Examination Paper

M.Sc. in Quantum Fields and Fundamental Forces

TP.4 Advanced Field Theory

14.00 - 17.00 Monday, June 12th, 2000

Answer **TWO** questions from Section A and **ONE** question from Section B
(Questions from Section A carry 30% each and questions from Section B carry 40%)

Use a separate booklet for each question. Make sure that each booklet carries your number, the course title, and the number of the question attempted.

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Q.A1

(i). Show that Maxwell's equations in empty space can be written as

$$\partial_\mu F^{\nu\mu} = 0 \tag{1}$$

and $F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, where $\mathbf{B} = \nabla \wedge \mathbf{A}$, $\mathbf{E} = -\dot{\mathbf{A}} - \nabla A^0$.

Further, show that they follow from the free action

$$S[A_\mu] = \int d^4x \, [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}]. \tag{2}$$

What are the equations of motion in the Lorenz gauge $\partial_\mu A^\mu = 0$. State why this condition cannot be implemented as an operator equation.

(ii). Explain why, as a path integral, the generating functional for the Green functions of the free electromagnetic field cannot be written simply as

$$Z[j_\mu] = \int \prod_\mu \mathcal{D}A_\mu \exp(iS[A_\mu] + i \int dx \, j_\mu A^\mu) \tag{3}$$

(iii). Show how the covariant gauge generating functional

$$Z_\xi[j_\mu] = \int \prod_\mu \mathcal{D}A_\mu \exp(iS_\xi[A_\mu] + i \int dx \, j_\mu A^\mu) \tag{4}$$

is constructed, where

$$S_\xi[A_\mu] = \int d^4x \, [-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2]. \tag{5}$$

What is the photon propagator in this gauge?

(iv). What are the Euler-Lagrange equations of motion $\delta S_\xi[A_\mu] = 0$, when $\xi = 1$? State briefly how this relates to the canonical Gupta-Bleuler formalism by which you were first introduced to Quantum electromagnetism.

Q.A2

(i). If q is a Grassmann variable, devise a definition of

$$J = \int dq\, f(q) \tag{1}$$

so that translation invariance of the 'measure' is preserved,

$$J = \int d(q + q_0)\, f(q) \tag{2}$$

for fixed q_0. Show that 'integration' and 'differentiation' are similar operations in this case.

(ii). If $q, \bar{q}, \eta, \bar{\eta}$ are Grassmann variables show that

$$\exp(\bar{\eta}q)\exp(\bar{q}\eta) = \exp(\bar{\eta}q + \bar{q}\eta), \tag{3}$$

and that

$$\frac{\delta}{\delta\bar{\eta}} \exp(\bar{\eta}q) = q\exp(\bar{\eta}q). \tag{4}$$

(iii). Let $q_1, \bar{q}_1, q_2, \bar{q}_2$ be Grassmann variables. If A is a 2x2 matrix show that the four-dimensional Grassmann integral

$$Z = \int dq_1\, d\bar{q}_1\, dq_2\, d\bar{q}_2\, e^{-\bar{q}Aq} \propto \ det\, A. \tag{5}$$

where $\bar{q}Aq = a_{11}\bar{q}_1 q_1 + \ldots$. Write down, without proof, the corresponding result for the 2N dimensional Grassmann integral

$$Z = \int \prod_1^N dq_i \prod_1^N d\bar{q}_i\; e^{-\bar{q}Aq}, \tag{6}$$

(iv). Consider the theory of a single fermion field ψ, interacting with a single scalar field ϕ, with action

$$S[\bar{\psi}, \psi, \phi] = \int d^4x \left(\bar{\psi}(i\gamma^\mu\partial_\mu - m - g\phi)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2\right) \tag{7}$$

The generating functional for ϕ-field Green functions is

$$Z[j] = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\phi\, \exp i\Big(S[\bar{\psi}, \psi, \phi] + \int d^4x\, j\phi\Big) \tag{8}$$

On integrating out the $\psi, \bar{\psi}$ fields, $Z[j]$ can be written as

$$Z[j] = \int \mathcal{D}\phi\, \exp i\Big(S_{eff}[\phi] + \int d^4x\, j\phi\Big). \tag{9}$$

What is $S_{eff}[\phi]$, and give a diagrammatic representation of it. [You may quote $\ln\det A = tr \ln A$ for relevant matrices A, without proof.]

(v). Use $S_{eff}[\phi]$ to explain the origin of the Feynman rule for fermions, that a diagram with n fermion loops acquires a factor $(-1)^n$ above and beyond that for a comparable diagram of scalar fields.

State any assumptions clearly.

Q.A3

A harmonic oscillator of unit mass moves in one dimension. Its energy eigenvalues are

$$E_n = \hbar\omega(n + \frac{1}{2}), \quad n = 0, 1, 2, \ldots. \tag{1}$$

(i). Consider an ensemble of such oscillators at temperature T. Using energy eigenstates evaluate the partition function $Z = tr(e^{-\beta \hat{H}})$ for the ensemble directly, where $\beta = (k_B T)^{-1}$ and k_B is Boltzmann's constant.

(ii). Show that the ensemble average of the harmonic oscillator energy can be written as

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta}, \tag{2}$$

and that it has the form

$$\bar{E} = \frac{1}{2}\hbar\omega\left[1 + \frac{2}{e^{\beta\hbar\omega} - 1}\right] \tag{3}$$

How are the two terms interpreted?

(iii). The probability amplitude $F(q_1, t_1; q_0, t_0)$ that the particle will be at q_1 at time t_1 if it was at q_0 at time t_0 can be represented by the path integral

$$F(q_1, t_1; q_0, t_0) = \int \mathcal{D}q \, \exp\left(\frac{i}{\hbar}\int_{t_0}^{t_1} dt \, \left[\frac{1}{2}\dot{q}^2(t) - \frac{1}{2}\omega^2 q^2(t)\right]\right) \tag{4}$$

with $q(t_1) = q_1$, $q(t_0) = q_0$. Use this expression to show that the partition function Z can be written as the path integral

$$Z = \int_{periodic} \mathcal{D}q \, \exp\left(-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \, \left[\frac{1}{2}\left(\frac{dq(\tau)}{d\tau}\right)^2 + \frac{1}{2}\omega^2 q^2(\tau)\right]\right) \tag{5}$$

In (4) $\tau = it$ denotes imaginary time and the integral is restricted to periodic paths, $q(\tau = \beta\hbar) = q(\tau = 0)$.

By expanding $q(\tau)$ as

$$q(\tau) = \sum_{n=-\infty}^{\infty} q_n \, e^{i\omega_n \tau} \tag{6}$$

where $\omega_n = 2\pi n/\beta\hbar$, show that

$$\ln Z = -\frac{1}{2}\sum_{n=-\infty}^{\infty} \ln[\beta^2\hbar^2(\omega_n^2 + \omega^2)] + constant. \tag{7}$$

(iv). It happens that the constant is β independent (you do not have to show this). Calculate the ensemble average $\bar{E}$ and show that it agrees with the result derived previously.

NOTE:

$$\int_{-\infty}^{\infty} dq \, e^{-Aq^2/2} = \sqrt{\frac{2\pi}{A}} \tag{8}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + (\Omega/2\pi)^2} = \frac{2\pi^2}{\Omega}\left[1 + \frac{2}{e^{\Omega} - 1}\right] \tag{9}$$

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Q.A4

(i). For a single scalar field ϕ, coupled to a source j, write down the relationship between the generating functional for n-field Green functions $Z[j]$, the generating functional for connected Green functions $W[j]$, and the effective action $\Gamma[\phi]$.

What sets of diagrams does

$$\frac{\delta^n \Gamma}{\delta\phi(x_1)....\delta\phi(x_n)} \tag{1}$$

represent?

(ii). Derive the relationship between

$$\frac{\delta^2 W}{\delta j(x_1)\delta j(x_2)} \text{ and } \frac{\delta^2 \Gamma}{\delta\phi(x_1)\delta\phi(x_2)} \tag{2}$$

(iii). Define the effective potential $V(\phi)$ and give a physical interpretation of it. Show that

$$V(\phi) = V_{cl}(\phi) + \text{terms of order } \hbar, \tag{3}$$

where

$$V_{cl}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{4}$$

is the classical potential in the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_{cl}(\phi). \tag{5}$$

(iv). To $O(\hbar)$, $V(\phi)$ is given formally (to an additive constant) as

$$V_\Lambda(\phi) = V_{cl}(\phi) + \frac{1}{2}\hbar\int_{|\bar{k}|<\Lambda} d^4\bar{k}\ \ln\left(\bar{k}^2 + m^2 + \frac{1}{2}\lambda\phi^2\right), \tag{6}$$

where we have imposed a cutoff $|\bar{k}| < \Lambda$ on the Euclidean momenta $\bar{k}$ to render the integral finite.

Explain briefly how the renormalisation programme enables us to construct a finite $V(\phi)$ from $V_\Lambda(\phi)$ that satisfies the definition given earlier. You do not have to construct $V(\phi)$ explicitly.

Q.B1

(i). What is the *Casimir Effect*. and how is it understood today? In what sense is it more mysterious now than when first proposed? What implications does this have for the existence of quantum fields, rather than the existence of quantised relativistic point particles?

(ii). Briefly describe experiments in which it is observed.

(iii). Calculate its magnitude for the simple case of two parallel conducting sheets, stating your assumptions. Is the force due to this effect attractive or repulsive?

NOTE: You may quote the result

$$\int \not{d}k_x \not{d}k_y \left[\sum_{n=-\infty}^{\infty} \sqrt{k_x^2 + k_y^2 + 4\pi^2 n^2/a^2} - \int_{-\infty}^{\infty} dn \sqrt{k_x^2 + k_y^2 + 4\pi^2 n^2/a^2} \right] = \frac{\pi^2}{45a^3} \tag{1}$$

without proof.

Q.B2

Consider a complex scalar field $\Phi(x) \in \mathbb{C}$, with conjugate momentum $\Pi(x) \in \mathbb{C}$, whose dynamics is controlled by the Hamiltonian

$$H = \int d^3\mathbf{x}\,\left(\Pi^\dagger\Pi + |\nabla\Phi|^2 + V(\Phi)\right), \quad V(\Phi) = m_0^2|\Phi|^2 + \lambda|\Phi|^4$$

(i). What are the classical ground states if (a) $m_0^2 > 0$, (b) $m_0^2 < 0$?

Why do you expect $\lambda > 0$ to be required for useful solutions?

(ii). By considering the thermal expectation value of the Hamiltonian $\langle H\rangle$, derive the gap equation in the form

$$m_0^2 + \lambda v^2 + 3\lambda\langle\xi^2\rangle + \lambda\langle\chi^2\rangle = 0$$

where

$$\Phi(x) = \frac{1}{\sqrt{2}}(v + \xi(x) + i\chi(x)), \quad \partial_\mu v = 0.$$

In your answer you should note what v is, the reasons why your representations for the fields are valid in this context, and how you treat $\langle\xi\rangle$ and similar expectation values which do not appear in the gap equation.

(iii). Solve the gap equation using the high temperature approximation, where all temperature independent contributions to the expectations values (including infinite ones) are ignored. You may find the following result useful

$$\int \frac{d^3\mathbf{k}}{(2\pi^3)}\frac{1}{\omega}\frac{1}{(\exp\{\omega/T\} - 1)} \simeq \frac{T^2}{12} + O(Tm_0), \qquad \omega = (\mathbf{k}^2 + m_0^2)^{1/2}$$

Find an expression for a critical temperature in terms of m_0^2 and λ.

Sketch a phase diagram in terms of $v/|m_0|$ vs. $T/|m_0|$ indicating what the different phases are.

(iv). Comment on the similarities between this model, Landau theory, the Ising model and the behaviour of magnetisation seen in real magnets. In particular comment on the behaviour of v near the critical temperature.

Q.B3

Explain how topological defects can form at a symmetry-breaking phase transition, using the formation of strings in a U(1)-breaking model as an example. Describe briefly how ideas about cosmic string formation in the early Universe may be tested in low-temperature laboratory experiments. Discuss the kind of information that has been or can be obtained from such experiments.