Exam

M.Sc. in Quantum Fields and Fundamental Forces

TP.4 — Advanced Field Theory

2:00 – 5:00, Friday May 9, 2008

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Answer **THREE** out of the four questions

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

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You may use the following results without proof:

- Loop integral in d dimensions (Minkowskian):

$$I_n(m^2) \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^n} = (-1)^n \frac{i m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

- Gamma functions: $z\Gamma(z) = \Gamma(z+1)$ and

$$\Gamma\left(-m + \frac{\epsilon}{2}\right) = \frac{(-1)^m}{m!}\left(\frac{2}{\epsilon} + \sum_{p=1}^{m} \frac{1}{p} - \gamma + O(\epsilon)\right) \quad \text{for integer } \ m \geq 0$$

- Feynman parameters:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[xa + (1-x)b]^2}$$

- Gaussian two-point function

$$\frac{\int d^N q \, q_i q_j e^{-\frac{1}{2} q^T M q}}{\int d^N q e^{-\frac{1}{2} q^T M q}} = \left(M^{-1}\right)_{ij} .$$

- Gaussian Grassmann integral

$$\int \left(\prod_i d\theta_i^* d\theta_i\right) \exp\left(-\theta_i^* B_{ij} \theta_j\right) = \det \mathbf{B}.$$

Question (1)

Consider a simple harmonic oscillator with action

$$S = \int_{-\infty}^{\infty} dt \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega_0^2 x^2 \right).$$

The amplitude for the oscillator to move from point x_a at time t_a to point x_b at time t_b is given by the path integral

$$U(x_a, x_b; t_b - t_a) \equiv \langle x_b; t_b | x_a; t_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t) e^{iS},$$

where $|x;t\rangle = \exp(i\hat{H}t)|x\rangle$ is an eigenstate of the coordinate operator $\hat{x}(t)$ with eigenvalue x. Furthermore, we have for a time-ordered operator

$$\langle x_b; t_b | \hat{\mathcal{O}} | x_a; t_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t)\, \mathcal{O}[x(t)] e^{iS}.$$

(a) Show that you can write the ground state $|0\rangle$ as

$$|0\rangle \propto \lim_{T\to\infty} |x_a; -T\rangle,$$

with an appropriate rotation of the time coordinate on the complex plane.

[4 marks]

(b) Show that the expectation value of any operator $\mathcal{O}$ in the ground state can be expressed in terms of path integrals as

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}x \mathcal{O}[x(t)] e^{iS}}{\int \mathcal{D}x e^{iS}}.$$

What are the boundary conditions in these integrals?

[5 marks]

(c) Calculate the two-point function $\langle x(\omega) x(\omega') \rangle$ of the Fourier transformed coordinate variable

$$x(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} x(t).$$

[6 marks]

(d) Take the Fourier transform of your result to show that you obtain the standard result for zero-point fluctuations

$$\langle x(t)^2 \rangle = \frac{1}{2m\omega_0}.$$

Pay particular attention to the poles of the integrand and the integration path on the complex plane.

[5 marks]

[TOTAL 20 marks]

Question (2)

Consider the d-dimensional (Minkowskian) integral

$$I^\mu(k^\nu, m) = \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu}{p^2(p+k)^2}.$$

(a) Explain why the integral is parallel to k^μ, i.e.,

$$I^\mu(k^\nu, m) = A(k^2, m)k^\mu,$$

where $A(k^2, m)$ is a scalar function.

[2 marks]

(b) Show that

$$I^\mu(k^\mu, m) = -\frac{k^\mu}{2} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2(p+k)^2}.$$

[6 marks]

(b) Show that

$$I^\mu(k^\nu, m) = -\frac{k^\mu}{2} \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{1}{[p^2 + x(1-x)k^2]^2}.$$

[5 marks]

(d) Show that for $d = 4 - \epsilon$ and $\epsilon \ll 1$, this becomes

$$I^\mu(k^\nu, m) = -\frac{k^\mu}{2} \frac{i}{16\pi^2} \left(\frac{2}{\epsilon} + \log \frac{4\pi}{-k^2} - \gamma - 2 \right) + O(\epsilon).$$

You can assume that $k^2 < 0$ and use the result

$$\int_0^1 dx \log x(1-x) = -2.$$

[7 marks]

[TOTAL 20 marks]

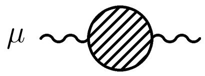
Question (3)

The bare Lagrangian of Quantum Electrodynamics is

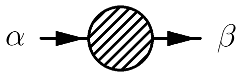
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu,$$

where the symbols have their usual meanings. In particular, ψ is the electron field, A_μ is the U(1) gauge field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

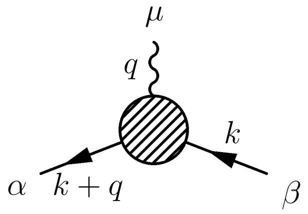
In *bare* perturbation theory, the superficially divergent one-particle irreducible correlation functions are



$$= -\frac{ie^2}{12\pi^2}(k^2 g^{\mu\nu} - k^\mu k^\nu)\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + O(k^4).$$



$$= \frac{ie^2}{16\pi^2}\left(\xi k\!\!\!/ + (3+\xi)m\right)_{\beta\alpha}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + O(k^2).$$



$$= -ie\gamma^\mu_{\alpha\beta} - \frac{ie^3\xi}{16\pi^2}\gamma^\mu_{\alpha\beta}\left(\frac{2}{\epsilon} + \log\frac{4\pi\mu^2}{m^2} - \gamma\right) + O(q). \qquad (3.1)$$

(a) Let us define the renormalised fields by $\psi = Z_\psi^{1/2}\psi_R$ and $A^\mu = Z_A^{1/2}A^\mu_R$, and renormalised parameters e_R, m_R and ξ_R by

$$\begin{aligned} Z_A^{1/2}Z_\psi e &= e_R + \delta e, \\ Z_\psi m &= m_R + \delta m, \\ Z_A/\xi &= Z_\xi/\xi_R. \end{aligned}$$

Let us further write $Z_A = 1+\delta Z_A$, $Z_\psi = 1+\delta Z_\psi$ and $Z_\xi = 1+\delta Z_\xi$. Write down the Lagrangian in terms of the renormalised fields and parameters.

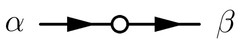
[3 marks]

(b) Identify the terms that are treated as interactions in *renormalised* perturbation theory.

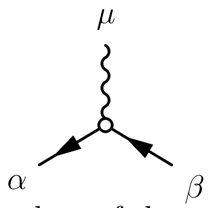
[3 marks]

(c) The Feynman rules for the counterterm vertices are

$$\leftrightarrow \quad -\frac{i}{2}\left[\delta Z_A\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) - \frac{\delta Z_\xi}{\xi_R}k^\mu k^\nu\right]$$



$$\leftrightarrow \quad i\left(\delta Z_\psi k\!\!\!/ - \delta m\right)_{\beta\alpha}$$



$$\leftrightarrow \quad -i\delta e\gamma^\mu_{\alpha\beta}$$

What are the values of the counterterms in the $\overline{\text{MS}}$ scheme?

[7 marks]

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(d) Write down the correlation functions (3.1) in renormalised perturbation theory as functions of renormalised parameters.

[4 marks]

(e) Show that to the leading order, the beta function of the theory is

$$\beta(e) \equiv M \left. \frac{\partial e_R}{\partial M} \right|_B = \frac{e^3}{12\pi^2}.$$

[3 marks]

[TOTAL 20 marks]

Question (4)

The Lagrangian of the SU(N) Yang-Mills field coupled to a complex scalar (Higgs) field in the fundamental representation is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\ &+\frac{g}{2} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\,\mu} A^{e\,\nu} \\ &+\partial_\mu \phi^*_i \partial^\mu \phi_i + i g t^a_{ij} A^a_\mu \left[(\partial^\mu \phi^*_i)\phi_j - \phi^*_i(\partial^\mu \phi_j)\right] + g^2 t^a_{ij} t^b_{jk} g^{\mu\nu} \phi^*_i A^a_\mu A^b_\nu \phi_j,\end{aligned}$$

where g is the gauge coupling constant, f^{abc} are the structure constants and t^a_{ij} are the group generators. The colour indices are $i, j \in \{1, \ldots, N\}$ and $a, b, c, d, e \in \{1, \ldots, N^2 - 1\}$.

(a) Find the dimensionalities of the fields ϕ_i and A^a_μ and the coupling g in d spacetime dimensions.

[3 marks]

(b) Draw all the interaction vertices, and indicate the terms in the Lagragian they correspond to. (You don't need to write down the corresponding Feynman rules.)

[4 marks]

(c) Explain what is meant by gauge fixing and show that it is necessary for deriving the gauge field propagator.

[5 marks]

(d) Besides the gauge fixing term $\mathcal{L}_{gf} = -(\partial^\mu A^a_\mu)^2/2\xi$, gauge fixing also introduces an extra factor

$$\det\left[i\partial^\mu(\delta^{ab}\partial_\mu + g f^{abc} A^c_\mu)\right]$$

in the path integral. Show how this determinant can be represented by a ghost field c. Write down the new terms that appear in the Lagrangian, and draw the corresponding new vertex/vertices.

[3 marks]

(e) Draw all the one-particle irreducible one-loop diagrams that contribute to the gluon three-point functions $\langle A^a_\mu A^b_\nu A^c_\rho \rangle$.

[5 marks]

[TOTAL 20 marks]