of [8] give only that $||\mathbf{X}||_{p\text{-var};[0,T]}$ has a Gaussian tail, hence the right hand side of (1.2) is not integrable in general. Worse still, the work Oberhauser and Friz [7] shows that the inequality (1.2) can actually be saturated for a (deterministic) choice of differential equation and driving rough path. However, for random paths that have enough structure to them (in particular for Gaussian paths) only a set of small (or zero) measure comes close to equality in (1.2). What is therefore needed (and what we provide!) is to recast the deterministic estimate in a form that allows us to more strongly interrogate the underlying probabilistic structure

Our results will allows us to deduce the existence of moments of all orders for $J^{\mathbf{x}}_{t\leftarrow 0}(y_0)$ for RDEs driven by a class of Gaussian processes (including, but not restricted to, fBm with Hurst index $H > 1/4$). In fact, our main estimate shows much more than simple moment estimates, namely that the logarithm of the Jacobian has a tail that decays faster than an exponential (to be a little more precise: we will show that

$$P\left(\log\left[\sup_{t\in[0,T]}\left|J^{\mathbf{X}}_{t\leftarrow 0}(y_0)\right|\right] > x\right) \lesssim \exp\left(-x^r\right) \tag{1.3}$$

for any $r < r_0 \in (1,2]$, and we describe r_0 in terms of the regularity properties of the Gaussian path.)

The results are relevant to a number of important problems. Firstly, they are a necessary ingredient if one wants to extend the work of [17] and [18] on the ergodicity of non-Markovian systems. Secondly, they are also an important ingredient in a Malliavin calculus proof on the smoothness of the density for RDEs driven by rough Gaussian noise in the elliptic setting. Furthermore, it allows one to achieve an analogue of Hörmander's Theorem on the smoothness of the density for Gaussian RDEs in conjunction with a suitable version of Norris's Lemma (see [29],[30]). In this context, we remark that Hu and Tindel [21] have recently obtained a Norris Lemma for fBm with $H > 1/3$ and proved smoothness-of-density results for a class of nilpotent RDEs. Hairer and Pillai [19] have also proved Hörmander-type theorems for a general class of RDEs; their results are predicated on the assumption that the Jacobian has finite moments of all order. Hence, one application of this paper is to use the tail estimate (1.3) together with the results in [21] or [19] to conclude that for $t > 0$ the law of Y_t (the solution to (1.1)) will, under Hörmander's condition, have a smooth density w.r.t. Lebesgue measure on $\mathbb{R}^e$ for a rich classes of Gaussian processes X including fBm $H > 1/3$. All of these problems (and many more besides) require, in one way or another, the L^q integrability of the Malliavin covariance matrix of the Wiener functional $Y(\omega)$, which is itself expressed in terms of the Jacobian.

The techniques developed in this paper are relevant to the study of more general RDEs and not just the one solved by the Jacobian. For example, the deterministic estimates we derive can also be obtained in the following cases (cf. Friz, Victoir [10]):

(1) RDEs driven along linear vector fields of the form $V_i(z) = A_i z + b_i$ for $e \times e$ matrices A_i and b_i in $\mathbb{R}^e$;
(2) Higher order derivatives of the flow (subject to suitably enhance regularity on the vector fields defining the flow);
(3) The inverse of the Jacobian of the flow;
(4) Situations where one wants to control the distance between two RDE solution in the (inhomogeneous) rough path metric (for example in stochastic fixed point theorems)

Recent work ([14])has extended the class of linear-growth RDEs for which we have non-explosion and there may be scope to extend our results to this setting. In this paper we focus only on the Jacobian because of its central role in the wide range of problems we have outlined.

We finally remark that the recent preprint [9] (The authors have informed us that there is a gap in their argument that they are presently dealing with.) is also concerned with finiteness of the moments in the special case of linear RDEs. Our analysis can be adapted to establish the moment bounds in [9] for this linear case with the Gaussian driving noises considered in our paper.

We now outline the structure of the paper. In section 2 we introduce some important notation and concepts on the theory of rough paths. Because this is now standard and there are many references available (e.g., [25], [26], [10], [27]) we keep the detail to a minimum. In section 3 we derive a quantitative bound on the growth of $J^{\mathbf{x}}_{t\leftarrow 0}$; the estimates we derive here are based very closely on [10]. We end up with a control on $J^{\mathbf{x}}_{t\leftarrow 0}$ in terms of a function on the space on (rough) path space which we (suggestively) name the accumulated α local p-variation (denoted by $M_{\alpha,I,p}(\cdot)$). When $\mathbf{X}$ is taken to be a Gaussian rough path the integrability properties of $M_{\alpha,I,p}(\mathbf{X})$ are not immediately obvious or easy to study therefore, we spend section 4 deriving a relationship between $M_{\alpha,I,p}(\cdot)$ and another function on path space- which we denote $N_{\alpha,p,I}(\cdot)$ (the analysis at this stage remains entirely deterministic). Section 5 records some facts about Gaussian rough paths including the crucial embedding theorems for the associated Cameron-Martin spaces that have been derived in [10]. We then present the main tail estimate on $N_{\alpha,p,I}(\mathbf{X})$ – our analysis is based on Gaussian isoperimetry (more specifically Borell's inequality, which we recall). Once this is achieved we can use the relationship between $J^{\mathbf{X}}_{t\leftarrow 0}$ and $N_{\alpha,p,I}(\mathbf{X})$ to exhibit the stated tail behaviour of $J^{\mathbf{X}}_{t\leftarrow 0}$, which then constitutes our main result.

2. Rough path concepts and notation

There are now many articles and texts providing an overview on rough path theory (for example [26] and [10] to name just two) so we will focus on establishing the notation we need for the current application. We will study continuous $\mathbb{R}^d-$valued paths x parameterised by time on a compact interval I (sometimes I will be taken to be $[0,T]$) and we denote the space of such functions by $C\left(I,\mathbb{R}^d\right)$ .We write $x_{s,t} = x_t - x_s$ as a shorthand for the increments of a path and for x in $C\left(I,\mathbb{R}^d\right)$ we have

$$|x|_\infty := \sup_{t\in I}|x_t|, \qquad |x|_{p\text{-var};I} := \left(\sup_{D[I]=(t_j)}\sum_{j:t_j\in D[I]}\left|x_{t_j,t_{j+1}}\right|^p\right)^{1/p},$$

for $p\geq 1$ (we refer to these quantities both symbolically and by name, i.e. the uniform norm and the p-variation semi-norm). We denote by $C^{p-var}\left(I,\mathbb{R}^d\right)$ the linear subspace of $C\left(I,\mathbb{R}^d\right)$ consisting of path of finite $p-$variation. In the case where x is in $C^{p-var}\left(I,\mathbb{R}^d\right)$ and p is in $[1,2)$, the iterated integrals of x are canonically defined by Young integration and the collection of all these iterated integrals together gives the signature: for $s<t$ in I

$$S\left(x\right)_{s,t}:=1+\sum_{k=1}^{\infty}\int_{s<t_1<t_2<....<t_k<t}dx_{t_1}\otimes dx_{t_2}\otimes...\otimes dx_{t_k}\in T\left(\mathbb{R}^d\right).$$

By writing $S\left(x\right)_{\inf I,\cdot}$ we can regard the signature as a path (on I) with values in the tensor algebra; similarly, the truncated signature

$$S_N\left(x\right)_{s,t}:=1+\sum_{k=1}^{N}\int_{s<t_1<t_2<....<t_k<t}dx_{t_1}\otimes dx_{t_2}\otimes...\otimes dx_{t_k}\in T^N\left(\mathbb{R}^d\right)$$

is a path in the truncated tensor algebra $T^N\left(\mathbb{R}^d\right)$. It is a well-known fact that the path $S_N\left(x\right)_{\inf I,\cdot}$ takes values in the step-N free nilpotent group with d generators, which we denote $G^N\left(\mathbb{R}^d\right)$. More generally, if $p\geq 1$ we can consider the set of such group-valued paths

$$\mathbf{x}_t=\left(1,\mathbf{x}_t^1,...,\mathbf{x}_t^{\lfloor p\rfloor}\right)\in G^{\lfloor p\rfloor}\left(\mathbb{R}^d\right).$$

The advantage this offers is that the group structure provides a natural notion of increment, namely $\mathbf{x}_{s,t}:=\mathbf{x}_s^{-1}\otimes\mathbf{x}_t$, and we can describe the set of "norms" on $G^{\lfloor p\rfloor}\left(\mathbb{R}^d\right)$ which are homogeneous with respect to the natural scaling operation on the tensor algebra (see [10] for definitions and details). One such example is the Carnot-Caratheodory norm (see [10]), which we denote by $||\cdot||_{CC}$. The subset of these so-called homogeneous norms which are symmetric and sub-additive ([10]) give rise to genuine metrics on $G^{\lfloor p\rfloor}\left(\mathbb{R}^d\right)$, which in turn give rise to a notion of homogenous p-variation metrics $d_{p\text{-var}}$ on the $G^{\lfloor p\rfloor}\left(\mathbb{R}^d\right)$-valued paths. Let

$$||\mathbf{x}||_{CC,p\text{-var};[0,T]}=\max_{i=1,...,\lfloor p\rfloor}\left(\sup_{D[0,T]=(t_j)}\sum_{j:t_j\in D[0,T]}\left|\left|\mathbf{x}_{t_j,t_{j+1}}\right|\right|_{CC}^p\right)^{1/p}\tag{2.1}$$

and note that if (2.1) is finite then $\omega_{CC}\left(s,t\right):=||\mathbf{x}||_{CC,p\text{-var};[s,t]}^p$ is a control (i.e. it is a continuous, non-negative, super-additive function on the simplex $\Delta_T=\left\{(s,t):0\leq s\leq t\leq T\right\}$ which vanishes on the diagonal.)

The space of weakly geometric $p-$rough paths (denote $WG\Omega_p\left(\mathbb{R}^d\right)$) is the set of paths (parameterised over I, although this is often implicit) with values in $G^{\lfloor p\rfloor}\left(\mathbb{R}^d\right)$ such that (2.1) is finite. A refinement of this notion is the space of geometric $p-$rough paths (denoted $G\Omega_p\left(\mathbb{R}^d\right)$) which is the closure of

$$\left\{S_{\lfloor p\rfloor}\left(x\right)_{\inf I,\cdot}:x\in C^{1-var}\left(I,\mathbb{R}^d\right)\right\}$$

with respect to $d_{p-\text{var}}$.

We will often end up considering an RDE driven by a path $\mathbf{x}$ in $WG\Omega_p\left(\mathbb{R}^d\right)$ along a collection of vector fields $V=\left(V^1,...,V^d\right)$ on $\mathbb{R}^e$. And from the point of view of existence and uniqueness results, the appropriate way to measure the regularity of the V_is results turns out to be the notion of Lipschitz-γ (short: Lip-γ) in the sense of Stein (see [10] and [26] and note the contrast with classical Lipschitzness). This notion provides a norm on the space of such vector fields (the Lip-γ norm), which we denote $|\cdot|_{Lip-\gamma}$, and for the collection of vector fields V we will often make use of the shorthand

$$|V|_{Lip-\gamma} = \max_{i=1,\ldots,d} |V_i|_{Lip-\gamma},$$

and refer to the quantity $|V|_{Lip-\gamma}$ as the Lip$-\gamma$ norm of V.

3. The derivative of the flow for ODEs and RDEs

Consider a solution to an ODEs driven by a path x in $C^{1-var}\left([0,T],\mathbb{R}^d\right)$ along some vector fields $V=(V_1,...,V_d)$ in $\mathbb{R}^e$, viz.

$$dy_t = V(y_t)\,dx_t,\ \ y(0)=y_0, \tag{3.1}$$

we call the map $y_0 \mapsto U^x_{\cdot\leftarrow 0}(y_0) \equiv y_\cdot \in C([0,T],\mathbb{R}^e)$ the (solution) flow. If the vector fields are sufficiently smooth then the map $y_0 \mapsto U^x_{t\leftarrow 0}(y_0)$ is differentiable for every t in I, moreover the derivative $J^x_{t\leftarrow 0}(y_0)$ ("the Jacobian of flow") is a path in $C([0,T],\mathbb{R}^{e\times e})$ which satisfies the ODE obtained by formal differentiating (3.1) w.r.t. y_0. Taken together with the solution to (3.1) this gives a path $z_t = (y_t, J^x_{t\leftarrow 0}(y_0)) \in \mathbb{R}^e \oplus \mathbb{R}^{e\times e}$, which is the solution to

$$dz_t = \hat{V}(z_t)\,dx_t,\ \ z(0) = (y_0, I),$$

where $\left(\hat{V}_i\right)_{1\le i\le d}$ is a collection of vector fields on $\mathbb{R}^e\oplus\mathbb{R}^{e\times e}$ defined by

$$\hat{V}_i(y,J) = (V_i(y), DV_i(y)\,J).$$

We now prove the necessary technical results that will allow us to obtain growth estimates for the Jacobian of the flow for an RDE; the culmination of this effort will be Lemma 3.1 which then forms the bedrock for the subsequent work. To describe the context for these results suppose we are interested in driving an RDE by a path $\mathbf{x}$ in $WG\Omega_p\left(\mathbb{R}^d\right)$ parameterised over I. Then, for any fixed a sub-interval $[s,t]\subset I$ Chow's Theorem (chapter 2 of [28]) shows the existence of a path $x^{s,t}$ in $C^{1-var}\left([s,t],\mathbb{R}^d\right)$ whose signature matches the increments of $\mathbf{x}$ over (s,t), i.e.

$$\mathbf{x}_{s,t} = S_{\lfloor p\rfloor}\left(x^{s,t}\right)_{s,t}. \tag{3.2}$$

Motivated by the discussion in [10] and [6], we can then consider the solution to the RDE over $[s,t]$ by the solution of the ODE driven by $x^{s,t}$; the advantage in doing this is that (3.2) ensures that the $\lfloor p\rfloor^{th}$-order Euler approximations to these two solutions agree, and therefore one can reasonably expect the two solutions to be close over small time intervals. Furthermore, Chen's identity (Theorem 2.9 of [26]) allows us to relate the signatures of $x^{s,t}$ and $x^{t,u}$over two consecutive intervals $[s,t]$ and $[t,u]$ via

$$S_{\lfloor p\rfloor}\left(x^{s,t}*x^{t,u}\right)_{s,u} = S_{\lfloor p\rfloor}\left(x^{s,t}\right)_{s,t}\otimes S_{\lfloor p\rfloor}\left(x^{t,u}\right)_{t,u} = \mathbf{x}_{s,t}\otimes\mathbf{x}_{t,u} = \mathbf{x}_{s,u}$$

(here $*$ denotes concatenation), preserving the relationship between the signature of the concatenation over $[s,u]$ and the increment $\mathbf{x}_{s,u}$. The following lemma, although classical, is an important step in making these ideas more precise. It provides quantitative error estimates for the approximation of a classical Jacobian ODE via its Euler scheme, and it should be compared to Proposition 10.3 of [10] which achieves something similar for ODEs driven along bounded vector fields.

Lemma 3.1. *Let $1 \leq p < \gamma < \lfloor p \rfloor + 1$ and $x \in C^{1-var}\left([0,T], \mathbb{R}^d\right)$ be a continuous path of finite $1-$variation on a compact interval I. Suppose that $V = (V_i)_{i=1}^d$ is collection of Lip-γ vector fields on $\mathbb{R}^e$, that y is the solution to the ODE*

$$dy_t = V(y_t)\, dx_t, \quad y(0) = y_0$$

and let $J^x_{t\leftarrow 0}(y_0)$ be the Jacobian of the corresponding flow. Then the path $z_t = (y_t, J^x_{t\leftarrow 0}(y_0)) \in \mathbb{R}^e \oplus \mathbb{R}^{e\times e}$ is the solution to the ODE

$$dz_t = \hat{V}(z_t)\, dx_t, \quad z(0) = (y_0, I),$$

where $\left(\hat{V}_i\right)_{1\leq i\leq d}$ is a collection of vector fields on $\mathbb{R}^e \oplus \mathbb{R}^{e\times e}$ defined by

$$\hat{V}_i(y, J) = (V_i(y), DV_i(y)\, J). \tag{3.3}$$

Moreover, if $I : \mathbb{R}^e \oplus \mathbb{R}^{e\times e} \to \mathbb{R}^e \oplus \mathbb{R}^{e\times e}$ is the identity function and the $\lfloor p \rfloor$-th level Euler approximation is denoted by

$$\xi_{(\hat{V})}\left(z_s, S_{\lfloor p \rfloor}(x)\right)_{s,t} = \sum_{k=1}^{\lfloor p \rfloor} \sum_{i_1,\ldots,i_k \in \{1,\ldots,d\}} \hat{V}_{i_1}\hat{V}_{i_1}....\hat{V}_{i_k} I(y)\, S_{\lfloor p \rfloor}(x)_{i_1,\ldots,i_k},$$

then for some finite constant c (depending only on p) we have that

$$\left|z_{s,t} - \xi_{(\hat{V})}\left(z_s, S_{\lfloor p \rfloor}(x)\right)_{s,t}\right| \leq c(1 + |z_s|)\left(|V|_{Lip-\gamma}|x|_{1-var;[s,t]}\right)^{\gamma} \exp\left(c|V|_{Lip-\gamma}|x|_{1-var;[s,t]}\right). \tag{3.4}$$

Proof. The c_is will denote constants depending on p and γ . The fact that z solves the stated ODE is classical. An elementary application of Gronwall's inequality shows that for all $0 \leq s < t \leq T$

$$|J^x_{\cdot\leftarrow 0}(y_0)|_{\infty;[s,t]} \leq |J^x_{s\leftarrow 0}(y_0)| \exp\left(c_1|V|_{Lip-\gamma}|x|_{1-var;[s,t]}\right). \tag{3.5}$$

Using this together with standard ODE estimates gives for any u in $[s,t]$

$$\begin{aligned} |J^x_{u\leftarrow 0}(y_0) - J^x_{s\leftarrow 0}(y_0)| &\leq |V|_{Lip-\gamma}|x|_{1-var;[s,t]}|J^x_{\cdot\leftarrow 0}(y_0)|_{\infty;[s,t]} \\ &\leq |J^x_{s\leftarrow 0}(y_0)||V|_{Lip-\gamma}|x|_{1-var;[s,t]} \exp\left(c_2|V|_{Lip-\gamma}|x|_{1-var;[s,t]}\right) \\ &\leq |z_s|\left(|V|_{Lip-\gamma}|x|_{1-var;[s,t]}\right)^{\gamma-\lfloor\gamma\rfloor} \exp\left(c_3|V|_{Lip-\gamma}|x|_{1-var;[s,t]}\right) \end{aligned} \tag{3.6}$$

and, at the same time, we observe that

$$|y_u - y_s| \leq |V|_{Lip-\gamma}|x|_{1-var;[s,t]}. \tag{3.7}$$

By imitating Proposition 10.3 of [10] we can obtain

$$\begin{aligned} &\left|z_{s,t} - \xi_{(\hat{V})}\left(z_s, S_{\lfloor p \rfloor}(x)\right)_{s,t}\right| \\ &\leq \sum_{i_1,\ldots,i_{\lfloor p \rfloor} \in \{1,\ldots,d\}} \left|\int_{s<u_1<\ldots<u_{\lfloor p \rfloor}<t} \left[\hat{V}_{i_1}....\hat{V}_{i_{\lfloor p \rfloor}} I(z_{u_1}) - \hat{V}_{i_1}....\hat{V}_{i_{\lfloor p \rfloor}} I(z_s)\right] dx^{i_1}_{u_1}...dx^{i_{\lfloor p \rfloor}}_{u_{\lfloor p \rfloor}}\right|, \end{aligned} \tag{3.8}$$

Using the Lip-γ regularity of the vector fields $(V_i)_{i=1}^d$, the linearity of $\left(\hat{V}_i(y,\cdot)\right)_{i=1}^d$ and the fact that $\lfloor p \rfloor = \lfloor \gamma \rfloor$, we can easily show that for all (y, J) in $\mathbb{R}^e \oplus \mathbb{R}^{e\times e}$ the function $\hat{V}_{i_1}....\hat{V}_{i_{\lfloor p \rfloor}} I(y, J)$ is

$(\gamma-\lfloor\gamma\rfloor)$ −Hölder in y and linear in J, with both the Hölder constant and norm of the linear map bounded by $c_4 |V|_{Lip-\gamma}^{\lfloor\gamma\rfloor}$. We therefore deduce immediately that

(3.9)

$$\left|\hat{V}_{i_1}....\hat{V}_{i_{\lfloor p\rfloor}} I\left(\tilde{y},\tilde{J}\right)-\hat{V}_{i_1}....\hat{V}_{i_{\lfloor p\rfloor}} I\left(y,J\right)\right|\leq c_4 |V|_{Lip-\gamma}^{\lfloor\gamma\rfloor}\left[\left(1+|(y,J)|\right)|y-\tilde{y}|^{\gamma-\lfloor\gamma\rfloor}+\left|J-\tilde{J}\right|\right].$$

Finally, we can use (3.6), (3.7) and (3.9) to bound the integrand in (3.8) uniformly over $[s,t]$ by

$$c_4 |V|_{Lip-\gamma}^{\gamma}\left(1+|z_s|\right)|x|_{1-var;[s,t]}^{\gamma-\lfloor\gamma\rfloor}\exp\left(c_5 |V|_{Lip-\gamma}|x|_{1-var;[s,t]}\right),$$

the result follows by classical estimates on the integral in (3.8). □

We now prepare the ground for the main growth estimate on the Jacobian $J_{t\leftarrow 0}^{\mathbf{x}}(y_0)$ of a RDE flow. In [10] the authors derive the bound

$$\left|J_{\cdot\leftarrow 0}^{\mathbf{x}}(y_0)\right|_{\infty;[0,T]}\leq C\exp\left(C\,||\mathbf{x}||_{CC,p\text{-var};[0,T]}^{p}\right)$$

but, as we remarked in the introduction, this is not useful for addressing the problem of the integrability of the Jacobian because, when $\mathbf{x}$ is replaced by a Gaussian rough path $\mathbf{X}$, the random variable $||\mathbf{X}||_{p\text{-var};[0,T]}$ only has a Gaussian tail. Nonetheless, the next theorem shows how we can use this as an initial estimate to bootstrap our way back to some bound which, as we will see, is more sensitive to the fine structure of the path.

Theorem 3.2. *For some $\gamma>p\geq 1$ suppose that $\mathbf{x}$ is a weakly geometric p-rough path over $[0,T]$ and $V=\left(V^1,...,V^d\right)$ a collection of Lip-γ vector fields on $\mathbb{R}^e$. Then there is a unique solution to the RDE:*

$$dy_t=V(y_t)\,d\mathbf{x}_t,\;\; y(0)=y_0 \tag{3.10}$$

which induces a differentiable flow $U_{\cdot\leftarrow 0}^{\mathbf{x}}:\mathbb{R}^e\to C^{p-var}\left([0,T],\mathbb{R}^e\right)$ such that $U_{\cdot\leftarrow 0}^{\mathbf{x}}(y_0)=y$. If the derivative of this flow is denoted $J_{\cdot\leftarrow 0}^{\mathbf{x}}(y_0)$ then $z_t=\left(y_t,J_{\cdot\leftarrow 0}^{\mathbf{x}}(y_0)\right)$ satisfies the non-explosive RDE

$$dz_t=\hat{V}(z_t)\,dx_t,\;\; z(0)=(y_0,I),$$

where $\left(\hat{V}_i\right)_{1\leq i\leq d}$ is the collection of vector fields on $\mathbb{R}^e\oplus\mathbb{R}^{e\times e}$ defined by

$$\hat{V}_i(y,J)=\left(V_i(y),DV_i(y)\,J\right).$$

Moreover, if ω_{CC} is the control $\omega_{CC}(s,t)\equiv||\mathbf{x}||_{CC,p\text{-}var;[s,t]}^{p}$ then for any $\alpha>0$ the Jacobian satisfies the growth estimate

(3.11)

$$\left|J_{\cdot\leftarrow 0}^{\mathbf{x}}(y_0)\right|_{\infty;[0,T]}=:\left|\pi_{\mathbb{R}^{e\times e}}z\right|_{\infty;[0,T]}\leq C\exp\left[C\,|V|_{Lip-\gamma}^{p}\sup_{\substack{D[0,T]=(t_i),\\ \omega_{CC}(t_i,t_{i+1})\leq|V|_{Lip-\gamma}^{-p}\alpha}}\sum_{i:t_i\in D}\omega_{CC}(t_i,t_{i+1})\right]$$

for some constant C depending only on p,γ and α.

Proof. The differentiability under the hypothesis of Lip-γ regularity on V is proved in [10]; they also prove that $z_t=\left(y_t,J_{t\leftarrow 0}^{\mathbf{x}}(y_0)\right)$ satisfies (locally) the RDE

$$dz_t=\hat{V}(z_t)\,d\mathbf{x}_t,\;\; z(0)=(y_0,I). \tag{3.12}$$

The fact, that the (a priori only local) solution is global can be seen from the estimate

$$|J^{\mathbf{x}}_{\cdot\leftarrow 0}(y_0)|_{\infty;[0,T]} \leq C \exp\left(C\,||\mathbf{x}||^p_{CC,p\text{-var};[0,T]}\right),$$

for $C > 1$, which follows from Exercise 11.10 of [10] and which precludes (3.12) from having explosive solutions.

To improve this growth estimate and arrive at (3.11) we first note that if

$$R := C \exp\left[C\left(1 \vee ||\mathbf{x}||^p_{CC,p\text{-var};[0,T]}\right)\right] \tag{3.13}$$

then we can localise the vector fields so that

$$\hat{V} \equiv \hat{W} \text{ on } \mathbb{R}^e \times \left\{A \in \mathbb{R}^{e\times e} : |A| \leq R^2\right\}$$

for some (globally!) Lip-$(\gamma-1)$ vector fields $\hat{W}$. It follows that z is a solution to the RDE

$$dz_t = \hat{W}(z_t)\,d\mathbf{x}_t.$$

We now proceed as in the proof of Lemma 10.52 of [10]. Thus, for each $s < t$ in $[0,T]$ let $x^{s,t}$ denote a path in $C^{1-\text{var}}\left([0,T],\mathbb{R}^d\right)$ such that

$$S_{\lfloor p\rfloor}\left(x^{s,t}\right)_{s,t} = \mathbf{x}_{s,t} \quad \text{and} \quad \left|x^{s,t}\right|_{1-var;[s,t]} \leq c_1\,||\mathbf{x}||^p_{CC,p\text{-var};[s,t]} \tag{3.14}$$

for c_1 depending only on p (the existence of such paths follows from Proposition 7.64 of [10]). By defining

$$\Gamma_{s,t} := z_{s,t} - z^{(s,t)}_{s,t}$$

where $z^{(s,t)}$ is the solution to the ODE

$$dz^{(s,t)}_u = \hat{V}\left(z^{(s,t)}_u\right)dx^{s,t}_u,\;\; z^{(s,t)}_s = z_s$$

we can mimic the proof of Lemma 10.52 of [10], replacing applications of their Lemma A_{linear} with our Lemma 3.1. But we need to record some changes to the proof given there; firstly, the simple Gronwall estimate (3.5) together with (3.14) shows (at the cost of possibly increasing the constant C in (3.13))

$$\left|\pi_{\mathbb{R}^{e\times e}} z^{(s,t)}_{\cdot}\right|_{\infty;[s,t]} \leq |z_s| \exp\left(c\,|V|_{Lip-\gamma}\,|x|_{1-var;[s,t]}\right) \leq R^2.$$

Thus, on $[s,t]$ we have that $z^{(s,t)}$ also solves the ODE

$$dz^{(s,t)}_u = \hat{W}\left(z^{(s,t)}_u\right)dx^{s,t}_u,\;\; z^{(s,t)}_s = z_s.$$

The estimates in Theorem 10.14 of [10] then show that

$$|\Gamma_{s,t}| \leq c_2\left(\left|\hat{W}\right|_{Lip-(\gamma-1)}||\mathbf{x}||_{CC,p\text{-var};[s,t]}\right)^{\gamma} =: \hat{\omega}(s,t)^{\gamma/p}$$

and hence

$$\lim_{r\to 0}\sup_{s<t,\hat{\omega}(s,t)\leq r}\frac{|\Gamma_{s,t}|}{r} = 0. \tag{3.15}$$

It is crucial to note that the control $\hat{\omega}$ is only used to verify (3.15) and does not feature in the final estimate. We can deduce the required estimate by applying Remark 10.64 of [10] once the argument there has been adapted to consider partitions with $|V|^p_{Lip-\gamma}\,\omega_{CC}(s,t)$ truncated at level α rather than at 1. Since these details are straight forward we omit them. □

Remark 3.3. *By reading very carefully the proof of Lemma 10.63 and Remark 10.64 of* [10] *it is possible to be a little more precise about the dependence of the constant C on the truncation parameter α. In fact we can show that*

$$\left|J^{\mathbf{x}}_{\cdot\leftarrow 0}(y_0)\right|_{\infty;[0,T]} \leq C_1 \exp\left[C_2 \left|V\right|^p_{Lip-\gamma} \sup_{\substack{D[0,T]=(t_i),\\ \omega_{CC}(t_i,t_{i+1})\leq |V|^{-p}_{Lip-\gamma}\alpha}} \sum_{i:t_i\in D} \omega_{CC}(t_i,t_{i+1})\right]$$

for two finite constant C_1 and C_2, where C_1 depends on p and γ and C_2 depends on p,γ and α in such a way that

$$C_2(p,\gamma,\alpha) = C_3(p,\gamma)\max\left(1,\alpha^{-1}\right) \tag{3.16}$$

for some finite $C_3(p,\gamma) > 1$. By replacing α by $|V|^p_{Lip-\gamma}\,\alpha$ this gives rise to the estimate

$$\left|J^{\mathbf{x}}_{\cdot\leftarrow 0}(y_0)\right|_{\infty;[0,T]} \leq C_1 \exp\left[C_2 \max\left(|V|^p_{Lip-\gamma}, \frac{1}{\alpha}\right) \sup_{\substack{D[0,T]=(t_i),\\ \omega_{CC}(t_i,t_{i+1})\leq\alpha}} \sum_{i:t_i\in D} \omega_{CC}(t_i,t_{i+1})\right]$$

for all $\alpha > 0$.

It will be convenient later on to use the estimates of the previous theorem with reference to controls other than those derived from p-variation norm based on the CC-metric. The following is simple corollary of Theorem 3.2 and Remark 3.3, it gathers together the appropriate assumptions we will need in subsequent sections.

Corollary 3.4. *Let the assumptions of Theorem 3.2 hold. Suppose $\omega : \Delta_{[0,T]} \to \mathbb{R}_+$ is a control which, for some finite $D > 1$, is equivalent to ω_{CC} in the sense that*

$$D^{-1}\omega(s,t) \leq \omega_{CC}(s,t) \leq D\omega(s,t). \tag{3.17}$$

Then we have the following growth estimate for the Jacobian:

$$\left|J^{\mathbf{x}}_{\cdot\leftarrow 0}(y_0)\right|_{\infty;[0,T]} \leq C \exp\left[C \max\left(D\left|V\right|^p_{Lip-\gamma}, \frac{1}{\alpha}\right) \sup_{\substack{D[0,T]=(t_i),\\ \omega(t_i,t_{i+1})\leq\alpha}} \sum_{i:t_i\in D} \omega(t_i,t_{i+1})\right] \tag{3.18}$$

for any $\alpha > 0$ and some constant C which depends only on p and γ.

Proof. From Theorem 3.2 and Remark 3.3 we already have that for any $\beta > 0$

$$\sup_{0\leq t\leq T}\left|J^{\mathbf{x}}_{t\leftarrow 0}(y_0)\right| \leq C_1 \exp\left[C_2 \max\left(|V|^p_{Lip-\gamma}, \frac{1}{\beta}\right) \sup_{\substack{D[0,T]=(t_i),\\ \omega_{CC}(t_i,t_{i+1})\leq\beta}} \sum_{i:t_i\in D} \omega_{CC}(t_i,t_{i+1})\right] \tag{3.19}$$

Therefore if we let $\alpha > 0$ and set $\beta = D\alpha$, we see from (3.17) that if $\omega_{CC}(s,t) \leq \beta$ then $\omega(s,t) \leq \alpha$. Hence applying (3.19) it is easy to deduce (3.18). □

Remark 3.5. *One control that satisfies the equivalence condition (3.17) and hence bounds the Jacobian in the manner described above is*

$$\omega_{\mathbf{x},p}(s,t)=||\mathbf{x}||^p_{p-var;[s,t]}:=\sum_{i=1}^{\lfloor p\rfloor}\sup_{D[s,t]=(t_j)}\sum_{j:t_j\in D[s,t]}\left|\mathbf{x}^i_{t_j,t_{j+1}}\right|^{p/i}_{(\mathbb{R}^d)^{\otimes i}}. \tag{3.20}$$

To see this we can exploit the fact that all homogenous norms on $G^N(\mathbb{R}^d)$ are equivalent (refer to [10] *and note that this is an intrinsically finite dimensional result). This, in particular gives that for some $c>0$*

$$c^{-1}\max_{i=1,\ldots,N}|\pi_i g|^{1/i}_{(\mathbb{R}^d)^{\otimes i}}\leq ||g||_{CC}\leq c\max_{i=1,\ldots,N}|\pi_i g|^{1/i}_{(\mathbb{R}^d)^{\otimes i}}$$

for all g in $G^N(\mathbb{R}^d)$. For any dissection $\{t_j:j=1,2,...,n\}$ of $[s,t]$ we then have on the one hand that

$$\begin{aligned}\sum_{j:t_j\in D[s,t]}||\mathbf{x}_{t_j,t_{j+1}}||^p&\leq c\sum_{j:t_j\in D[s,t]}\max_{i=1,\ldots,\lfloor p\rfloor}\left|\mathbf{x}^i_{t_j,t_{j+1}}\right|^{p/i}_{(\mathbb{R}^d)^{\otimes i}}\\&\leq c\sum_{i=1}^{\lfloor p\rfloor}\sum_{j:t_j\in D[s,t]}\left|\mathbf{x}^i_{t_j,t_{j+1}}\right|^{p/i}_{(\mathbb{R}^d)^{\otimes i}}\\&\leq c\omega_{\mathbf{x},p}(s,t).\end{aligned}\tag{3.21}$$

On the other hand for every $i=1,2,...,\lfloor p\rfloor$

$$\begin{aligned}\sum_{j:t_j\in D[s,t]}||\mathbf{x}_{t_j,t_{j+1}}||^p&\geq c^{-1}\sum_{j:t_j\in D[s,t]}\max_{i=1,\ldots,\lfloor p\rfloor}\left|\mathbf{x}^i_{t_j,t_{j+1}}\right|^{p/i}_{(\mathbb{R}^d)^{\otimes i}}\\&\geq c^{-1}\sum_{j:t_j\in D[s,t]}\left|\mathbf{x}^i_{t_j,t_{j+1}}\right|^{p/i}_{(\mathbb{R}^d)^{\otimes i}}.\end{aligned}$$

Taking the supremum over all dissections of $[s,t]$ and summing over $i=1,2,...,\lfloor p\rfloor$ we get

$$\lfloor p\rfloor\,\omega_{CC}(s,t)\geq c^{-1}\omega_{\mathbf{x},p}(s,t). \tag{3.22}$$

The desired relation follows from (3.21) and (3.22).

We will use the control $\omega_{\mathbf{x},p}$ extensively in the ensuing calculations. It therefore makes sense to distinguish it amongst the family of controls which are related to $\mathbf{x}$.

Definition 3.6. *Let $\mathbf{x}$ be a weakly geometric p-rough path over $[0,T]$. Define the control $\omega_{\mathbf{x},p}$ and the function $||\cdot||_{p-var}:WG\Omega_p(\mathbb{R}^d)\to\mathbb{R}_+$ by*

$$\omega_{\mathbf{x},p}(s,t)=||\mathbf{x}||^p_{p-var;[s,t]}:=\sum_{i=1}^{\lfloor p\rfloor}\sup_{D=(t_j)}\sum_{j:t_j\in D}\left|\mathbf{x}^i_{t_j,t_{j+1}}\right|^{p/i}_{(\mathbb{R}^d)^{\otimes i}}.$$

And refer to $\omega_{\mathbf{x},p}$ as the ***control induced by*** $\mathbf{x}$.

Remark 3.7. *The fact that $\omega_{\mathbf{x},p}$ induced in this way is a control is standard and can be found in several references (for example in Lyons, Caruana, Levy* [26], *p.6)*

4. Deterministic estimates for solutions to RDEs

In this section we will develop the pathwise estimate obtained in the previous section. To assist with the clarity of the presentation it will be important to first introduce some definitions of the main objects featuring in our discussion.

Definition 4.1. *Let $\alpha > 0$, $p \geq 1$ and $I \subseteq \mathbb{R}$ be a compact interval. We define the **accumulated** $\alpha-$**local** $p-$**variation** to be the non-negative function $M_{\alpha,I,p}$ acting on weakly geometric $p-$rough paths (parameterised over I) by*

$$M_{\alpha,I,p}(\mathbf{x}) = \sup_{\substack{D(I)=(t_i),\\ \omega_{\mathbf{x},p}(t_i,t_{i+1})\leq\alpha}} \sum_{i:t_i\in D} \omega_{\mathbf{x},p}(t_i,t_{i+1}). \tag{4.1}$$

Remark 4.2. *$M_{\alpha,I,p}$ is well-defined because the super-additivity of the control $\omega_{\mathbf{x},p}$ ensures that*

$$M_{\alpha,I,p}(\mathbf{x}) \leq ||\mathbf{x}||^p_{p-var;I} < \infty$$

for any weakly geometric rough path $\mathbf{x}$ (again, parameterised over I). The function is increasing in α, equals $\omega_{\mathbf{x},p}$ for $\alpha \geq ||\mathbf{x}||^p_{p-var;I}$ and is continuous in α on $(0,\infty)$. Finally for $p=1$ $M_{\alpha,I,p}$ coincides with $\omega_{\mathbf{x},p}$.

From Theorem 3.2 it is already evident that the structure of $M_{\alpha,I,p}$ makes it particularly suited to controlling differential equations in Gronwall-type estimates. The following lemma exhibits the key relationship between $M_{\alpha,I,p}(\mathbf{x})$ and $|J^{\mathbf{x}}_{\cdot\leftarrow 0}(y_0)|_{\infty;I}$.

Lemma 4.3. *Let $\gamma > p \geq 1$ and suppose $\mathbf{x}$ is a weakly geometric $p-$rough path over $I=[0,T]$. Let $V=(V^1,...,V^d)$ be a collection of Lip-γ vector fields on $\mathbb{R}^e$ and $J^{\mathbf{x}}_{t\leftarrow 0}(\cdot)$ be the derivative of the solution flow of the RDE driven by $\mathbf{x}$ along V. For any y_0 in $\mathbb{R}^e$ and $\alpha > 0$ there exists a finite constant $C>0$, which depends only on p and γ, such that*

$$\sup_{0\leq t\leq T} |J^{\mathbf{x}}_{t\leftarrow 0}(y_0)| \leq C \exp\left[C \max\left(|V|^p_{Lip-\gamma}, \frac{1}{\alpha}\right) M_{\alpha,I,p}(\mathbf{x})\right].$$

Proof. This can be deduced immediately from Corollary 3.4 and the Remark following it. □

We have successfully shown how we can control the derivative of the flow by using the function $M_{\alpha,I,p}(\cdot)$. But it is still not obvious how to get a handle on the tail behaviour of $M_{\alpha,I,p}(\cdot)$ when we evaluate it at a Gaussian p-rough path. To expose the structure further, we will now consider another function $N_{\alpha,I,p}(\cdot)$ on $WG\Omega_p(\mathbb{R}^d)$ which is closely related to $M_{\alpha,I,p}(\cdot)$. To this end let $\mathbf{x} \in WG\Omega_p(\mathbb{R}^d)$ and inductively define a non-decreasing sequence of times $(\tau_i(\alpha,p,\mathbf{x}))_{i=0}^{\infty} = (\tau_i(\alpha))_{i=0}^{\infty}$ in I by letting

$$\tau_0(\alpha) = \inf I \tag{4.2}$$

$$\tau_{i+1}(\alpha) = \inf\left\{t : ||\mathbf{x}||^p_{p-\text{var};[\tau_i,t]} \geq \alpha, \tau_i(\alpha) < t \leq \sup I\right\} \wedge \sup I,$$

with the convention that $\inf\emptyset = +\infty$. For $\tau_i(\alpha) < \sup I$ and $||\mathbf{x}||^p_{p-\text{var};[\tau_i(\alpha),\sup I]} \geq \alpha$, $\tau_{i+1}(\alpha)$ is intuitively the first time $||\mathbf{x}||^p_{p-\text{var};[\tau_i(\alpha),\cdot]}$ reaches α (recall that the $p-$variation is a continuous function). We then let $N_{\alpha,I,p} : WG\Omega_p(\mathbb{R}^d) \to \mathbb{R}_+$ be given by

$$N_{\alpha,I,p}(\mathbf{x}) := \sup\{n \in \mathbb{N}\cup\{0\} : \tau_n(\alpha) < \sup I\} \tag{4.3}$$

and we note that $N_{\alpha,I,p}$ describes the size of the non-trivial part of the sequence $(\tau_i(\alpha))_{i=0}^{\infty}$, i.e. the number of distinct terms in the sequence $(\tau_i(\alpha))_{i=0}^{\infty}$ equals $N_{\alpha,I,p}(\mathbf{x})+1$. The partition of the interval given by the times $(\tau_i(\alpha))_{i=0}^{N_{\alpha,I,p}(\mathbf{x})+1}$ can now heuristically be thought of as a "greedy" approximation to the supremum in identity (4.1) in the definition of the accumulated $\alpha-$local $p-$variation.

Lemma 4.4. *For any $\alpha>0$, $p\geq 1$ and any compact interval I the function $N_{\alpha,I,p}: WG\Omega_p(\mathbb{R}^d)\to\mathbb{R}_+$ is well defined; that is, $N_{\alpha,I,p}(\mathbf{x})<\infty$ whenever $\mathbf{x}$ is in $WG\Omega_p(\mathbb{R}^d)$.*

Proof. From the continuity of $||\mathbf{x}||_{p-\text{var};[s,\cdot]}$ we can deduce that

$$||\mathbf{x}||_{p-\text{var};[\tau_{i-1}(\alpha),\tau_i(\alpha)]}^p=\alpha \text{ for } i=1,2,...,N_{\alpha,I,p}(\mathbf{x}).$$

Thus, the super-additivity of $\omega_{\mathbf{x},p}$ implies that if $\mathbf{x}$ is in $WG\Omega_p(\mathbb{R}^d)$ then

$$\alpha N_{\alpha,I,p}(\mathbf{x})=\sum_{i=1}^{N_{\alpha,I,p}(\mathbf{x})}\omega_{\mathbf{x},p}(\tau_{i-1}(\alpha),\tau_i(\alpha))\leq\omega_{\mathbf{x},p}\left(0,\tau_{N_{\alpha,I,p}(\mathbf{x})}(\alpha)\right)\leq||\mathbf{x}||_{p-\text{var};[0,T]}^p<\infty.$$

□

Corollary 4.5. *Let $\mathbf{x}$ be a path in $WG\Omega_p(\mathbb{R}^d)$ and suppose $\alpha>0$. Define the sequence $(\tau_i(\alpha))_{i=0}^{\infty}$ by (4.2) and let $N_{\alpha,I,p}(\mathbf{x})$ be given by (4.3). Then the set*

$$D_\tau=\{\tau_i(\alpha):i=0,1,...,N_{\alpha,I,p}(\mathbf{x})+1\}$$

is a dissection of I.

Proof. This now follows immediately from the definition of $(\tau_i(\alpha))_{i=0}^{\infty}$ and the fact that $N_{\alpha,I,p}(\mathbf{x})$ is finite. □

Proposition 4.6. *Let $p\geq 1$ and suppose $\mathbf{x}$ is a path in $WG\Omega_p(\mathbb{R}^d)$ parameterised over the compact interval I, then for every $\alpha>0$*

$$M_{\alpha,I,p}(\mathbf{x})\leq(2N_{\alpha,I,p}(\mathbf{x})+1)\,\alpha$$

Proof. Let $D=\{t_i:i=0,1...,n\}$ be any dissection of I with the property that

$$\omega_{\mathbf{x},p}(t_{i-1},t_i)\leq\alpha \text{ for all } i=1,...,n. \tag{4.4}$$

Corollary 4.5 ensures that D_τ is a dissection of $[0,T]$. We re-label the points in D with reference to the dissection D_τ by writing $t_i=t_j^l$ for $i=1,2,...,n$, where l indicates which of disjoint subintervals $\{(\tau_i(\alpha),\tau_{i+1}(\alpha)]:i=0,1,...,N_{\alpha,I,p}(\mathbf{x})\}$ contains t_i, and j orders the t_is within each of these subintervals. More precisely, $l\in\{0,1,...,N_{\alpha,I,p}(\mathbf{x})\}$ is the unique natural number such that

$$\tau_l(\alpha)<t_i\leq\tau_{l+1}(\alpha);$$

and then $j\geq 1$ is well-defined by

$$j=i-\max_{t_r\leq\tau_l(\alpha)}r.$$

For each $l\in\{0,1,...N_{\alpha,I,p}(\mathbf{x})\}$ let n_l denote the number of elements of D in $(\tau_l(\alpha),\tau_{l+1}(\alpha)]$. Suppose now for a contradiction that $n_l=0$. In this case, $t_{n_{l-1}}^{l-1}$ and t_1^{l+1} are two consecutive points of D with $t_{n_{l-1}}^{l-1} \leq \tau_l(\alpha) < \tau_{l+1}(\alpha) < t_1^{l+1}$ and since the $(\tau_i(\alpha))_{i=0}^{\infty}$ are defined to be maximal (recall (4.2)) we have

$$\omega_{\mathbf{x},p}\left(t_{n_{l-1}}^{l-1}, t_1^{l+1}\right) > \omega_{\mathbf{x},p}\left(\tau_l(\alpha), \tau_{l+1}(\alpha)\right) = \alpha.$$

This contradicts the assumptions on D (4.4) and we deduce that $n_l \geq 1$.

We observe that if $n_l \geq 2$ then the super-additivity of $\omega_{\mathbf{x},p}$ results in

$$\sum_{j=1}^{n_l-1} \omega_{\mathbf{x},p}\left(t_j^l, t_{j+1}^l\right) \leq \omega_{\mathbf{x},p}\left(t_1^l, t_{n_l}^l\right) \text{ for } l = 0, 1, ..., N_{\alpha,I,p}(\mathbf{x});$$

thus, by a simple calculation we have

$$\begin{aligned} &\sum_{j=1}^{n} \omega_{\mathbf{x},p}\left(t_{j-1}, t_j\right) \\ &\leq \sum_{l=0}^{N_{\alpha,I,p}(\mathbf{x})-1} \left\{\left[\omega_{\mathbf{x},p}\left(t_{n_l}^l, t_1^{l+1}\right) + \omega_{\mathbf{x},p}\left(t_1^{l+1}, t_{n_{l+1}}^{l+1}\right)\right] 1_{\{n_{l+1} \geq 2\}} + \omega_{\mathbf{x},p}\left(t_{n_l}^l, t_{n_{l+1}}^{l+1}\right) 1_{\{n_{l+1}=1\}}\right\} \\ &+ \omega_{\mathbf{x},p}\left(0, t_{n_0}^0\right). \end{aligned} \tag{4.5}$$

To conclude the proof we note that $\omega_{\mathbf{x},p}\left(t_1^{l+1}, t_{n_{l+1}}^{l+1}\right) \leq \alpha$ and $\omega_{\mathbf{x},p}\left(0, t_{n_0}^0\right) \leq \alpha$ by the definition of the sequence $\left(t_j^l\right)$, and $\omega_{\mathbf{x},p}\left(t_{n_l}^l, t_1^{l+1}\right) \leq \alpha$ because $t_{n_l}^l$ and t_1^{l+1} are two consecutive points in D. Hence, we may deduce from (4.5) that

$$\sum_{j=1}^{n} \omega_{\mathbf{x},p}\left(t_{j-1}, t_j\right) \leq \left(2N_{\alpha,I,p}(\mathbf{x}) + 1\right)\alpha.$$

Because the right hand side of the last inequality does not depend on D, optimising over all such dissections gives the stated result. □

5. Gaussian Rough Paths

The previous section developed the key pathwise estimate on $J_{t\leftarrow 0}^{\mathbf{x}}(y_0)$ in terms of $N_T(\mathbf{x})$, but the importance of controlling $J_{t\leftarrow 0}^{\mathbf{x}}(y_0)$ using this, as opposed to simpler alternatives, is best appreciated when the driving rough path is taken to be random (we distinguish situations where the path is random by writing it in upper-case: $\mathbf{X}$). Of special interest is when $\mathbf{X}$ is the lift of some continuous $\mathbb{R}^d-$valued Gaussian process $(X_t)_{t\in I}$ (by lifting X we mean that the projection of $\mathbf{X}$ to the first tensor level is X). A theory of such Gaussian rough paths has been developed by a succession of authors ([5],[12],[4][8]) and we will mostly work within their framework.

To be more precise, we will assume that $X_t = \left(X_t^1, ..., X_t^d\right)$ is a continuous, centred (i.e. mean zero) Gaussian process with independent and identically distributed components. Let $R : I \times I \to \mathbb{R}$ denote the covariance function of any component, i.e.:

$$R(s,t) = E\left[X_s^1 X_t^1\right].$$

Throughout we will assume that this process is realised on the abstract Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$ where $\mathcal{W} = C_0\left(I, \mathbb{R}^d\right)$ (the space of continuous $\mathbb{R}^d$-valued functions on I); more precisely we mean that X is the canonical process on $\mathcal{W}$, i.e. $X_t(\omega) = \omega(t)$, and $(X_t)_{t\in I}$ has the required Gaussian distribution under μ. We recall the notion of the "rectangular increments of R" from [13], these are defined by

$$R\begin{pmatrix} s,t \\ u,v \end{pmatrix} := E\left[\left(X_t^1 - X_s^1\right)\left(X_v^1 - X_u^1\right)\right].$$

The existence of a lift for X is guaranteed by insisting on a sufficient rate of decay on the correlation of the increments. And this is captured, in a very general way, by the following two-dimensional $\rho-$variation constant on the covariance function.

Condition 1. *There exists of $1 \leq \rho < 2$ such that R has finite ρ-variation in the sense*

$$V_\rho\left(R; I \times I\right) := \left(\sup_{\substack{D=(t_i)\in\mathcal{D}(I) \\ D'=\left(t_j'\right)\in\mathcal{D}(I)}} \sum_{i,j} \left| R\begin{pmatrix} t_i, t_{i+1} \\ t_j', t_{j+1}' \end{pmatrix}\right|^\rho\right)^{\frac{1}{\rho}} < \infty. \tag{5.1}$$

Remark 5.1. *Under Condition 1 Theorem 35 of* [12] *shows that $(X_t)_{t\in[0,T]}$ lifts to a geometric p-rough path for any $p > 2\rho$. Moreover, there is a unique* ***natural lift*** *which is the limit (in the d_{p-var}-induced rough path topology) of the canonical lift of piecewise linear approximations to X.*

The following theorem appears in [12] as Proposition 17 (cf. also the recent note [13]); it shows how the assumption $V_\rho\left(R; [0,T]^2\right) < \infty$ allows us to embed $\mathcal{H}$ in the space of continuous paths with finite ρ variation. As the result is stated in [12] the proof applies to one dimensional Gaussian processes, but the generalisation to arbitrary finite dimensions is straight forward and we will not elaborate on the proof.

Theorem 5.2 ([12])**.** *Let $(X_t)_{t\in I} = \left(X_t^1, ..., X_t^d\right)_{t\in I}$ be a continuous, mean-zero Gaussian process with independent and identically distributed components. Let R denote the covariance function of (any) one of the components. Then if R is of finite ρ-variation for some $\rho \in [1,2)$ we can embed $\mathcal{H}$ in the space $C^{\rho-var}\left(I, \mathbb{R}^d\right)$, in fact*

$$|h|_{\mathcal{H}} \geq \frac{|h|_{\rho-var;I}}{\sqrt{V_\rho\left(R; I\times I\right)}}. \tag{5.2}$$

Remark 5.3 ([11])**.** *Writing $\mathcal{H}^H$ for the Cameron-Martin space of fBm for H in $(1/4, 1/2)$, the variation embedding in* [11] *gives the stronger result that*

$$\mathcal{H}^H \hookrightarrow C^{q\text{-}var}\left(I, \mathbb{R}^d\right) \text{ for any } q > (H + 1/2)^{-1}.$$

Once we have established a lift $\mathbf{X}$ of X we will often want to make sense of $\mathbf{X}(\omega + h)$. The main technique used for achieving this is to relate it to the translated rough path $T_h\mathbf{x}$ (the definition is standard see, for example, [10] or [27]). We recall that if $\mathbf{x} = \left(1, \mathbf{x}^1, ..., \mathbf{x}^{\lfloor p\rfloor}\right)$ is a weakly geometric p rough path and h is in $C^{q-\text{var}}\left(I, \mathbb{R}^d\right)$ such that $p^{-1} + q^{-1} > 1$ then the terms of $T_h\mathbf{x}$ at the first two non-trivial tensor levels are given by

$$\begin{aligned} (T_h\mathbf{x})^1 &= \mathbf{x}^1 + h \\ (T_h\mathbf{x})^2 &= \mathbf{x}^2 + \int h \otimes d\mathbf{x}^1 + \int \mathbf{x}^1 \otimes dh + \int h \otimes dh. \end{aligned}$$

The higher order terms become increasingly tiresome to write down, we will not go beyond the level $(\mathbb{R}^d)^{\otimes 3}$ so we simply record for reference that this can be written as

$$
\begin{aligned}
(T_h\mathbf{x})^3_{s,t} &= \mathbf{x}^3_{s,t} + \int_s^t \int_s^v h_{s,u} \otimes dh_u \otimes dh_v \\
&+ \int_s^t \mathbf{x}^2{}_{s,u} \otimes dh_u + \int_s^t \int_s^v \mathbf{x}^1_{s,u} \otimes dh_u \otimes d\mathbf{x}^1_v - \int_s^t h_{s,u} \otimes d\mathbf{x}^2{}_{u,t} \qquad (5.3)\\
&+ \int_s^t \int_s^v h_{s,u} \otimes d\mathbf{x}^1_u \otimes dh_v + \int_s^t \int_s^v \mathbf{x}^1{}_{s,u} \otimes dh_u \otimes dh_v + \int_s^t \int_s^v h_{s,u} \otimes dh_u \otimes d\mathbf{x}^1_v
\end{aligned}
$$

The the following result appeared in [4] and demonstrates that $\mathbf{X}(\omega + h)$ and $T_h\mathbf{X}(\omega)$ are in fact equal for all h in $\mathcal{H}$ on a set of $\mu-$full measure under certain conditions.

Lemma 5.4. *Let $(X_t)_{t\in I} = (X^1_t, ..., X^d_t)_{t\in I}$ be a mean-zero Gaussian process with i.i.d. components. If X has a natural lift to a geometric $p-$rough path and $\mathcal{H} \hookrightarrow C^{q\text{-}var}(I, \mathbb{R}^d)$ such that $1/p + 1/q > 1$ then*

$$\mu\{\omega : T_h\mathbf{X}(\omega) = \mathbf{X}(\omega + h) \text{ for all } h \in \mathcal{H}\} = 1.$$

From the different choices of the parameters p and q (such that X lifts path in $G\Omega_p(\mathbb{R}^d)$ and $\mathcal{H}$ continuously embeds in $C^{q\text{-var}}(I, \mathbb{R}^d)$) it will often prove useful to work with a particular choice that satisfy certain constraints. The purpose of the next lemma is to show that these constraints can always be satisfied (for some choice of p and q) for the examples of Gaussian processes that will interest us most.

Lemma 5.5. *Let $(X_t)_{t\in I} = (X^1_t, ..., X^d_t)_{t\in I}$ be a continuous, mean-zero Gaussian process with i.i.d. components on $(\mathcal{W}, \mathcal{H}, \mu)$. Suppose that at least one of the following holds:*

(1) *For some ρ in $[1, \frac{3}{2})$ the covariance function of X has finite ρ-variation in the sense of Condition 1 .*
(2) *X is a fractional Brownian motion for H in $(1/4, 1/2)$.*

Then there exist real numbers p, q such that the following three statements are true simultaneously:

(1) *X has a natural lift to a geometric p-rough path;*
(2) *$\mathcal{H} \hookrightarrow C^{q\text{-}var}(I, \mathbb{R}^d)$ where $1/p + 1/q > 1$;*
(3) *$p > q\lfloor p\rfloor$.*

Proof. If Condition 1 is satisfied with $\rho \in [1, 3/2)$ then (taking $\frac{1}{0} = \infty$)

$$2\rho < 3 < \frac{\rho}{\rho - 1},$$

thus if we take $q = \rho$ and choose p in $(2q, 3)$ Remark 5.1 guarantees the existence of a natural lift for X; furthermore, Theorem 5.2 ensures that $\mathcal{H} \hookrightarrow C^{q\text{-var}}(I, \mathbb{R}^d)$ and we also have $p > q\lfloor p\rfloor = 2q$.

In the case where X is fBm note first that if $H \in (1/3, 1/2)$ then $(H + 1/2)^{-1} < 1/(2H)$ so any choice of p and q satisfying

$$2\left(H + \frac{1}{2}\right)^{-1} < 2q < \frac{1}{H} < p < 3$$

will do the job by using Remark 5.1 and Remark 5.3. Finally, if $H \in (1/4, 1/3]$ then we notice

$$\frac{1}{3H} < H^{-1}\left(1 + \frac{1}{2H}\right)^{-1} = \left(H + \frac{1}{2}\right)^{-1} < \frac{4}{3}.$$

Hence, by choosing p and q such that

$$3\left(H+\frac{1}{2}\right)^{-1} < 3q < p < 4$$

we can verify each of the conclusions by once more referring to Remark 5.1, Remark 5.3 or by direct calculation as appropriate. □

6. The tail behaviour of $N_{\alpha,I,p}(\mathbf{X}(\cdot))$ via Gaussian isoperimetry

We continue to work in the setting of an abstract Wiener space $(\mathcal{W},\mathcal{H},\mu)$. If $\mathcal{K}$ denotes the unit ball in $\mathcal{H}$ then for any $A\subseteq\mathcal{W}$ we can consider the Minkowski sum:

$$A+r\mathcal{K}:=\{x+ry:x\in A,y\in\mathcal{K}\}.$$

We then recall the following isoperimetric inequality of C.Borell (cf. Theorem 4.3 of [23]).

Theorem 6.1 (Borell)**.** *Let $(\mathcal{W},\mathcal{H},\mu)$ be an abstract Wiener space and $\mathcal{K}$ denote the unit ball in $\mathcal{H}$. Suppose A is a Borel subset of $\mathcal{W}$ such that $\mu(A)\geq\Phi(a)$ for some real number a. Then, for every $r\geq 0$*

$$\mu_*(A+r\mathcal{K})\geq\Phi(a+r),$$

where μ_* is the inner measure of μ and Φ denotes the standard normal cumulative distribution function.

Theorem 6.2. *Let $(X_t)_{t\in I}=\left(X_t^1,...,X_t^d\right)_{t\in I}$ be a continuous, mean-zero Gaussian process, parameterised over a compact interval I on the abstract Wiener space $(\mathcal{W},\mathcal{H},\mu)$. Suppose that p and q are real numbers such that $p<4$ and*

1. *X has a natural lift to a geometric p-rough path $\mathbf{X}$;*
2. *$\mathcal{H}\hookrightarrow C^{q\text{-}var}\left(I,\mathbb{R}^d\right)$ where $1/p+1/q>1$;*
3. *$p>q\lfloor p\rfloor$.*

Then there exists a set $E\subseteq\mathcal{W}$, of $\mu-$full measure with the property that, for all ω in E,h in $\mathcal{H}$ and $\alpha>0$, if

$$||\mathbf{X}(\omega-h)||_{p-var;I}\leq\alpha$$

then

$$|h|_{q-var;I}\geq\frac{\alpha}{(2c_{p,q})^2}N_{\widetilde{\alpha},I,p}(\mathbf{X}(\omega))^{1/q},$$

where $\widetilde{\alpha}=3(2\alpha)^p$, $c_{p,q}=2\cdot 4^{1/p+1/q}\zeta\left(\frac{1}{p}+\frac{1}{q}\right)$ and ζ is the classical Riemann zeta function.

Remark 6.3. *Lemma 5.5 ensures that if X satisfies Condition 1 or if X is fBm with Hurst index H in $(1/4,1/2)$, then it is always possible to find real numbers p and q satisfying (simultaneously) condition 1-3 in Theorem 6.2.*

Proof. From the definition of the sequence $(\tau_i(\widetilde{\alpha}))_{i=0}^{\infty}$ and the integer $N_{\widetilde{\alpha},I,p}(\mathbf{X}(\omega))$ we have for $i=0,1,2,....,N_{\widetilde{\alpha},I,p}(\mathbf{X}(\omega))-1$

(6.1)

$$||\mathbf{X}(\omega)||_{p\text{-var};[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}=\left(\sum_{j=1}^{\lfloor p\rfloor}\sup_{D=(t_l)\in\mathcal{D}[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}\sum_{l:t_l\in D}\left|\mathbf{X}^j(\omega)_{t_l,t_{l+1}}\right|_{(\mathbb{R}^d)^{\otimes j}}^{p/j}\right)^{1/p}=3^{1/p}2\alpha,$$

which implies (recall: $p<4$) for some integer $k_i \in \{1,2,...,\lfloor p \rfloor\}$ that

$$\sup_{D=(t_l)\in\mathcal{D}[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \sum_{l:t_l\in D} \left|\mathbf{X}^{k_i}(\omega)_{t_l,t_{l+1}}\right|_{(\mathbb{R}^d)^{\otimes k_i}}^{p/k_i} \geq (2\alpha)^p . \tag{6.2}$$

We fix k_i by taking it to be the least such integer. Consider the subset of $\mathcal{W}$

$$E := \{\omega \in \mathcal{W} : T_h\mathbf{X}(\omega) = \mathbf{X}(\omega+h) \ \ \forall h \in \mathcal{H}\}$$

(recall from Lemma 5.4 that $\mu(E)=1$) and for every ω in E define a subset of $\mathcal{H}$ by

$$F_{\alpha,\omega} := \left\{h \in \mathcal{H} : ||\mathbf{X}(\omega-h)||_{p-\text{var};I} \leq \alpha\right\}.$$

For each ω in E we will show that

$$|h|_{q-\text{var};[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \geq \frac{\alpha}{(2c_{p,q})^2} \text{ for all } i=0,1,...,N_{\widetilde{\alpha},I,p}(\mathbf{X}(\omega))-1 \text{ and } h \in F_{\alpha,\omega}; \tag{6.3}$$

the required result will then follow from the calculation

$$|h|^q_{q-\text{var};I} \geq \sum_{i=0}^{N_{\widetilde{\alpha},I,p}(\mathbf{X}(\omega))-1} |h|^q_{q-\text{var};[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \geq \frac{\alpha^q}{(2c_{p,q})^{2q}} N_{\widetilde{\alpha},I,p}(\mathbf{X}(\omega)).$$

We now prove that (6.3) holds by studying three separate cases. Thus, let ω be in E and h in $F_{\alpha,\omega}$ then we have:

Case 1: $k_i = 1$ Under this assumption

$$\sup_{D=(t_l)\in\mathcal{D}[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \sum_{l:t_l\in D} \left|\mathbf{X}^1(\omega)_{t_l,t_{l+1}}\right|^p \geq (2\alpha)^p .$$

Since $T_h\mathbf{X}(\cdot) = \mathbf{X}(\cdot+h)$ on E we have

$$\begin{aligned}
|h|_{q-\text{var};[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} &\geq |h|_{p-\text{var};[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \\
&\geq \left|\mathbf{X}^1(\omega)\right|_{p-\text{var};[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} - \left|\mathbf{X}^1(\omega)-h\right|_{p-\text{var};[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \\
&\geq 2\alpha - ||\mathbf{X}(\omega-h)||_{p-\text{var};[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \\
&\geq \alpha \\
&\geq \frac{\alpha}{(2c_{p,q})^2}.
\end{aligned}$$

Case 2: $k_i = 2$ This can only happen when $p \geq 2$. Let $\{u_j : j=0,1,...,m\}$ be a dissection of $[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]$. Exploiting (again) the fact that $T_h\mathbf{X}(\omega) = \mathbf{X}(\omega+h)$ we have for any $r \in \{1,2....,m\}$

$$\begin{aligned}
&\int_{u_{r-1}}^{u_r} h_{u_{r-1},s}\otimes dh_s \\
&= \mathbf{X}^2(\omega)_{u_{r-1},u_r} - \mathbf{X}^2(\omega-h)_{u_{r-1},u_r} + \int_{u_{r-1}}^{u_r} h_{u_{r-1},s}\otimes d\mathbf{X}^1(\omega)_s + \int_{u_{r-1}}^{u_r} \mathbf{X}^1(\omega)_{u_{r-1},s}\otimes dh_s;
\end{aligned} \tag{6.4}$$

the cross-integrals are well-defined Young integrals by the hypotheses on p and q. We consider the terms on both sides of equation (6.4): on the one hand Young's inequality ([26]) and the super-additivity of the $q-$variation provides the upper bound

$$\begin{aligned}\sum_{r=1}^{m}\left|\int_{u_{r-1}}^{u_r} h_{u_{r-1},s}\otimes dh_s\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2} &\leq c_{p,q}^{p/2}\sum_{r=1}^{m}|h|_{q-var;[u_{r-1},u_r]}^{p} \\ &\leq c_{p,q}^{p/2}\left(\sum_{r=1}^{m}|h|_{q-var;[u_{r-1},u_r]}^{q}\right)^{p/q} \\ &\leq c_{p,q}^{p/2}\,|h|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}^{p}. \qquad (6.5)\end{aligned}$$

On the other hand, we have have the lower bound

$$\begin{aligned}&\left(\sum_{r=1}^{m}\left|\int_{u_{r-1}}^{u_r} h_{u_{r-1},s}\otimes dh_s\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2}\right)^{2/p} \qquad (6.6)\\ &\geq\left(\sum_{r=1}^{m}\left|\mathbf{X}^2(\omega)_{u_{r-1},u_r}\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2}\right)^{2/p}-\left(\sum_{r=1}^{m}\left|\mathbf{X}^2(\omega-h)_{u_{r-1},u_r}\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2}\right)^{2/p}\\ &-\left(\sum_{r=1}^{m}\left|\int_{u_{r-1}}^{u_r} h_{u_{r-1},s}\otimes d\mathbf{X}^1(\omega)_s+\int_{u_{r-1}}^{u_r}\mathbf{X}^1(\omega)_{u_{r-1},s}\otimes dh_s\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2}\right)^{2/p}.\end{aligned}$$

We the estimate the terms on the right hand side of this inequality by noticing

$$\left(\sum_{r=1}^{m}\left|\mathbf{X}^2(\omega-h)_{u_{r-1},u_r}\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2}\right)^{2/p}\leq||\mathbf{X}(\omega-h)||_{p-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}^{2}\leq\alpha^2 \qquad (6.7)$$

and, using Young's inequality,

$$\left|\int_{u_{r-1}}^{u_r} h_{u_{r-1},s}\otimes d\mathbf{X}^1(\omega)_s+\int_{u_{r-1}}^{u_r}\mathbf{X}^1(\omega)_{u_{r-1},s}\otimes dh_s\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}\leq 2c_{p,q}\,|h|_{q\text{-}var;[u_{r-1},u_r]}\left|\mathbf{X}^1(\omega)\right|_{p\text{-}var;[u_{r-1},u_r]}$$

Since k_i is defined to be the least integer for which we have (6.2) we must have $\left|\mathbf{X}^1(\omega)\right|_{p\text{-}var;[u_{r-1},u_r]}\leq=2\alpha$. Using this together with $p>2q$ we arrive at

$$\begin{aligned}&\sum_{r=1}^{m}\left|\int_{u_{r-1}}^{u_r} h_{u_{r-1},s}\otimes d\mathbf{X}^1(\omega)_s+\int_{u_{r-1}}^{u_r}\mathbf{X}^1(\omega)_{u_{r-1},s}\otimes dh_s\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2}\\ &\leq(4\alpha c_{p,q})^{p/2}\sum_{r=1}^{m}|h|_{q\text{-}var;[u_{r-1},u_r]}^{p/2}\\ &\leq(4\alpha c_{p,q})^{p/2}\,|h|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}^{p/2}. \qquad (6.8)\end{aligned}$$

Substituting (6.7), (6.8) into (6.6), taking the supremum over all dissections of $[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]$ and using the fact that

$$\sup_{D=(u_r)\in\mathcal{D}[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}\left(\sum_{l:u_r\in D}\left|\mathbf{X}^2(\omega)_{u_{r-1},u_r}\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2}\right)^{1/p}\geq 2\alpha$$

gives
(6.9)

$$\sup_{D=(u_r)\in\mathcal{D}[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}\left(\sum_{r:u_r\in D}\left|\int_{u_{r-1}}^{u_r} h_{u_{r-1},s}\otimes dh_s\right|_{\mathbb{R}^d\otimes\mathbb{R}^d}^{p/2}\right)^{2/p} \geq 3\alpha^2-4\alpha c_{p,q}\left|h\right|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}.$$

Using this in (6.5) gives

$$c_{p,q}\left|h\right|^2_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \geq 3\alpha^2-4\alpha c_{p,q}\left|h\right|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}$$

and by re-arranging we see that

$$\left[\left|h\right|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}+2\alpha\right]^2 \geq (2\alpha)^2+\frac{3\alpha^2}{c_{p,q}}.$$

Because $c_{p,q}>1$ and $\left|h\right|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}\geq 0$ we can deduce that

$$\left|h\right|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]} \geq 2\alpha\left(-1+\sqrt{1+\frac{3}{4c_{p,q}}}\right) \geq \frac{2\alpha}{2\left(1+\frac{3}{4c_{p,q}}\right)^{1/2}}\frac{3}{4c_{p,q}} \geq \frac{\alpha}{\left(2c_{p,q}\right)^2}.$$

Case 3: $k_i=3$ In this case we must have $3\leq p<4$.We recall the form of the third level of the translation $\left(T_h\mathbf{X}\left(\omega\right)\right)^3$ from (5.3) and proceed as in Case 2. First, using Young's inequality we have that

$$\begin{aligned}\sum_{i=1}^m\left|\int_{u_{r-1}}^{u_r}\int_{u_{r-1}}^{u} h_{u_{r-1},s}\otimes dh_s\otimes dh_s\right|^{p/3}_{(\mathbb{R}^d)^{\otimes 3}} &\leq c_{p,q}^{2p/3}\sum_{i=1}^m\left|h\right|^p_{q-var;[u_{r-1},u_r]}\\ &\leq c_{p,q}^{2p/3}\left|h\right|^p_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}.\end{aligned}$$

Then using Young's inequality repeatedly we can show

$$\begin{aligned}&\left|\int_{u_{r-1}}^{u_r}\mathbf{X}^2\left(\omega\right)_{u_{r-1},u}\otimes dh_u+\int_{u_{r-1}}^{u_r}\int_{u_{r-1}}^{u}\mathbf{X}^1\left(\omega\right)_{u_{r-1},s}\otimes dh_s\otimes d\mathbf{X}^1\left(\omega\right)_u-\int_{u_{r-1}}^{u_r}h_{u_{r-1},u}\otimes d\mathbf{X}^2\left(\omega\right)_{u,u_r}\right|_{(\mathbb{R}^d)^{\otimes 3}}\\ &\leq 2c_{p,q}\left|\mathbf{X}^2\left(\omega\right)\right|_{p-var;[u_{r-1},u_r]}\left|h\right|_{q-var;[u_{r-1},u_r]}+c_{p,q}^2\left|\mathbf{X}^1\left(\omega\right)\right|^2_{p-var;[u_{r-1},u_r]}\left|h\right|_{q-var;[u_{r-1},u_r]}\\ &\leq 3c_{p,q}^2\left(2\alpha\right)^2\left|h\right|_{q-var;[u_{r-1},u_r]}\end{aligned}$$

and, similarly,

$$\begin{aligned}&\left|\int_{u_{r-1}}^{u_r}\int_{u_{r-1}}^{u}\left(h_{u_{r-1},s}\otimes d\mathbf{X}^1\left(\omega\right)_s+\mathbf{X}^1\left(\omega\right)_{u_{r-1},s}\otimes dh_s\right)\otimes dh_u+\int_{u_{r-1}}^{u_r}\int_{u_{r-1}}^{u}h_{u_{r-1},s}\otimes dh_s\otimes d\mathbf{X}^1\left(\omega\right)_u\right|_{(\mathbb{R}^d)^{\otimes 3}}\\ &=3c_{p,q}^2\left|h\right|^2_{q-var;[u_{r-1},u_r]}\left|\mathbf{X}^1\left(\omega\right)\right|_{p-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}\\ &\leq 3c_{p,q}^2\left(2\alpha\right)\left|h\right|^2_{q-var;[u_{r-1},u_r]}\end{aligned}$$

Using the fact that $p>3q$ we deduce

$$\left(\sum_{r=1}^m\left|h\right|^{p/3}_{q-var;[u_{r-1},u_r]}\right)^{3/p} \leq \left|h\right|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}$$

and also

$$\left(\sum_{r=1}^{m}|h|_{q-var;[u_{r-1},u_r]}^{2p/3}\right)^{3/p}\leq |h|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}^{2}.$$

Under the assumption that $k_i=3$ we have that

$$\sup_{D=(u_r)\in\mathcal{D}[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}\left(\sum_{l:u_r\in D}\left|\mathbf{X}^3(\omega)_{u_{r-1},u_r}\right|_{(\mathbb{R}^d)^{\otimes 3}}^{p/3}\right)^{1/p}\geq 2\alpha$$

and hence

$$c_{p,q}^2|h|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}^3\geq 7\alpha^3-3c_{p,q}^2(2\alpha)^2|h|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}-3c_{p,q}^2(2\alpha)|h|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}^2.$$

By rearranging this we see that

$$\left[|h|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}^3+2\alpha\right]^3\geq(2\alpha)^3+\frac{7\alpha^3}{c_{p,q}^2},$$

thus by a simple calculation

$$|h|_{q-var;[\tau_i(\widetilde{\alpha}),\tau_{i+1}(\widetilde{\alpha})]}\geq 2\alpha\left(-1+\sqrt[3]{1+\frac{7}{8c_{p,q}^2}}\right)\geq\frac{2\alpha}{3\left(1+\frac{7}{8c_{p,q}^2}\right)^{2/3}}\frac{7}{8c_{p,q}^2}\geq\frac{\alpha}{(2c_{p,q})^2}$$

and the proof is complete. □

By using these estimates in concert with Borell's inequality we are lead directly to the following theorem which describes the needed tails estimate on the random variable $N_{\widetilde{\alpha},I,p}(\mathbf{X}(\cdot))$.

Theorem 6.4. *Let $(X_t)_{t\in I}=\left(X_t^1,...,X_t^d\right)_{t\in I}$ be a continuous, mean-zero Gaussian process with i.i.d. components on the abstract Wiener space $(\mathcal{W},\mathcal{H},\mu)$. Let $c_{p,q}=2\cdot 4^{1/p+1/q}\zeta\left(\frac{1}{p}+\frac{1}{q}\right)$, where ζ is the Riemann zeta function. If:*

(1) *For some ρ in $[1,\frac{3}{2})$ the covariance function of X has finite ρ-variation in the sense of Condition 1 , then for any p in $(2\rho,3)$ the natural lift $\mathbf{X}$ of X to a geometric $p-$rough path satisfies*

$$\mu\left\{\omega:N_{\widetilde{\alpha},I,p}(\mathbf{X}(\omega))>n\right\}\leq C_1\exp\left[\frac{-\alpha^2n^{2/\rho}}{2^7c_{p,\rho}^4V_\rho(R;I\times I)}\right]\tag{6.10}$$

for all $n\geq1,\alpha>0$ and where $\tilde{\alpha}=3(2\alpha)^p$. The constant C_1, which depends only on α, is given explicitly by

$$C_1=\exp\left[2\Phi^{-1}(\mu(A_\alpha))^2\right],\tag{6.11}$$

where Φ^{-1} is the inverse of the standard normal cumulative distribution function and

$$A_\alpha:=\left\{\omega\in\mathcal{W}:||\mathbf{X}(\omega)||_{p-var;I}\leq\alpha\right\}.$$

(2) *X is a fractional Brownian motion for H in $(1/4,1/2)$, then for any two real numbers p and q in simultaneously satisfying the inequalities*

(a) $p>H^{-1}$

(b) $\left(H+\frac{1}{2}\right)^{-1}<q<\min\left(\frac{p}{\lfloor p\rfloor},\frac{p}{p-1}\right)=\frac{p}{\lfloor p\rfloor}$

the natural lift $\mathbf{X}$ *of* X *to a geometric* $p-$*rough path satisfies*

$$\mu\left\{\omega : N_{\widetilde{\alpha},I,p}\left(\mathbf{X}\left(\omega\right)\right) > n\right\} \leq C_1 \exp\left[-\frac{\alpha^2 n^{2/q}}{2^7 c_{p,q}^4 C_2^2}\right]$$

for all $n \geq 1, \alpha > 0$ *and where* $\tilde{\alpha} = 3\left(2\alpha\right)^p$. *The constant* C_1 *defined by (6.11) and* $C_2 > 0$ *depends only on* q.

Proof. We deal with case 1. Notice from Lemma 5.5 that $q = \rho$ and p satisfy the hypotheses of Theorem 6.2. Hence by applying Theorem 6.2 together with Theorem 5.2 we can deduce that

$$\left\{\omega : N_{\widetilde{\alpha},I,p}\left(\mathbf{X}\left(\omega\right)\right) > n\right\} \cap E \subset \mathcal{W}\backslash\left(A_\alpha + r_n\mathcal{K}\right) \tag{6.12}$$

where $E \subseteq \mathcal{W}$ with $\mu\left(E\right) = 1$ and

$$r_n := \frac{\alpha n^{1/\rho}}{\left(2c_{p,\rho}\right)^2 \sqrt{V_\rho\left(R; I \times I\right)}}.$$

Noticing that $\mu\left(A_\alpha\right) =: \Phi\left(a_\alpha\right)$ is in $(0,1)$ (i.e. a_α is in $(-\infty,\infty)$) an application of Borell's inequality then gives that

$$\mu\left\{\omega : N_{\widetilde{\alpha},I,p}\left(\mathbf{X}\left(\omega\right)\right) > n\right\} \leq 1 - \Phi\left(a_\alpha + r_n\right) \leq \exp\left[-\frac{\left(a_\alpha + r_n\right)^2}{2}\right]. \tag{6.13}$$

If $a_\alpha > -r_n/2$ then (6.13) implies

$$\mu\left\{\omega : N_{\widetilde{\alpha},I,p}\left(\mathbf{X}\left(\omega\right)\right) > n\right\} \leq \exp\left(-\frac{r_n^2}{8}\right),$$

alternatively if $a_\alpha \leq -r_n/2$ then $a_\alpha^2 + 2a_\alpha r_n \geq -r_n^2$, and also (obviously) $r_n^2 \leq 4a_\alpha^2$ so we have that

$$\mu\left\{\omega : N_{\widetilde{\alpha},I,p}\left(\mathbf{X}\left(\omega\right)\right) > n\right\} \leq \exp\left(-\frac{a_\alpha^2 + 2a_\alpha r_n}{2}\right)\exp\left(-\frac{r_n^2}{2}\right) \leq \exp\left(2a_\alpha^2\right)\exp\left(-\frac{r_n^2}{2}\right).$$

Since $a_\alpha = \Phi^{-1}\left(\mu\left(A_\alpha\right)\right)$ we have shown the required estimate (6.10).

The fractional Brownian case is similar; from Remark 5.3 we see that for p and q as stated we have

$$|h|_{q-var;I} \leq C_2 |h|_{\mathcal{H}}$$

for some $C_2 = C_2\left(q\right)$. Then we can conclude by observing that

$$\left\{\omega : N_{\widetilde{\alpha},I,p}\left(\mathbf{X}\left(\omega\right)\right) > n\right\} \cap E \subset \mathcal{W}\backslash\left(A_\alpha + s_n\mathcal{K}\right),$$

where this time

$$s_n := \frac{\alpha n^{1/q}}{\left(2c_{p,q}\right)^2 C_2},$$

and applying Borell's inequality exactly as in the first case. □

Remark 6.5. *Note that under the assumption of finite* $\rho-$*variation on the covariance, the tail estimates just proved lead to moment estimates on* $N_{\alpha,I,p}\left(\mathbf{X}\left(\omega\right)\right)$ *in the usual way. More exactly, for any* $\alpha > 0$ *and* η *satisfying*

$$\eta < \frac{\alpha^{2/p}}{3^{2/p} 2^9 c_{p,\rho}^4 V_\rho\left(R; I \times I\right)}$$

a simple calculation shows that

$$\int_{\mathcal{W}} \exp\left[\eta N_{\alpha,I,p}\left(\mathbf{X}\left(\omega\right)\right)^{2/\rho}\right]\mu\left(d\omega\right) < \infty. \tag{6.14}$$

For the Brownian rough path $(\rho = 1)$ *this shows that* $N_{\alpha,I,p}(\mathbf{X}(\omega))$ *has a Gaussian tail (since* $\log\left|J_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0)\right| \lesssim N_{\alpha,I,p}(\mathbf{X}(\omega))$*, rudimentary Itô or Stratonovich calculus tells us that we cannot expect the tail of* $N_{\alpha,I,p}(\mathbf{X}(\omega))$ *to decay any faster than Gaussian). By a similar argument we can show that for any* $r < 2/\rho$

$$\exp\left[N_{\alpha,I,p}\left(\mathbf{X}\left(\cdot\right)\right)^{r}\right] \text{ is in } \bigcap_{q>0} L^{q}\left(\mu\right);$$

and we can perform similar calculations in the fractional Brownian setting too.

Theorem 6.6 (Moment estimates on the Jacobian)**.** *Let* $(X_t)_{t\in[0,T]} = \left(X_t^1,...,X_t^d\right)_{t\in[0,T]}$ *be a continuous, mean-zero Gaussian process with i.i.d. components on the abstract Wiener space* $(\mathcal{W},\mathcal{H},\mu)$*. If for some* $p \geq 1$*,* X *lifts to a geometric* $p-$*rough path* $\mathbf{X}$ *then for any collection of Lip*$-\gamma$ *vector fields* $V = \left(V^1,...,V^d\right)$ *on* $\mathbb{R}^e$ *with* $\gamma > p$ *the solution to the RDE*

$$dY_t = V\left(Y\right)d\mathbf{X},\; Y\left(0\right) = y_0$$

induces a flow $U_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0)$ *which is differentiable. When the derivative exists let*

$$J_{t\leftarrow 0}^{\mathbf{X}(\omega)}\left(y_0\right)\cdot a := \left\{\frac{d}{d\varepsilon}U_{t\leftarrow 0}^{\mathbf{X}(\omega)}\left(y_0+\varepsilon a\right)\right\}_{\varepsilon=0}$$

and let $M_{\mathbf{X}(\cdot)}^{(y_0,V)} : \mathcal{W} \to \mathbb{R}_+$ *denote the function*

$$M_{\mathbf{X}(\cdot)}^{(y_0,V)}\left(\omega\right) = M_{\mathbf{X}(\omega)}^{(y_0,V)} := \sup_{t\in[0,T]}\left|J_{t\leftarrow 0}^{\mathbf{X}(\omega)}\left(y_0\right)\right|.$$

Suppose that for some ρ *in* $[1,\frac{3}{2})$ *the covariance function of* X *has finite* ρ*-variation (in the sense of Condition 1). Then, for any* p *in* $(2\rho, 3)$*, the natural lift of* X *to a geometric* p*-rough path* $\mathbf{X}$ *is such that for all* $r < 2/\rho$ *and* $\gamma > p$

$$\exp\left[\left(\log M_{\mathbf{X}(\cdot)}^{(y_0,V)}\right)^{r}\right] \text{ is in } \bigcap_{q>0} L^{q}\left(\mu\right),$$

for all y_0 *in* $\mathbb{R}^e$ *and all collections of Lip*$-\gamma$ *vector fields* $V = \left(V^1,...,V^d\right)$ *on* $\mathbb{R}^e$*.*

Proof. Fix $p > 2\rho$ then Remark 5.1 guarantees the existence of a unique natural lift $\mathbf{X}$ for X. Furthermore, we know that if $V = \left(V^1,...,V^d\right)$ is any collection of Lip-γ vector fields (and $\gamma > p$) then the solution flow obtained by driving $\mathbf{X}$ along V is differentiable. Lemma 4.3 and Proposition 4.6 together show that that for any $\alpha > 0$ and y_0 in $\mathbb{R}^e$

$$M_{\mathbf{X}(\cdot)}^{(y_0,V)} \leq c_1 \exp\left[c_1 N_{\alpha,I,p}\left(\mathbf{X}\left(\omega\right)\right)\right]$$

where $I = [0,T]$ and c_1 is a non-random constant which depends on α, p, γ and $|V|_{Lip\text{-}\gamma}$. Without loss of generality we take $c_1 > 1$ then for two further (again non-random) constants c_2 and c_3 an easy calculation gives

$$\left(\log M_{\mathbf{X}(\cdot)}^{(y_0,V)}\right)^{r} \leq c_2 + c_3 N_{\alpha,I,p}\left(\mathbf{X}\left(\omega\right)\right)^{r}.$$

Hence, we have

$$\exp\left[\left(\log M_{\mathbf{X}(\cdot)}^{(y_0,V)}\right)^{r}\right] \leq c_4 \exp\left[c_4 N_{\alpha,I,p}\left(\mathbf{X}\left(\omega\right)\right)^{r}\right]; \tag{6.15}$$

and by Theorem 6.4 and the remark following it the random variable on the right hand side of this inequality is $L^q(\mu)$ for all $q > 0$ provided $r < 2/\rho$. □

The above result applies (in particular) to fractional Brownian motion, $H > 1/3$, but we can state an alternative version of the theorem based on the second part of Theorem 6.4 which works specifically for fBm and applies when $H > 1/4$.

Theorem 6.7 (Fractional Brownian motion)**.** *Let $(X_t)_{t\in[0,T]} = \left(X_t^1, ..., X_t^d\right)_{t\in[0,T]}$ be fractional Brownian motion on the abstract Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$ with Hurst parameter $H > 1/4$. For some $p > 1/H$, X lifts to a geometric $p-$rough path $\mathbf{X}$ and for any collection of Lip$-\gamma$ vector fields $V = \left(V^1, ..., V^d\right)$ on $\mathbb{R}^e$ with $\gamma > p$ the solution to the RDE*

$$dY_t = V(Y)\, d\mathbf{X},\ Y(0) = y_0$$

induces a flow $U_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0)$ which is differentiable. When the derivative exists let

$$J_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0)\cdot a := \left\{\frac{d}{d\varepsilon} U_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0 + \varepsilon a)\right\}_{\varepsilon=0}$$

and let $M_{\mathbf{X}(\cdot)}^{(y_0,V)} : \mathcal{W} \to \mathbb{R}_+$ denote the function

$$M_{\mathbf{X}(\cdot)}^{(y_0,V)}(\omega) = M_{\mathbf{X}(\omega)}^{(y_0,V)} := \sup_{t\in[0,T]} \left| J_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0) \right|.$$

If H is in $(1/3, 1/2)$ then, for any p in $\left(H^{-1}, 3\right)$,the natural lift of X to a geometric p-rough path $\mathbf{X}$ is such that for all $r < 2H + 1$ and $\gamma > p$

$$\exp\left[\left(\log M_{\mathbf{X}(\cdot)}^{(y_0,V)}\right)^r\right] \text{ is in } \bigcap_{q>0} L^q(\mu)$$

for all y_0 in $\mathbb{R}^e$ and all collections of Lip$-\gamma$ vector fields $V = \left(V^1, ..., V^d\right)$ on $\mathbb{R}^e$. On the other hand, if H is in $(1/4, 1/3]$ then the same conclusion holds for any p satisfying

$$3(H + 1/2)^{-1} < p < 4$$

and any $\gamma > p$.

Proof. The argument is the same as the last theorem; we perform the same estimates used there but instead use the second conclusion in Theorem 6.4. Let us explain the origin of the constraint on the value of r. Firstly, if H is in $(1/3, 1/2)$ then we can apply the second part of Theorem 6.4 for p as given and any q satisfying

$$\frac{1}{H + 1/2} < q < \frac{1}{2H}.$$

to deduce that for any such $r < 2/q$ the random variable

$$\exp\left[N_{\alpha,I,p}(\mathbf{X}(\omega))^r\right] \tag{6.16}$$

is $\mu-$integrable. Similarly, if H is in $(1/4, 1/3]$ then for any p in $\left(3(H + 1/2)^{-1}, 4\right)$ we can apply Theorem 6.4 for any q satisfying

$$\frac{1}{H + 1/2} < q < \frac{p}{3}$$

to deduce again that for any $r < 2/q$ the random variable (6.16) is $\mu-$integrable. The result then follows from the relation(6.15). □

Remark 6.8. *In particular these results imply (under the stated conditions) that* $\sup_{t\in[0,T]}\left|J_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0)\right|$ *has finite moments of all order.*

References

[1] Baudoin, F.; Hairer, M. A version of Hörmander's theorem for the fractional Brownian motion. Probab. Theory Related Fields 139 (2007), no. 3-4, 373–395.

[2] Cass, T., Lyons T.: Evolving communities and individual preferences (Preprint, 2010)

[3] Cass, T.; Friz, P.: Densities for rough differential equations under Hörmander's condition, Ann. of Math. (2), 171(3):2115–2141,201

[4] Cass, T.; Friz, P.; Victoir, N.: Non-degeneracy of Wiener functionals arising from rough differential equations. Trans. Amer. Math. Soc., 361(6):3359–3371, 2009

[5] Coutin, L., Qian, Z. : Stochastic analysis, rough path analysis and fractional Brownian motion Probab. Theory Related Fields 122 (2002), 108–140

[6] Davie, A.M.: Differential equations driven by rough paths: an approach via discrete approximation. Appl. Math. Res. Express. AMRX, (2): Art. ID abm009, 40, (2007)

[7] Friz, P., Oberhauser, H.: Rough path limits of the Wong-Zakai type with a modified drift term. J. Funct. Anal. 256 (2009), no. 10, 3236–3256

[8] Friz, P., Oberhauser, H.: A generalized Fernique theorem and applications, Proceedings of the American Mathematical Society. 138, no. 10, 3679-3688, (2010)

[9] Friz, P., Riedel, S.: Integrability of linear rough differential equations arXiv:1104.0577

[10] Friz, P., Victoir, N.: Multidimensional stochastic processes as rough paths. Theory and applications. Cambridge Studies in Advanced Mathematics, 120. Cambridge University Press, Cambridge, 2010

[11] Friz, P., Victoir, N.: A Variation Embedding Theorem and Applications, Journal of Functional Analysis, Vol 239 Nr 2 (2006) 631-637

[12] Friz, P., Victoir, N.: Differential equations driven by Gaussian signals. Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 2, 369–41

[13] Friz, P., Victoir, N.: A note on higher dimensional p-variation arXiv:1102.4587v1

[14] Gubinelli M., Lejay A. , Antipolis S. Global existence for rough differential equations under linear growth condition. arXiv:0905.2399v1

[15] Guasoni, P. :No Arbitrage with Transaction Costs, with Fractional Brownian Motion and Beyond, Mathematical Finance, 16 (2006), no. 3, 569-582

[16] Ganesh, A.; O'Connell, N.; Wischik, Damon. Big queues. Lecture Notes in Mathematics, 1838. Springer-Verlag, Berlin, 2004

[17] Hairer, M., Mattingly, J.C. : Ergodcity of the 2D Navier-Stokes Equations with Degenerate Stochastic Forcing, Annals of Mathematics, vol. 164, (2006), no. 3

[18] Hairer, M., Pillai N.S. : Ergodicity of hypoelliptic SDEs driven by fractional Brownian motion, Ann. Inst. H. Poincaré Probab. Statist. Volume 47, Number 2 (2011), 601-628.

[19] Hairer, M., Pillai, N.: Regularity of Laws and Ergodicity of Hypoelliptic SDEs Driven by Rough Paths. arXiv:1104.5218v1

[20] Hambly, B. M.; Lyons, T. J. Stochastic area for Brownian motion on the Sierpinski gasket. Ann. Probab. 26 (1998), no. 1, 132–148.

[21] Hu Y., Tindel, S.: Smooth density for some nilpotent rough differential equations. ArXiv 1104.1972.

[22] Inahama, Y.: A moment estimate of the derivative process in rough path theory. ArXiv e-prints 1007.4651. To appear in Proc. Amer. Math. Soc..

[23] Ledoux, Michel: Isoperimetry and Gaussian analysis. Lectures on probability theory and statistics (Saint-Flour, 1994), 165–294, Lecture Notes in Math., 1648, Springer, Berlin, 1996.I

[24] Ledoux, M.; Qian, Z.; Zhang, T. Large deviations and support theorem for diffusion processes via rough paths. Stochastic Process. Appl. 102 (2002), no. 2, 265–283

[25] Lyons, T.: Differential equations driven by rough signals, Rev. Mat. Iberoamericana 14 (1998), no. 2, 215–310.

[26] Lyons, T.; Caruana, M.; Levy, T.: Differential Equations Driven by Rough Paths, Differential Equations Driven by Rough Paths, Ecole d'Eté de Probabilités de Saint-Flour XXXIV-2004, Lecture Notes in Mathematics Vol. 1908, Springer, 2007

[27] Lyons, T.; Qian, Z.: System Control and Rough Paths, Oxford University Press, 2002.

[28] Montgomery, R.: A Tour of Subriemannian Geometries, Their Geodesics and Applications, American Mathematical Society, 2002
[29] Norris, J.: Simplified Malliavin calculus. Séminaire de Probabilités, XX, 1984/85, 101–130, Lecture Notes in Math., 1204, Springer, Berlin, 1986
[30] Nualart, D.: The Malliavin calculus and related topics. Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006

Mathematical Institute and Oxford-Man Institute of Quantitative Finance, University of Oxford

Institut für Mathematik, Technische Universität Berlin and Oxford-Man Institute of Quantitative Finance, University of Oxford

Mathematical Institute and Oxford-Man Institute of Quantitative Finance, University of Oxford