Examination Paper

M.Sc. in Quantum Fields and Fundamental Forces

TP.4 Advanced Field Theory

14:00 - 17:00 Monday, June 7th, 1999

Answer **TWO** questions from Section **A** and **ONE** question from Section **B**
(Questions from Section A carry 30% each and Questions from Section B carry 40% of the total)

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

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Section A

Q.A1

A particle of unit mass, coordinate q, moves in one dimension in a potential $V(q)$.

(i). By inserting a complete set of momentum eigenstates, in units in which $\hbar = 1$, show that

$$\langle q'|e^{-i\epsilon\hat{p}^2/2}\, e^{-i\epsilon V(\hat{q})}|q\rangle = \int \frac{dp}{2\pi} e^{i(q'-q)p}\, e^{-i\epsilon p^2/2}\, e^{-i\epsilon V(q)}. \tag{1}$$

(ii). By inserting complete sets of states judiciously, show that the probability amplitude

$$\langle q_1, t_1|q_0, t_0\rangle = \langle q_1|e^{-i\hat{H}(t_1-t_0)}|q_0\rangle \tag{2}$$

that the particle will be at q_1 at time t_1 if it was at q_0 at time t_0 can be represented by the path integral

$$\langle q_1, t_1|q_0, t_0\rangle = \int \mathcal{D}q \, \exp\left(iS[q]_{t_0}^{t_1}\right), \tag{3}$$

where $S[q]_{t_0}^{t_1}$ is the action

$$S[q]_{t_0}^{t_1} = \int_{t_0}^{t_1} dt \, \left[\frac{1}{2}\dot{q}^2 - V(q)\right] \tag{4}$$

with $q(t_1) = q_1$, $q(t_0) = q_0$. Define $\mathcal{D}q$.

(iii). In particle physics a more useful quantity is the generating functional of time-ordered products of the qs, the groundstate expectation value

$$Z[j] = \langle 0|T\left(e^{i\int dt\, j\hat{q}}\right)|0\rangle. \tag{5}$$

Show that, if $t_0 < t, t' < t_1$,

$$\langle q_1, t_1|T(\hat{q}(t)\hat{q}(t'))|q_0, t_0\rangle = \int \mathcal{D}q \, q(t)q(t') \exp\left(iS[q]_{t_0}^{t_1}\right) \tag{6}$$

List (but do not implement) the steps whereby you would arrive at the path integral for $Z[j]$,

$$Z[j] = \int \mathcal{D}q \, \exp\left(iS[q] + i\int dt\, jq\right). \tag{7}$$

Explain the nature of the time contour in eq.(7).

NOTE:

$$lim_{N\to\infty}\left(e^{(\hat{A}/N + \hat{B}/N)}\right)^N = lim_{N\to\infty}\left(e^{\hat{A}/N} e^{\hat{B}/N}\right)^N, \tag{8}$$

and

$$\int \frac{dp}{2\pi} e^{iqp}\, e^{-i\epsilon p^2/2} = \left(\frac{1}{2\pi i\epsilon}\right)^{1/2} e^{iq^2/2\epsilon}. \tag{9}$$

Q.A2

Maxwell's equations in empty space are

$$\begin{aligned} \nabla.\mathbf{E} &= 0 = \nabla.\mathbf{B}, \\ -\dot{\mathbf{E}} + \nabla \wedge \mathbf{B} &= 0 = \dot{\mathbf{B}} + \nabla \wedge \mathbf{E}. \end{aligned} \tag{1}$$

(i). Show that they are equivalent to

$$\partial_\mu F^{\nu\mu} = 0 \tag{2}$$

and $F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, where $\mathbf{B} = \nabla \wedge \mathbf{A}$, $\mathbf{E} = -\dot{\mathbf{A}} - \nabla A^0$.

Further, show that they follow from the free action

$$S[A_\mu] = \int d^4x \, [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}]. \tag{3}$$

(ii). Explain why the generating functional for the Green functions of the free electromagnetic field cannot be written simply as

$$Z[j_\mu] = \int \prod_\mu \mathcal{D}A_\mu \exp(iS[A_\mu] + i \int dx \, j_\mu A^\mu) \tag{4}$$

(iii). Show how the covariant gauge generating functional

$$Z_\xi[j_\mu] = \int \prod_\mu \mathcal{D}A_\mu \exp(iS_\xi[A_\mu] + i \int dx \, j_\mu A^\mu) \tag{5}$$

is constructed, where

$$S_\xi[A_\mu] = \int d^4x \, [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2]. \tag{6}$$

What is the photon propagator in this gauge?

(iv). Quoting Feynman rules without proof explain why, if we use this propagator in the lowest order diagrams for electron-electron scattering, the scattering amplitude is independent of ξ.

Q.A3

(i). Let A be a real nonsingular NxN symmetric matrix. Show that the integral over N complex variables q_i, $(i = 1, 2, .., N)$

$$Z = \int \prod_1^N dq_i \prod_1^N dq_i^* \; e^{-q^\dagger A q} \propto \frac{1}{det\,A}. \tag{1}$$

In eq.(1) q denotes the column vector with components q_i.

Hence show that, if j is a fixed column vector with components j_i,

$$Z[j, j^*] = \int \prod_1^N dq_i \prod_1^N dq_i^* \; e^{-q^\dagger A q + (j^\dagger q + q^\dagger j)} \propto \frac{1}{det\,A} e^{j^\dagger A^{-1} j}. \tag{2}$$

(ii). Now consider the case when the qs are Grassmann variables. Beginning with one q, devise a definition of

$$J = \int dq\, f(q) \tag{3}$$

so that translation invariance of the 'measure' is preserved,

$$J = \int d(q + q_0)\, f(q) \tag{4}$$

for fixed q_0. Show that 'integration' and 'differentiation' are similar operations in this case.

(iii). Let q_1, q_1^*, q_2, q_2^* be Grassmann variables. If A is a 2x2 matrix show that the four-dimensional Grassmann integral

$$Z = \int dq_1\, dq_1^*\, dq_2\, dq_2^*\, e^{-q^\dagger A q} \propto \; det\, A. \tag{5}$$

where $q^\dagger A q = a_{11} q_1^* q_1 + ...$ Write down, without proof, the corresponding result for the 2N dimensional Grassmann integral

$$Z = \int \prod_1^N dq_i \prod_1^N dq_i^* \; e^{-q^\dagger A q}, \tag{6}$$

Hence, by completing the square, determine

$$Z[\eta, \eta^*] = \int \prod_1^N dq_i \prod_1^N dq_i^* \; e^{-q^\dagger A q + (\eta^\dagger q + q^\dagger \eta)}, \tag{7}$$

where η is a Grassmann column vector, and $\eta^\dagger$ is defined accordingly.

(iv). What implication does this have for Feynman rules for fermions? (Only a brief discussion is required.)

Q.A4

A particle of unit mass, coordinate q, moves in one dimension in a potential $V(q)$. A canonical ensemble of such particles is held at temperature T in a heatbath.

(i). By adopting a basis of energy eigenstates $|E_n\rangle$ explain why the ensemble average $\bar{O}$ of an observable with operator $\hat{O}$ can be written as

$$\bar{O} = \frac{tr(\hat{O}\hat{\rho})}{tr\hat{\rho}} \qquad (1)$$

where $\hat{\rho} = e^{-\beta\hat{H}}$, $\beta = (k_B T)^{-1}$ where k_B is Boltzmann's constant, and $tr\hat{A} = \sum_n \langle E_n|\hat{A}|E_n\rangle$.

How does completeness allow us to write an equivalent expression in terms of position eigenstates $|q\rangle$?

(ii). We work in units in which $\hbar = 1$. The probability amplitude $F(q_1, t_1; q_0, t_0)$ that the particle will be at q_1 at time t_1 if it was at q_0 at time t_0 can be represented by the path integral

$$F(q_1, t_1; q_0, t_0) = \int \mathcal{D}q \, \exp\left(i \int_{t_0}^{t_1} dt \, \left[\frac{1}{2}\dot{q}^2(t) - V(q(t))\right]\right) \qquad (2)$$

with $q(t_1) = q_1$, $q(t_0) = q_0$. Use this expression to show that the partition function $Z = tr(e^{-\beta\hat{H}})$ for the ensemble can be written as

$$Z = \int_{periodic} \mathcal{D}q \, \exp\left(-\int_0^{\beta} d\tau \, \left[\frac{1}{2}\left(\frac{dq(\tau)}{d\tau}\right)^2 + V(q(\tau))\right]\right), \qquad (3)$$

where $\tau = it$ denotes imaginary time and the integral is restricted to periodic paths, $q(\tau = \beta) = q(\tau = 0)$.

(iii). Suppose that $q(t)$ is coupled to a *constant* source j to give a partition function

$$Z(j) = \int_{periodic} \mathcal{D}q \, \exp\left(-\int_0^{\beta} d\tau \, \left[\frac{1}{2}\left(\frac{dq(\tau)}{d\tau}\right)^2 + V(q(\tau)) + jq(\tau)\right]\right). \qquad (4)$$

Define $\mathcal{F}(j)$ by

$$\mathcal{F}(j) = -k_B T \ln Z(j), \qquad (5)$$

whereby $\bar{q}(j)$,

$$\bar{q}(j) = \frac{\partial \mathcal{F}(j)}{\partial j}, \qquad (6)$$

is the ensemble average of q in the presence of the source.

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If the relation between $\bar{q}$ and j is invertible, define $\Gamma(\bar{q})$ by

$$\Gamma(\bar{q}) = \mathcal{F}(j(\bar{q})) - q\, j(\bar{q}). \tag{7}$$

Show that

$$\frac{\partial \Gamma(\bar{q})}{\partial \bar{q}} = -j. \tag{8}$$

How do we understand the value of $\bar{q}$ for which $\partial\Gamma(\bar{q})/\partial\bar{q} = 0$?

(iv). In fact, this value is $\bar{q} = 0$, because of the peculiarities of quantum mechanics in one spatial dimension. We extend $Z(j)$ above to the partition function of a real scalar field ϕ at temperature T, Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4. \tag{9}$$

as

$$Z(j) = \int_{periodic} \mathcal{D}\phi\, \exp\left(-\int_0^{\beta} d\tau \int d^3x \left[\frac{1}{2}\left(\frac{d\phi}{d\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 + j\phi\right]\right). \tag{10}$$

For constant j we can proceed in a similar way to define a $\Gamma(\bar{\phi})$, now proportional to the spatial volume L^3 of the system as $\Gamma(\bar{\phi}) = L^3 V_{eff}(\bar{\phi})$.

$V_{eff}(\bar{\phi})$ is expressible to one loop as

$$V_{eff}(\bar{\phi}) = \frac{1}{2}m^2\bar{\phi}^2 + \frac{1}{4}\lambda\bar{\phi}^4 + \frac{1}{2}\beta^{-1}\sum_{-\infty}^{\infty}\int d^3k\, \ln(k^2 + m^2 + (\frac{2\pi n}{\beta})^2 + 3\lambda\bar{\phi}^2). \tag{11}$$

Explain briefly why the second term in eq.(11) takes the form it does.

(v). If $m^2 < 0$ we find that, when $T = 0$, $\partial V_{eff}(\bar{\phi})/\partial\bar{\phi} = 0$ has solutions with $\bar{\phi} \neq 0$. Sketch what happens to $V_{eff}(\bar{\phi})$ as T increases. How do we interpret this?

Section B

Q. B5

Write an essay on the interpretation of quantum mechanics. You may use the following questions as an outline.

Describe the Copenhagen Interpretation (CI) of quantum mechanics. What predictions can be made using the CI? Give examples of questions that the CI cannot be used to answer. Are these scientific questions? Is the CI precisely defined?

Describe the "Many Worlds Interpretation" (MWI). Is it precisely defined? What predictions can be made using the MWI? In what way(s) does it improve upon the CI? In what ways is it less successful than the CI?

Q. B6

Write down the Schrödinger equation for a particle of unit mass with electric cha
e in (a) the vacuum and (b) a static magnetic field $\mathbf{B} = \nabla \wedge \mathbf{A}$.

In the double slit experiment with electrons, fringes appear on a screen, S, due
interference between the parts of the wavefunction that pass through the upper and lo
of two slits in a previous screen in front of an electron gun. The wavefunction at a po
Q on the screen S is given by

$$\Psi(Q) = \Psi_+(Q) + \Psi_-(Q) = e^{i\int_{C_+} \mathbf{p}\cdot d\mathbf{s}}\,\Psi(P) + e^{i\int_{C_-} \mathbf{p}\cdot d\mathbf{s}}\,\Psi(P)$$

where $\mathbf{p}$ is the momentum of the electrons. What are C_+, C_- and P, illustrating y
answers on a diagram? Where, on your diagram, would you place a very long thin solen
in order to demonstrate the Aharonov-Bohm effect? What is the effect and why is i
surprise? Show that the effect is governed by the magnitude of the magnetic flux thro
the solenoid.

Show how the same mathematics can be reproduced by considering a particle mov
in vacuum on the two-dimensional plane with the origin removed.

Q. B7

Explain how *topological defects* can form at symmetry-breaking phase transitions. Describe some of the types of defects that may form in condensed-matter systems. How has it been possible to test ideas about cosmological defect formulation by doing laboratory experiments?