Advanced Field Theory 2001

14.00 - 17.00 hrs, June 12

Answer TWO questions from Section A and ONE question from Section B

Q.A2

Maxwell's equations in empty space are written in terms of the electromagnetic field A_μ as

$$\partial_\mu F^{\nu\mu} = 0, \tag{1}$$

where $F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

(i). Show that they follow from the free action

$$S[A_\mu] = \int d^4x \, [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}]. \tag{2}$$

How are the electric and magnetic fields defined in terms of the $F^{\mu\nu}$?

Give one reason why there are difficulties if we take this action as it stands in the canonical (operator) approach.

(ii). From the point of view of path integrals explain why the generating functional for the Green functions of the free electromagnetic field cannot be written simply as

$$Z[j_\mu] = \int \prod_\mu \mathcal{D}A_\mu \exp(iS[A_\mu] + i \int dx \, j_\mu A^\mu) \tag{3}$$

(iii). Show how the covariant gauge generating functional

$$Z_\xi[j_\mu] = \int \prod_\mu \mathcal{D}A_\mu \exp(iS_\xi[A_\mu] + i \int dx \, j_\mu A^\mu) \tag{4}$$

is constructed, where

$$S_\xi[A_\mu] = \int d^4x \, [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2]. \tag{5}$$

Calculate the photon propagator $D_{\mu\nu}(x)$ with this gauge choice.

(iv). Quoting Feynman rules without proof explain why, if we use this propagator in the lowest order diagrams for electron-electron scattering, the scattering amplitude is independent of ξ.

Q.A3

This question concerns Grassmann variables and Grassmannian path integration.

We would like to show that, if $Z[\eta^*, \eta]$ is the generating functional for the *free* electron field coupled to sources η and η^*, written as a Grassmannian path integral, then it satisfies the Dyson-Schwinger equation

$$\Big((i\partial\!\!\!/ - m)\frac{\delta}{i\delta\eta^*(x)} - \eta(x)\Big) Z[\eta^*, \eta] = 0. \tag{1}$$

However, this requires too much work and, instead, you are asked to do a truncated version of the problem with a finite number of degrees of freedom.

(i). Consider the case of a single Grassmann variable q. Devise a definition of

$$J = \int dq\, f(q) \tag{2}$$

so that translation invariance of the 'measure' is preserved,

$$J = \int dq\, f(q + q_0) \tag{3}$$

for fixed q_0. Further, show that

$$\int dq\, f'(q) = 0, \tag{4}$$

where the prime denotes differentiation.

(ii). Let q, q^*, η, η^* be Grassmann variables. We define a simplified generating function $Z(\eta^*, \eta)$ by

$$Z(\eta^*, \eta) = N \int dq^*\, dq\, e^{-S} \tag{5}$$

where $S = Aq^*q - \eta^*q - q^*\eta$, with A a c-number. N is chosen so that $Z[0, 0] = 1$.

Evaluate $Z(\eta^*, \eta)$ a) directly, by expanding the exponential, and integrating term by term b) by completing the square in the exponent and using the translation invariance of the measure.

What is N?

(iii). Show, by explicit calculation that

$$\int dq^*\, dq\, \frac{\partial S}{\partial q^*} e^{-S} = 0. \tag{6}$$

Show that

$$\frac{\partial}{\partial \eta^*} e^{\eta^* q} = q\, e^{\eta^* q} \tag{7}$$

and, therefore, that Eq.(6) can be written as

$$\Big(- A \frac{\partial}{\partial \eta^*} + \eta \Big) Z(\eta^*, \eta) = 0. \tag{8}$$

Using your solution for Z given earlier, check this by explicit calculation.

Section A

Q.A1

A particle of unit mass, coordinate q, moves in one dimension in a potential $V(q)$.

(i). By inserting complete sets of states judiciously, show that the probability amplitude

$$\langle q_1, t_1 | q_0, t_0 \rangle = \langle q_1 | e^{-i\hat{H}(t_1 - t_0)} | q_0 \rangle \tag{1}$$

that the particle will be at q_1 at time t_1 if it was at q_0 at time t_0 can be represented by the path integral, in units in which $\hbar = 1$,

$$\langle q_1, t_1 | q_0, t_0 \rangle = \int \mathcal{D}q \ \exp\Big(iS[q]_{t_0}^{t_1} \Big), \tag{2}$$

where $S[q]_{t_0}^{t_1}$ is the action

$$S[q]_{t_0}^{t_1} = \int_{t_0}^{t_1} dt \ \left[\frac{1}{2}\dot{q}^2 - V(q) \right] \tag{3}$$

with $q(t_1) = q_1$, $q(t_0) = q_0$. Define $\mathcal{D}q$.

State any assumptions that you make.

(ii). Show that the classical path for the particle to begin at q_0 at time t_0 and arrive at q_1 at time t_1 is the solution to the variational equation

$$\delta S[q] = 0. \tag{4}$$

What is the role of the classical path in the sum over paths and is it typical of the paths that contribute to the sum?

(iii). A canonical ensemble of such particles is held at temperature T in a heat-bath.

Show that the partition function $Z = tr(e^{-\beta\hat{H}})$ for the ensemble can be written as

$$Z = \int_{periodic} \mathcal{D}q \ \exp\left(-\int_0^\beta d\tau \ \left[\frac{1}{2}\left(\frac{dq(\tau)}{d\tau}\right)^2 + V(q(\tau)) \right] \right), \tag{5}$$

where $\tau = it$ denotes imaginary time and the integral is restricted to periodic paths, $q(\tau = \beta) = q(\tau = 0)$, where $\beta = 1/T$, in units in which $k_B = 1$.

NOTE: You may use the fact that

$$\langle q' | e^{-i\epsilon\hat{p}^2/2} \, e^{-i\epsilon V(\hat{q})} | q \rangle = \int \frac{dp}{2\pi} e^{i(q'-q)p} \, e^{-i\epsilon p^2/2} e^{-i\epsilon V(q)}, \tag{6}$$

without proof. Similarly, you can quote the result

$$\int \frac{dp}{2\pi} e^{iqp} \, e^{-i\epsilon p^2/2} = \left(\frac{1}{2\pi i\epsilon} \right)^{1/2} e^{iq^2/2\epsilon}. \tag{7}$$

Q.A4

Consider a real scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4. \tag{1}$$

The generating functional for its Green functions is

$$Z[j] = \int \mathcal{D}\phi \ \exp\left(i\int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 + j\phi\right]\right). \tag{2}$$

(i). Show that $iW[j] = \ln Z[j]$ generates connected Green functions. You may assume that Green functions for an odd number of fields are identically zero.

(ii). If $\bar{\phi}(x) = \delta W/\delta j(x)$ is invertible, we define the effective action $\Gamma[\bar{\phi}]$ by

$$\Gamma[\bar{\phi}] = W[j] - \int d^4x\, j\bar{\phi}, \tag{3}$$

where we eliminate j in favour of ϕ. Show that

$$\frac{\delta\Gamma}{\delta\bar{\phi}(x)} = -j(x). \tag{4}$$

(iii). For constant j we can proceed in a similar way to define a $\Gamma(\bar{\phi})$ for constant $\bar{\phi}$, now proportional to the space-time volume Ω of the system as

$$\Gamma(\bar{\phi}) = -\Omega V_{eff}(\bar{\phi}). \tag{5}$$

Show that, to one loop, $V_{eff}(\bar{\phi})$ takes the form

$$V_{eff}(\bar{\phi}) = \frac{1}{2}m^2\bar{\phi}^2 + \frac{1}{4}\lambda\bar{\phi}^4 + \frac{\hbar}{2}\int d\!\!^-{}^4\bar{k}\ \ln(\bar{k}^2 + m^2 + 3\lambda\bar{\phi}^2), \tag{6}$$

where $\bar{k}$ is Euclidean momentum. You can use any method. There is no need to prove every step e.g. you can take for granted that $\det K = \exp(tr\ln K)$. State your assumptions clearly.

(iv). As it stands, $V_{eff}(\bar{\phi})$ is UV divergent. Indicate the steps needed to renormalise $V_{eff}(\bar{\phi})$, but do not do so.

Section B

Q.B1

(i). What is the *Casimir Effect*, and how is it understood today? In what sense is it more mysterious now than when first proposed? What implications does this have for the existence of quantum fields, rather than the existence of quantised relativistic point particles?

(ii). Briefly describe experiments in which it is observed.

(iii). Calculate its magnitude for the simple case of two parallel conducting sheets, stating your assumptions. Is the force due to this effect attractive or repulsive?

NOTE: You may quote the result

$$\int \not{d}k_x \not{d}k_y \left[\sum_{n=-\infty}^{\infty} \sqrt{k_x^2 + k_y^2 + 4\pi^2 n^2/a^2} - \int_{-\infty}^{\infty} dn \, \sqrt{k_x^2 + k_y^2 + 4\pi^2 n^2/a^2} \right] = \frac{\pi^2}{45 a^3} \quad (1)$$

without proof.

Q.B2

(i). Define coherent states in quantum mechanics and discuss their main properties.

(ii). The quantum forced harmonic oscillator (with zero-point energy removed) has Hamiltonian

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2 - \hat{q}\,f(t) - \frac{1}{2}\omega. \tag{1}$$

Show that, if $|\alpha(t)\rangle$ is a coherent state, then

$$|\psi, t\rangle = |\alpha(t)\rangle e^{i\phi(t)} \tag{2}$$

is an eigenstate of $\hat{H}$ provided

$$\frac{d\alpha}{dt} = -i\omega\alpha + \frac{i f(t)}{\sqrt{2\omega}}. \tag{3}$$

Deduce that $\langle\psi, t|\hat{p}|\psi, t\rangle$ and $\langle\psi, t|\hat{q}|\psi, t\rangle$ satisfy the classical equations of motion for p and q in the presence of a source.

(iii). Use this result, and your knowledge of coherent states to explain (briefly) how Maxwell's classical equations are to be understood, given that electromagnetism is intrinsically quantum mechanical.