<u>Exam</u>

M.Sc. in Quantum Fields and Fundamental Forces

TP.4 — Advanced Field Theory

2:00 – 5:00, Friday May 11, 2007

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Answer **THREE** out of the five questions

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

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You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{i m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!} \left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon) \right) \quad \text{for integer } \ m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa + (1-x)b]^2}$$

- Dirac matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \mathrm{tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu},$$
$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4.$$

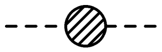
Question (1)

The Lagrangian of the pseudoscalar Yukawa theory is

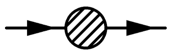
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - ig\bar{\psi}\gamma^5\psi\phi, \tag{1.1}$$

where ϕ is a real (pseudo)scalar field and ψ is a fermion field.

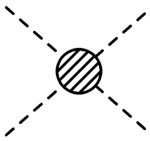
In *bare* perturbation theory, the superficially divergent correlation functions are



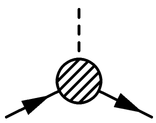
$$= \frac{i}{16\pi^2}\left(\lambda m_\phi^2 + 2g^2p^2\right)\frac{2}{\epsilon} + \text{finite terms.}$$



$$= \frac{i}{16\pi^2}\frac{g p\!\!\!/}{2}\frac{2}{\epsilon} + \text{finite terms.}$$



$$= \frac{i}{16\pi^2}\left(\frac{3}{2}\lambda^2 - 24g^4\right)\frac{2}{\epsilon} + \text{finite terms.}$$



$$= -\frac{g^3}{16\pi^2}\gamma^5\frac{2}{\epsilon} + \text{finite terms.}$$

(a) Let us define the renormalised fields by $\phi = Z_\phi^{1/2}\phi_R$ and $\psi = Z_\psi^{1/2}\psi_R$, and renormalised couplings by

$$\begin{aligned} Z_\phi^2\lambda &= \lambda_R + \delta\lambda, \\ Z_\phi^{1/2}Z_\psi g_B &= g_R + \delta g, \\ Z_\phi m_\phi^2 &= m_{\phi R}^2 + \delta m_\phi^2, \\ Z_\psi m &= m_R + \delta m. \end{aligned}$$

Let us further write $Z_\phi = 1 + \delta Z_\phi$ and $Z_\psi = 1 + \delta Z_\psi$. Write down the Lagrangian in terms of the renormalised fields and parameters.

[3 marks]

(b) Identify the terms that are treated as interactions in *renormalised* perturbation theory.

[2 marks]

(c) What is the contribution from the counterterms to each of the correlation functions in part (a)?

[3 marks]

(d) What are the values of the counterterms in the $\overline{\text{MS}}$ scheme?

[6 marks]

(e) Show that the beta functions of the theory are

$$\begin{aligned} \beta_g &\equiv M\left.\frac{\partial g_R}{\partial M}\right|_B = \frac{5g^3}{16\pi^2}, \\ \beta_\lambda &\equiv M\left.\frac{\partial \lambda_R}{\partial M}\right|_B = \frac{1}{16\pi^2}\left(3\lambda^2 + 8\lambda g^2 - 48g^4\right). \end{aligned}$$

[6 marks]

[TOTAL 20 marks]

Question (2)

The gauge-fixed Lagrangian of the Yang-Mills theory is

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\ & +\frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \\ & -\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \partial^\mu c^{a*}\left(\partial_\mu c^a - g f^{abc} A^b_\mu c^c\right). \end{aligned} \tag{2.1}$$

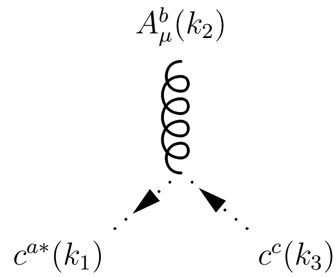
(a) Why are the terms on the last row of Eq. (2.1) needed? How do observable quantities depend on the value of ξ?

[2 marks]

(b) Explain in words how the ghost field arises in the gauge fixing procedure. What are its main properties?

[3 marks]

(c) Derive the Feynman rule for the ghost-gluon vertex



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[3 marks]

(d) Using the propagators

$A^a_\mu(k)$  $A^b_\nu(-k) \quad \leftrightarrow \quad \dfrac{-i\delta^{ab}}{k^2}\left[g^{\mu\nu} - (1-\xi)\dfrac{k^\mu k^\nu}{k^2}\right],$

$c^{b*}(k)$  $c^a(k) \quad \leftrightarrow \quad \dfrac{i\delta^{ab}}{k^2},$

show that the one-loop contribution to the ghost two-point function in dimensional regularisation is

$$g^2 C_2(G)\delta^{ab}\mu^\epsilon \int \frac{d^dp}{(2\pi)^d}\frac{1}{(k+p)^2p^2}\left[k^2 + k\cdot p - (1-\xi)\left(\frac{(k\cdot p)^2}{p^2} - k\cdot p\right)\right],$$

where $C_2(G) = f^{acd}f^{bcd}$.

[5 marks]

(e) Calculate the divergent part of this integral in the Feynman gauge ($\xi = 1$).

[6 marks]

(f) What kind of a counterterm will you need to deal with the divergence?

[1 marks]

[TOTAL 20 marks]

Question (3)

Consider the d-dimensional (Minkowskian) integral

$$I^\mu(k,m) = \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu}{p^2((p+k)^2 - m^2)}. \tag{3.1}$$

(a) Explain why the integral is parallel to k^μ, i.e.,

$$I^\mu(k,m) = A(k,m)k^\mu,$$

where $A(k,m)$ is a scalar function.

[2 marks]

(b) Show that

$$I^\mu(k,m) = \frac{1}{2}\left(\frac{m^2 - k^2}{k^2} I_2(k;0,m) - \frac{1}{k^2} I_1(m)\right) k^\mu, \tag{3.2}$$

where

$$I_1(m) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2}.$$

$$I_2(k;m_1,m_2) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m_1^2)((p+k)^2 - m_2^2)}.$$

[6 marks]

(c) Show that in the limit $k \to 0$, the integral is

$$I^\mu(k,m) = -\frac{i}{(4\pi)^{d/2}} \frac{2}{d} \Gamma\left(2 - \frac{d}{2}\right) m^{d-4} k^\mu + O(k^3).$$

[12 marks]

[TOTAL 20 marks]

Question (4)

Consider a real scalar field theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}g\phi^3.$$

(a) Write down the propagator and the Feynman rule for the interaction vertex.

[2 marks]

(b) What are the dimensionalities of ϕ, m and g in d dimensions?

[3 marks]

(c) If a diagram has E external legs and V vertices, how many internal lines does it have? How many loops does it have?

[2 marks]

(d) In four dimensions, is the theory non-renormalisable, renormalisable or superrenormalisable?

[1 marks]

(e) Draw all superficially divergent one-particle irreducible (1PI) Feynman diagrams that have at least one external leg.

[4 marks]

(f) For each diagram in part (e), write down the integral it corresponds to including all the prefactors

[6 marks]

(g) In renormalised perturbation theory, will you have to introduce a quartic (counter)term ϕ^4? Will you need a ϕ^4 term in Wilsonian renormalisation?

[2 marks]

[TOTAL 20 marks]

Question (5)

Let us consider a simplified version of the Standard Model of particle physics, which consists of the SU(3) gauge field, the top quark and the Higgs. The model has three coupling constants: the gauge coupling g_s, the Higgs self-coupling λ and the Yukawa coupling g_t. The couplings run as a function of the renormalisation scale M (when $M \gtrsim m_t$) as

$$\frac{dg_s^2}{d\ln M} = -\frac{14}{16\pi^2} g_s^4, \tag{5.1}$$

$$\frac{dg_t^2}{d\ln M} = \frac{g_t^2}{16\pi^2}\left(9g_t^2 - 16g_s^2\right), \tag{5.2}$$

$$\frac{d\lambda}{d\ln M} = \frac{1}{16\pi^2}\left(24\lambda^2 + 12g_t^2\lambda - 6g_t^4\right). \tag{5.3}$$

The mass of top quark is

$$m_t = \frac{g_t(v)v}{\sqrt{2}},$$

and the mass of the Higgs is

$$m_H = \sqrt{2\lambda(v)}v,$$

where $v \approx 246$ GeV is the Higgs expectation value, and the couplings are calculated at $M = v$. The top mass has been measured to be $m_t \approx 171$ GeV. The Higgs mass has a lower bound $m_H \gtrsim 115$ GeV. The gauge coupling is $g_s(v) \approx 1.2$.

(a) Consider first each of the three couplings on its own, setting the two others zero. State whether each of the couplings is relevant, marginally relevant, marginally irrelevant or irrelevant.

[3 marks]

(b) Still considering each of the couplings on its own, solve the renormalisation group equations to find $g_s^2(M)$, $g_t^2(M)$ and $\lambda(M)$. Based on this calculation, what is the maximum energy at which the theory could be valid?

[4 marks]

(c) Consider now the coupled equations (5.1) and (5.2). Use an ansatz $g_t^2 = \rho g_s^2$ with constant ρ to find an exact solution. Sketch the renormalisation group flow on the (g_s^2, g_t^2) plane. If we could ignore the Higgs coupling λ, would the theory have a continuum limit? Does the existence of the continuum limit depend on the couplings g_t^2 and g_s^2, and if so, how? Is this compatible with the measured values?

[7 marks]

(d) Consider now all three coupled equations (5.1), (5.2) and (5.3). Assuming the relationship you found in part (c), find a solution with $\lambda = \sigma g_s^2$ and $g_t^2 = \rho g_s^2$ with constant σ and ρ. If the couplings satisfied these relation, would the theory have a continuum limit? Given the measured values of g_s and v, what would the predicted top and Higgs masses be?

[6 marks]

[TOTAL 20 marks]