Imperial College London

MSc EXAMINATION May 2009

This paper is also taken for the relevant Examination for the Associateship

ADVANCED FIELD THEORY

For Students in Quantum Fields and Fundamental Forces

Wednesday, 13 May 2009: 14:00 to 17:00

Answer 3 out of the following 4 questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the THREE answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in THREE answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{i m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!} \left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon) \right) \quad \text{for integer } \ m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa + (1-x)b]^2}$$

- Gaussian two-point function

$$\frac{\int d^N q \, q_i q_j e^{-\frac{1}{2} q^T M q}}{\int d^N q e^{-\frac{1}{2} q^T M q}} = \left(M^{-1}\right)_{ij} .$$

- Gaussian Grassmann integral

$$\int \left(\prod_i d\theta_i^* d\theta_i \right) \exp\left(-\theta_i^* B_{ij} \theta_j\right) = \det \mathbf{B}.$$

1. This question deals with Feynman integrals in d-dimensional Minkowski space.

(i) (a) Show that, if Δ is a scalar quantity,

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{(p^2-\Delta)^2} = \frac{g^{\mu\nu}}{d} \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{(p^2-\Delta)^2}.$$

[4 marks]

(b) Using

$$I_1(\Delta) = \frac{2}{d-2} \Delta I_2(\Delta),$$

Show that

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{(p^2-\Delta)^2} = \frac{g^{\mu\nu}}{d-2} \Delta I_2(\Delta).$$

[3 marks]

(ii) Consider now the integral

$$I^{\mu\nu}(k^\rho) = \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{p^2(p+k)^2}.$$

(a) Show that

$$I^{\mu\nu}(k^\rho) = \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu + x^2 k^\mu k^\nu}{[p^2 + x(1-x)k^2]^2}.$$

[6 marks]

(b) Show further that

$$I^{\mu\nu}(k^\rho) = \int_0^1 dx \left(x^2 k^\mu k^\nu - \frac{x(1-x)}{d-2} k^2 g^{\mu\nu} \right) I_2(-x(1-x)k^2).$$

[5 marks]

(c) Show that in $d = 4 - \epsilon$ dimensions, the divergent piece is

$$I^{\mu\nu}(k^\rho) = \left(\frac{1}{3} k^\mu k^\nu - \frac{1}{12} g^{\mu\nu} k^2 \right) \frac{i}{16\pi^2} \frac{2}{\epsilon}.$$

[2 marks]

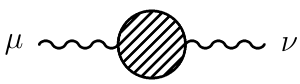
[Total 20 marks]

Please turn over

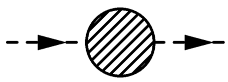
2. Scalar electrodynamics is a theory of an electrically charged (complex) scalar field ϕ. It has the bare Lagrangian

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_{B\nu} - \partial_\nu A_{B\mu})(\partial^\mu A_B^\nu - \partial^\nu A_B^\mu) \\ &\quad +(\partial_\mu \phi_B + ie_B A_{B\mu}\phi_B)^*(\partial^\mu \phi_B + ie_B A_B^\mu \phi_B) - m_B^2 \phi_B^* \phi_B - \frac{1}{4}\lambda_B(\phi_B^*\phi_B)^2.\end{aligned}$$

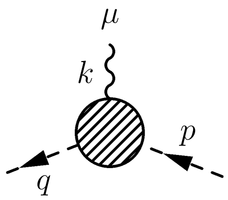
The superficially divergent 1PI correlation functions at one loop in naive perturbation theory (in the Lorentz gauge) are



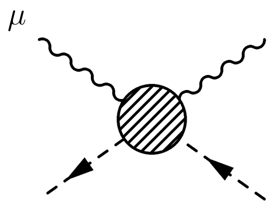
$$= -\frac{ie_B^2}{48\pi^2}(k^2 g^{\mu\nu} - k^\mu k^\nu)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),$$



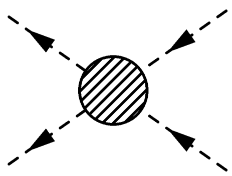
$$\begin{aligned}&= -\frac{3ie_B^2}{16\pi^2}k^2\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right) \\ &\quad +\frac{i\lambda_B}{16\pi^2}m_B^2\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),\end{aligned}$$



$$= ie_B(p^\mu + q^\mu) - \frac{3ie_B^3}{16\pi^2}(p^\mu + q^\mu)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),$$



$$= 2ie_B^2 g^{\mu\nu} - \frac{6ie_B^4}{16\pi^2}g^{\mu\nu}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),$$



$$= -i\lambda_B + \left(\frac{3ie_B^4}{4\pi^2} + \frac{5i\lambda_B^2}{32\pi^2}\right)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m_B^2} - \gamma + \text{finite}\right),$$

where γ is the Euler-Mascheroni constant.

(i) In renormalised perturbation theory, one defines

$$\begin{aligned} A_B^\mu &= Z_A^{1/2} A^\mu, \\ \phi_B &= Z_\phi^{1/2}\phi, \\ Z_\phi m_B^2 &= m_R^2 + \delta m^2, \\ Z_\phi^2 \lambda_B &= \lambda + \delta\lambda, \\ Z_\phi Z_A^{1/2} e_B &= Z_1 e, \\ Z_\phi Z_A e_B^2 &= Z_2 e^2. \end{aligned}$$

and $Z_X = 1 + \delta Z_X$ for any X. Write the Lagrangian in terms of renormalised fields, renormalised couplings and counterterms. Identify the piece that is treated as the free Lagrangian $\mathcal{L}_0$ in renormalised perturbation theory. [4 marks]

(ii) Draw all counterterm vertices. The Feynman rules for the two-point counterterms are

$$-\frac{i\delta Z_A}{(2!)}(k^2 g^{\mu\nu} - k^\mu k^\nu) \quad \text{and} \quad i(\delta Z_\phi k^2 - \delta m^2).$$

Write down the Feynman rules for the other counterterms. (Note that you can deduce them from the correlators shown above, by comparing the forms of different terms in the Lagrangian.) [4 marks]

(iii) What are the values of the same 1PI correlators as above in renormalised perturbation theory? [4 marks]

(iv) Explain how the values of the counterterms are determined in the $\overline{MS}$ scheme, and calculate them. [4 marks]

(v) Calculate the leading non-trivial terms in the beta functions of the two couplings

$$\beta_\lambda = \frac{d\lambda}{d\log M}, \quad \beta_e = \frac{de}{d\log M},$$

where M is the renormalisation scale. You should find

$$\beta_\lambda = \frac{1}{16\pi^2}\left(5\lambda^2 - 12\lambda e^2 + 24e^4\right), \quad \text{and} \quad \beta_e = \frac{e^3}{48\pi^2}.$$

[4 marks]

[Total 20 marks]

Please turn over

3. The action of the electromagnetic field can be written as

$$S = -\frac{1}{2}\int d^4q d^4k A_\mu(q)(2\pi)^4\delta(k+q)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)A_\nu(k).$$

(i) Explain how you would calculate the propagator $D_F^{\mu\nu}(k)$ defined by

$$\langle A^\mu(k)A^\nu(q)\rangle = (2\pi)^4\delta(k+q)D_F^{\mu\nu}(k),$$

and show that it cannot be done in this case. [3 marks]

(ii) Show that for a given gauge fixing function $G(A_\mu)$, the path integral can be written as

$$\int \mathcal{D}A_\mu \mathcal{O}[A_\mu]e^{iS[A_\mu]} \propto \int \mathcal{D}A_\mu \delta(G(A_\mu))\det\left(\frac{\delta G(A_\mu^\alpha)}{\delta\alpha}\right)\mathcal{O}[A_\mu]e^{iS[A_\mu]},$$

where $\mathcal{O}[A_\mu]$ is a gauge-invariant observable and A_μ^α is the gauge transform of A_μ,

$$A_\mu^\alpha = A_\mu + \frac{1}{e}\partial_\mu\alpha.$$

[5 marks]

(iii) Choosing $G(A_\mu) = \tilde{G}(A_\mu) - \omega(x)$, where $\omega(x)$ is an arbitrary function in spacetime and $\tilde{G}$ is independent of ω, show that you can write

$$\int \mathcal{D}A_\mu \mathcal{O}[A_\mu]e^{iS[A_\mu]} \propto \int \mathcal{D}A_\mu \det\left(\frac{\delta \tilde{G}(A_\mu^\alpha)}{\delta\alpha}\right)\mathcal{O}[A_\mu]e^{iS_\xi[A_\mu]},$$

where

$$S_\xi[A_\mu] = S[A_\mu] - \int d^4x \frac{1}{2\xi}\tilde{G}^2.$$

[4 marks]

(iv) Express the determinant as a path integral, and show that

$$\int \mathcal{D}A_\mu \mathcal{O}[A_\mu]e^{iS[A_\mu]} \propto \int \mathcal{D}A_\mu \mathcal{D}c^*\mathcal{D}c\mathcal{O}[A_\mu]e^{iS_{\rm gf}[A_\mu,c]},$$

where the full gauge-fixed action is

$$S_{\rm gf}[A_\mu,c] = S[A_\mu] - \int d^4x\left[\frac{1}{2\xi}\tilde{G}^2 + c^*\left(\frac{\delta\tilde{G}(A_\mu^\alpha)}{\delta\alpha}\right)c\right].$$

What are the properties of the new field c, which you introduced? [4 marks]

(v) Write down the gauge-fixed action $S_{\rm gf}$ for the gauge fixing function

$$\tilde{G}(A_\mu) = \partial_\mu A^\mu + \gamma A_\mu A^\mu,$$

where γ is a constant number. Draw the corresponding interaction vertices. Do they represent real, physical interactions? [4 marks]

[Total 20 marks]

4. The Lagrangian of the SU(N) Georgi-Glashow model is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \mathrm{Tr}[D_\mu, \Phi][D^\mu, \Phi] - m^2\mathrm{Tr}\Phi^2 - \lambda\mathrm{Tr}\Phi^4,$$

where $D_\mu = \partial_\mu + igA_\mu$ is the covariant derivative. The scalar field Φ, which is in the adjoint representation, and the SU(N) gauge field A_μ can be written as $\Phi = \Phi^a t^a$ and $A_\mu = A_\mu^a t^a$, where $a \in \{1, \ldots, N^2 - 1\}$.

(i) Draw the interaction vertices of this theory (in naive perturbation theory) and indicate which term in the Lagrangian each of them corresponds to. [3 marks]

(ii) Use the identities

$$\mathrm{Tr}t^a t^b = \frac{1}{2}\delta^{ab}, \quad [t^a, t^b] = if^{abc}t^c,$$

where f^{abc} are the SU(N) structure constants, to show that in terms of the field components, the scalar-gauge interaction terms are

$$\mathcal{L} = \ldots + gf^{abc}(\partial^\mu\Phi^a)\Phi^b A_\mu^c + \frac{g^2}{2}g^{\mu\nu}f^{ace}f^{bde}\Phi^a\Phi^b A_\mu^c A_\nu^d + \ldots.$$

[4 marks]

(iii) Derive the Feynman rules corresponding to these two terms and write them in their symmetric form. [4 marks]

(iv) Draw all one-loop 1PI diagrams that contribute to the scalar two-point correlator $\langle\Phi^a(k)\Phi^b(q)\rangle$. [3 marks]

(v) Using the propagators

$$\delta^{ab}D_F^{\mu\nu}(k) = \frac{-i\delta^{ab}}{k^2}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right],$$

and

$$\delta^{ab}D_F(k) = \frac{i\delta^{ab}}{k^2 - m^2},$$

and the identity $f^{acd}f^{bcd} = N\delta^{ab}$, show that the two diagrams that involve the gauge field correspond to the integrals

$$g^2 N\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\frac{d-1+\xi}{k^2},$$

and

$$-g^2 N\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\frac{(2p^\mu - k^\mu)(2p^\nu - k^\nu)}{k^2\left[(k-p)^2 - m^2\right]}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right].$$

[5 marks]

(vi) Explain why one of these integrals vanishes in dimensional regularisation. [1 mark]

[Total 20 marks]

End of examination paper