Advanced Field Theory 2002

14.00 - 17.00 hrs, May 24

Answer THREE questions

Q.1

In this question I would like you to implement renormalisation in $\lambda\phi^4$ theory in four dimensions to order λ^2. The Lagrangian density is, in terms of unrenormalised variables,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4. \tag{1}$$

It is sufficient to adopt a UV cutoff regularisation scheme, in which Euclidean momenta are cut off at value Λ, say. The diagrams that you will need are

$$\begin{aligned}
I_\Lambda(m_0^2) &= &&= \int_\Lambda \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m_0^2} \\
J_\Lambda(k^2, m_0^2) &= &&= \int_\Lambda \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+m_0^2)((k-q)^2+m_0^2)} \\
K_\Lambda(k^2, m_0^2) &= &&= \int_\Lambda \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+m_0^2)(q'^2+m_0^2)((k-q-q')^2+m_0^2)}
\end{aligned}$$

All momenta are Euclidean and $k^2 < \Lambda^2$ in each case.

To order λ^2, the Euclidean inverse propagator $\Gamma^{(2)}(k^2)$ and the 1PI four-point function $\Gamma^{(4)}(k_1, k_2, k_3, k_4)$ are given as

$$\begin{aligned}
\Gamma^{(2)}(k^2) &= k^2 + m_0^2 + \frac{1}{2}\lambda_0 I_\Lambda(m_0^2) - \frac{1}{4}I_\Lambda(m_0^2)J_\Lambda(0, m_0^2) - \frac{1}{6}K_\Lambda(k^2, m_0^2) \\
\Gamma^{(4)}(k_1, k_2, k_3, k_4) &= \lambda_0 - \frac{1}{2}\lambda_0^2[J_\Lambda((k_1+k_2)^2, m_0^2) + \text{permutations}]
\end{aligned}$$

(i). Give a diagrammatic representation of $\Gamma^{(2)}(k^2)$ and $\Gamma^{(4)}(k_1, k_2, k_3, k_4)$.

(ii). What renormalisation conditions do you impose on the renormalised $\Gamma_R^{(2)}(k^2)$ and $\Gamma_R^{(4)}(k_1, k_2, k_3, k_4)$? In particular, how is the field renormalisation constant Z_3 defined?

(iii). Without using counterterms, show how *mass renormalisation* is implemented at one loop in the limit $\Lambda \to \infty$. Is there field renormalisation at one loop?

(iv). Now show how *coupling constant renormalisation* is implemented at one loop in the limit $\Lambda \to \infty$.

(v). Finally, give an expression for Z_3 (or Z_3^{-1}) to order λ^2 in terms of m, λ and Λ. It is sufficient to give an answer in terms of such of the $I_\Lambda, J_\Lambda, K_\Lambda$ as are needed.

You may quote the behaviour $I_\Lambda \sim \Lambda^2$, $J_\Lambda \sim \ln\Lambda$ and $K_\Lambda \sim \Lambda^2$, but K_Λ contains terms $k^2\ln\Lambda$, without proof.

Q.2

Consider a real scalar field ϕ with Lagrangian density (omitting counterterms)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4. \tag{1}$$

The generating functional W[j] for its *connected* Green functions is defined (in units in which $\hbar = 1$) by

$$\exp\left(\frac{i}{\hbar}W[j]\right) = \int \mathcal{D}\phi\ \exp\left(\frac{i}{\hbar}\int d^4x\ \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 + j\phi\right]\right). \tag{2}$$

(i). If $\bar{\phi}(x) = \delta W/\delta j(x)$ is invertible, we define the effective action $\Gamma[\bar{\phi}]$ by

$$\Gamma[\bar{\phi}] = W[j] - \int d^4x\, j\bar{\phi}, \tag{3}$$

where we eliminate j in favour of ϕ. Show that

$$\frac{\delta\Gamma}{\delta\bar{\phi}(x)} = -j(x). \tag{4}$$

(ii). For constant j we can proceed in a similar way to define a $\Gamma(\bar{\phi})$ for constant $\bar{\phi}$, now proportional to the space-time volume Ω of the system as

$$\Gamma(\bar{\phi}) = -\Omega V_{eff}(\bar{\phi}). \tag{5}$$

Show that, to order $\hbar$ (i.e. one loop), $V_{eff}(\bar{\phi})$ takes the form (omitting counterterms)

$$V_{eff}(\bar{\phi}) = \frac{1}{2}m^2\bar{\phi}^2 + \frac{1}{4}\lambda\bar{\phi}^4 + \frac{\hbar}{2}\int d\!\!\!{}^-{}^4\bar{k}\ \ln(\bar{k}^2 + m^2 + 3\lambda\bar{\phi}^2), \tag{6}$$

where $\bar{k}$ is Euclidean momentum. You can use any method. There is no need to prove every step e.g. you can quote the results for Gaussian integrals and take for granted that $\det K = \exp(tr \ln K)$. State your assumptions clearly.

(iii). As it stands, $V_{eff}(\bar{\phi})$ is UV divergent and needs to be renormalised. Explain (briefly) how counterterms are accommodated in the path integral and how they need to be included in $V_{eff}(\bar{\phi})$. What renormalisation conditions do you impose on V_{eff} and its derivatives?

Briefly indicate the steps needed to renormalise $V_{eff}(\bar{\phi})$, but do not do so.

Q.3

This question concerns Grassmann variables and Grassmannian integration.

(i). Consider the case of a single Grassmann variable q. Devise a definition of

$$J = \int dq\, f(q) \tag{1}$$

so that translation invariance of the 'measure' is preserved,

$$J = \int dq\, f(q + q_0) \tag{2}$$

for fixed q_0.

Show that 'integration' and 'differentiation' are similar operations in this case.

(ii). Let $q_1, \bar{q}_1, q_2, \bar{q}_2$ be Grassmann variables. If A is a 2x2 matrix show that the four-dimensional Grassmann integral

$$Z = \int dq_1\, dq_2\, d\bar{q}_1\, d\bar{q}_2\, e^{-\tilde{\bar{q}}Aq} \propto \; det\, A. \tag{3}$$

where the *tilde* denotes *transpose* i.e. $\tilde{\bar{q}}Aq = a_{11}\bar{q}_1 q_1 + \ldots$. Write down, without proof, the corresponding result for the 2N dimensional Grassmann integral

$$Z = \int \prod_1^N dq_i \prod_1^N d\bar{q}_i \; e^{-\tilde{\bar{q}}Aq}, \tag{4}$$

where, for simplicity, we take A to be real, symmetric, and non-singular. Hence determine

$$Z[\eta, \bar{\eta}] = N \int \prod_1^N dq_i \prod_1^N d\bar{q}_i \; e^{-\tilde{\bar{q}}Aq + (\tilde{\bar{\eta}}q + \tilde{\bar{q}}\eta)}, \tag{5}$$

where η is a Grassmann column vector, and $\tilde{\bar{\eta}}$ is the corresponding row vector. N is chosen so that $Z[0,0] = 1$.

(iii). Using the fact that, for arbitrary Grassmann variables $\bar{\eta}_i, Q_i$, $(\bar{\eta}_i Q_i)(\bar{\eta}_j Q_j) = (\bar{\eta}_j Q_j)(\bar{\eta}_i Q_i)$, etc., show explicitly that

$$\frac{\partial}{\partial \bar{\eta}_i} \exp(\sum_i \bar{\eta}_i Q_i) = Q_i \, \exp(\sum_i \bar{\eta}_i Q_i). \tag{6}$$

Assuming a similar expression on differentiation by η_j, evaluate

$$\frac{\partial^2}{\partial \eta_j \partial \bar{\eta}_i} Z[\eta, \bar{\eta}] \Big|_{\eta = \bar{\eta} = 0}. \tag{7}$$

Q.4

Maxwell's equations in empty space are written in terms of the electromagnetic field A_μ as

$$\partial_\mu F^{\nu\mu} = 0, \tag{1}$$

where $F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

They follow from the free action

$$S[A_\mu] = \int d^4x \, [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}]. \tag{2}$$

(i). From the point of view of path integrals explain why the generating functional for the Green functions of the free electromagnetic field cannot be written simply as (in units in which $\hbar = 1$)

$$Z[j_\mu] = \int \prod_\mu \mathcal{D}A_\mu \exp(iS[A_\mu] + i \int dx \, j_\mu A^\mu) \tag{3}$$

(ii). Show how the covariant gauge generating functional

$$Z_\xi[j_\mu] = \int \prod_\mu \mathcal{D}A_\mu \exp(iS_\xi[A_\mu] + i \int dx \, j_\mu A^\mu) \tag{4}$$

is constructed, where

$$S_\xi[A_\mu] = \int d^4x \, [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2]. \tag{5}$$

Calculate the photon propagator $D_{\mu\nu}(x)$ with this gauge choice.

In deriving (4) it is sufficient, for examination purposes, to establish the identity between (3) and (4) for $j_\mu = 0$.

(iii). With this gauge choice the Ward identity for the QED effective action $\Gamma_\xi(\tilde{\bar{\psi}}, \tilde{\psi}, \bar{A}_\mu)$ (in an obvious notation) takes the form

$$0 = \frac{1}{\xi} \Box \partial_\mu \bar{A}^\mu(x) - \partial_\mu \frac{\delta \Gamma_\xi}{\delta A_\mu}(x) - ie_0 \tilde{\psi} \frac{\delta \Gamma_\xi}{\delta \tilde{\psi}}(x) + ie_0 \tilde{\bar{\psi}} \frac{\delta \Gamma_\xi}{\delta \tilde{\bar{\psi}}}(x). \tag{6}$$

Use this to explain why the renormalised photon is *massless*. State any assumptions.

Q.5

(i) The renormalization group equation for the proper n–point vertex of a massless scalar theory reads:

$$\left[\beta(g)\frac{\partial}{\partial g} + \mu\frac{\partial}{\partial \mu} - n\gamma(g)\right]\Gamma^{(n)}(g,\mu) = 0\ .$$

Explain **briefly** the origin of this equation and the significance of the symbols β and γ.

(ii) Show that the equation

$$\left[\beta(g)\frac{\partial}{\partial g} + \mu\frac{\partial}{\partial \mu}\right] G(g,\mu) = 0$$

has contours of constant G given by $G(\bar{g}(t),\mu) = G(g,\mu_0)$, where $t := \ln(\mu/\mu_0)$ and $\bar{g}(t)$ is the solution of

$$\frac{\partial \bar{g}(t)}{\partial t} = \beta(\bar{g}(t))\ .$$

(iii) Explain how the nature of the renormalized theory is determined by the beta function, and in particular its zeros.

(iv) In QCD the lowest-order perturbative expression for $\beta(g)$ is $\beta(g) = -bg^3$.
Use this to find the form of $\alpha_s(Q^2) := (\bar{g}(t))^2/4\pi$, with $Q^2 = Q_0^2 e^{2t}$. What is the significance of the scale parameter Λ?

(v) The solution of the full renormalization group equation involves the factor $\exp[\int_0^t dt'\{4 - n - n\gamma(\bar{g}(t'))\}]$. Evaluate this factor to lowest order, given that $\gamma(g) = cg^2$.